The most degenerate $H$ and $I$-surfaces have LMHS of type I,

which means they are of Hodge-Tate type — only $I^{\text{top}} \neq 0$. In this case it is

$$h^0 \rightarrow \times (\mathcal{O}(-2)) \rightarrow \times (\mathcal{O}(-2)) \rightarrow \times \mathcal{O}$$

The $\text{Gr}(\text{LMHS})$ is rigid; the corresponding period domain is just points (0-dimensional). However (11) has parameters given by the extension data. Recall that the form $G$ induces a polarization on the LMHS (2). Using
the polarization we have a duality between the first and second pieces of (2). Hence the parameters are in $\text{Ext}^2(\Omega^1, \Omega^1(1,2))$.

The 2nd order parameters are given by the "extension upon extension" data in (2) if this is only well-defined when the 1st order parameters vanish.

As a general principle, on a boundary component in moduli, a fibre of the extended period map to the period domains for $\mathcal{M}(\text{LMHS})$ maps to extension data. Geometrically, if we have an equi-singular deformation such that
the map to $\text{Gr} (\text{MHS's})$ is constant, then there is an induced map to the extension data in the MHS's. The simplest example of this is furnished by the moduli of the degenerate genus 2 curve

with normalization given by a $\mathbb{P}^1$ with 4 marked points $(p_1, q_1; p_2, q_2)$.

The moduli of the equisingular deformations is given by the cross-ratio of the 4 points. If $\omega_R$ is the logarithmic differential on $\mathbb{P}^1$ with (opposite) residues at $p_1, q_1$, then
the "period matrix" of the LMHS is given by

$$\int \frac{w_{\gamma_2, \delta_2}}{\delta_2} = \pm \int \frac{w_{\gamma_2, \delta_2}}{\delta_2}$$

where $\delta_2 = \gamma_2 - \varphi_2$.

Of particular interest are the 0-dimensional components of moduli, or rigid degenerate variations. For curves these correspond to trivalent graphs, the simplest one of which corresponds to the curve

[Diagram]
The "building blocks" of the curves corresponding to the weight graphs are the rigid variety \(\mathbb{P}^3\) (or \(\mathbb{P}^2\)). It is of interest to find analogues of this for at least some surfaces. Here we shall carry this for \(I\) and \(H\)-surfaces (I don't know the analogue of

\[
\begin{array}{c}
\end{array}
\]

for \(H\) surfaces). With the notations to be explained below we shall prove the
Theorem: There exists a (unique?) rigid $E$-surface $\mathcal{X}$ whose LMHS is of type $V$.

Its normalization $(\widetilde{\mathcal{X}}, \widetilde{\Theta}) = (\mathcal{X}^2, \mathcal{P}_2, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4)$

Here we recall that a smooth $E$-surface $\mathcal{X}$ is a minimal surface with

- $\rho_2(\mathcal{X}) = 2$, $\rho(\mathcal{X}) = 0$
- $K^2 = 1$
- $\mathcal{X}$ is of general type

Then

$$\phi : \mathcal{X} \to Q_0 = P_3$$

with branch locus $P + \Gamma \cap Q_0$, where $V \subset 10^2(5)$ is a general quintic. The vertex $P$ is the image of the base point of $|K^2\mathcal{X}|$, and the curves $C \subset |K^2\mathcal{X}|$ map via $\phi$ as double covers of the rulings of $Q_0$ branched at $P + \Gamma \cap Q_0$ (intersections with $V$).
The idea is to construct $X_0$ via the operation: Given 2 disjoint curves $C_1, C_2$ on a smooth surface $Y$ and an isomorphism $C_2 \cong C_2$, we define

$I/f = \{ \text{surface with a double curve} \}$ obtained by identifying $C_2$ and $C_2$.

If $E$ is a smooth curve meeting $C_1, C_2$ and with $f(E \cap C_i) = E \cap C_2$, then $E$ induces a curve on $Y/f$. If $w$ is a form in $H^0(S^2, (C_1 + C_2 + E))$, then $w$ induces a form on $Y/f$.

$\text{Res}_{C_1} w = -f^*(\text{Res}_{C_2} w)$

and if the double residues of $w$ at $C_2 \cap E$ are the same as those of $C_2 \cap E$. 
Following Liu–Rolleffski we start with $\mathbb{P}^2$ and 4 lines in general position

Then $H^0(\mathbb{P}^2, (L_1 + \cdots + L_4)) \cong \mathbb{C}^4$. We want to construct a quotient surface by identifying $L_3 \leftrightarrow L_4$, $L_2 \leftrightarrow L_3$. There are two issues in trying to do this:

(i) the lines are not disjoint, so we have to blow up the $P_{ij} = L_i \cap L_j$

(ii) the identifications $(\mathbb{P}^2, 3\text{ points}) \cong (\mathbb{P}^2, 3\text{ points})$ require labelling the identifications of the 3 points.

First we blow up $\mathbb{P}^2$ at the $P_{ij}$ to obtain exceptional curves $E_{ij}$. For the next step...
Here "a" on L_1 means \( P_{1a} \), "1" on L_2 means \( P_{21} \) and the line \( a \longrightarrow \) means \( E_{12} = E_{21} \). The identifications are:

\[
\begin{align*}
L_1 & \quad L_2 \\
1 & \leftrightarrow 2 \\
2 & \leftrightarrow 3 \\
3 & \leftrightarrow 1 \\
1 & \leftrightarrow 4 \\
4 & \leftrightarrow 3
\end{align*}
\]
When $L_3$ is identified with $L_4$ we obtain

and then identifying $L_2$ and $L_3$ gives the double curve

One may identify $|K_{\mathcal{X}_0}|$ with the pencil of lines thru $P$ in the figure

Thus $K_{\mathcal{X}_0}^2 = 1$
Any construction using tiles, i.e. collections of $(\mathbb{P}^3, 4$ lines) glued together, has $K_{\mathbb{Z}_0}^3 = \# \text{ of tiles}$.

For the $H$-surface there are $n$ constructions, one of which is

![Diagram](image)

It is clear that these constructions are rigid. One may suspect that
\[(\star)\quad T_{\Sigma_0} \text{Def}^a (\Sigma_0) = 0 \quad (\text{like } H^2(\Sigma_0) = 0)\]

- for the $I$-surface, any degeneration where the LMHS has type $I$
  and where $(\star)$ is satisfied must be the one given above (and
  maybe the analogous result for $H$-surfaces)

Regarding the first point $(\star)$, the bi-canonical model of $\Sigma_0$ maps to
$\mathbb{P}(1, 3, 2) \cong \mathbb{G}_0 \subset \mathbb{P}^3$ with branch
locus $D = [0, 0, 2]$ and the quintic
\[(x = 0) \cap (y(\lambda x-y)^2) = (\lambda x-y)^2)\]

This quintic is rigid (any degeneration...
deformation is induced by \( \text{Aut}(\mathbb{P}(2,2,2)) \), so one has that

\[
\text{if an equisingular deformation of } X_0 \text{ induces one of the branches of its bi-canonical model, then (2) holds.}
\]

Returning to the central problem (cf. page 117), the story is in its earliest stage. We consider a HSBA degeneration

\[
X_0 \to X
\]

and make the following assumptions:

(i) \( X \) is irreducible and the desingularization \( \tilde{X} \) is regular,

(ii) the singularities of \( X \) are only those that affect the LMHS.
(iii) the LMHS is of type I
(iv) $\Sigma_2$ is either an $H_I$-surface

Theorem: The only singularities that can appear are

(a) simple elliptic
(b) those of the form $(\mathcal{X}, D,E)$ where $D$ is a smooth, irreducible curve of genus $2$
(c) two disjoint curves $C_1, C_2$ with $g(C_1) = 2$ and $C_2$ with $g(C_2) = 2$ joined to form a double curve on $\mathcal{X}$

For $\mathcal{X}$ an $I$-surface only the first can occur. For $\mathcal{X}$ an $H$-surface all the possibilities can and do occur.
Discussion: The main assumption is (ii).

One might conceptually separate the classification question into 3 parts

(a) analyze the classification when the singularities do not affect
the LHS - e.g., assume that the monodromy in $\mathcal{X} \to A^1$ is a finite

(b) make the assumption (ii)

(c) combine (a) and (b)

The assumption that $\mathcal{X}$ is regular can
most likely be removed

As will be seen from what follows,
the main missing ingredient is to use
the hypotheses to analyze the full
pluri-canonical ring; here one needs
the usual tools of a vanishing theorem and (relatedly) the Riemann-Roch theorem.

The above result is of course far from definitive; it is offered as a step towards what one would like to have.

Proof (Step 2): As background, not presented here, is an analysis of the semi-stable-reduction for the singularities in Kollár's list, as well as for the singularities that arise from the LR-type construction above (and those are related in that degenerations of type (3,3,4) in Kollár's list may be treated by the method used for the LR constructions).
With this in mind, the contributions to $I_{2,0} \cong C$ in the LMHS are:

- 1 for each simple elliptic singularity (which is not the limit of a base point of $\mathbb{I}_{2,0}$, a case we will ignore)

- $g(\mathcal{D}) - g(\mathcal{D}/\mathcal{D})$

- $g(C_2)$

**Step 2:** If $C_1, \ldots, C_n$ are the disjoint curves that arise in the desingularization $\tilde{\mathbb{X}}$ of $\mathbb{X}$ so that

$$k_{\tilde{\mathbb{X}}} = k_{\mathbb{X}} + C_1 + \ldots + C_n,$$

then

$$k_{\tilde{\mathbb{X}}}^2 = (k_{\mathbb{X}} + C_1 + \ldots + C_n)^2 = k_{\mathbb{X}}^2 + \sum_{i} 2k_{\tilde{\mathbb{X}}} \cdot C_i \cdot C_i$$

so that

$$2k_{\tilde{\mathbb{X}}} \cdot C_i \cdot C_i = \text{contribution of } C_i \text{ to } k_{\tilde{\mathbb{X}}}^2 - k_{\mathbb{X}}^2.$$
The contributions are:

- \( K_{\tilde{X}} \cdot E = -\, E^2 > 0 \) (using adjunction)
- \( 2K_{\tilde{X}} \cdot \tilde{B} + \tilde{B}^2 = K_{\tilde{X}} \cdot \tilde{B} + 2g(\tilde{B}) - 2 \)
- \( 2K_{\tilde{X}} \cdot (C_2 + C_3) + C_2^2 + C_3^2 + 2C_2 \cdot C_3 = \)

\( K_{\tilde{X}}(C_2 + C_3) + 4g(C_2) - 4 + 2C_2 \cdot C_3 \)

**Step 3:** For any smooth curve \( C = \tilde{X} \), since \( \tilde{X} \) is assumed regular,

\[ 0 \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow \mathcal{O}_{\tilde{X}}(C) \rightarrow \mathcal{O}_C(C) \rightarrow 0 \]

gives

\( h^0(\mathcal{O}_{\tilde{X}}(C)) = h^0(\mathcal{O}_C(C)) + 2 \)

If \( h^0(\mathcal{O}_C(C)) \geq 0 \) this implies that \( C \) moves in a pencil. Using \( \mu_g(\tilde{X}) = 2 \) we have

\[ K_{\tilde{X}} \cdot C \geq 0 \]

If
$K_{\tilde{X}} \cdot \tilde{E} < 0$, then $\tilde{D}^2 > 2g(\tilde{E})-2$, which implies that $h^0(\mathcal{O}_{\tilde{X}}(\tilde{D})) > 0$ and hence

$$K_{\tilde{X}} \cdot \tilde{E} \geq 0$$

Similarly, $K_{\tilde{X}} \cdot \mathcal{C}_2 < 0$ gives

$$C_2^2 > 2g(\mathcal{C}_2) - 2 \Rightarrow h^0(\mathcal{O}_{\tilde{X}}(C_2)) > 0 \Rightarrow h^0(\mathcal{O}_{\tilde{X}}(\mathcal{C}_2)) > 0$$

Thus we have in cases (a), (c)

$$K_{\tilde{X}} \cdot \tilde{E} \geq 0, \ K_{\tilde{X}} \cdot \mathcal{C}_2 > 0.$$  

**Step 4**: We now restrict to the case

$K_{\tilde{X}} \cdot \tilde{E} = 1$ of $I$-surfaces. Referring to Step 2, the contribution of $\tilde{D}$ to $K_{\tilde{X}}^2 - K_{\tilde{X}}^2$ is

$$K_{\tilde{X}} \cdot \tilde{D} + 2g(\tilde{E}) - 2 \geq 2g(\tilde{E}) - 2 = 2.$$  

Since $\tilde{X}$ is assumed to be regular, by the classification of surfaces $K_{\tilde{X}}^2 \geq 0$. From this we have
\( K_B^2 \geq 2 \), so that the possibility of 
(\( \tilde{S}, \tilde{S}, 2 \)) is ruled out. For (c) we have 
\[ 2c_2 \cdot c_2 + K_\tilde{B} \cdot (c_2 + c_3) + 4g(c_2) - 4 \geq 1 \]
Thus \( c_2 \rightarrow c_2 \) is ruled out (for an 
I-surface).

Step 5: We are reduced to the case of 
one simple elliptic singularity, then 
\[ K_B^2 = K_\tilde{B}^2 + K_\tilde{B} \cdot E, \quad K_\tilde{B} \cdot E = -E^2 > 0 \]
which gives 
\[ 1 = K_B^2 \Rightarrow K_\tilde{B}^2 = 0 \text{ and } K_\tilde{B} \cdot E = 2. \]
By the classification of surfaces, the conditions
\[ \tilde{S} \text{ regular, } K_\tilde{S}^2 = 0, \quad K_\tilde{B} \cdot E = 2, \quad g_\tilde{B}(\tilde{S}) = 1 \]
\[ \tilde{S} \text{ is an elliptic surface} \]
(the possibility \( m \mathcal{K} \mathcal{X} = \mathcal{O} \mathcal{X} \) for some \( m > 1 \) is ruled out by \( \mathcal{K} \mathcal{X} \cdot \mathcal{E} > 0 \)).

**Step 6.** We have \( \mathcal{X} \xrightarrow{\pi} \mathbb{P}^2 \) with

\[
\pi^* \mathcal{O}_\mathbb{P}^2 = \mathcal{L}^{-1}, \quad \mathcal{L} = \text{line bundle on } \mathbb{P}^2,
\]

say \( \deg \mathcal{L} = d \).

This gives

\[
\mathcal{K}_\mathcal{X} = \pi^*(\mathcal{K}_\mathbb{P}^2 \mathcal{O}_\mathcal{L}) + \sum (m_i - 1) \mathcal{F}_i
\]

where the \( m_i \mathcal{F}_i \) are the multiple fibres.

Moreover, the restriction of \( \pi \) to \( E \)

\[
E \xrightarrow{\pi} \mathbb{P}^2
\]

is a branched cover. Using

- \( \mathcal{L} = \mathcal{K}_\mathcal{X} \cdot \mathcal{E} = (d - 2) \deg(\mathcal{L}|_\mathcal{E}) + \sum (m_i - 1) \mathcal{E} \cdot \mathcal{F}_i \)
- \( \mathcal{E} \cong H^2(\mathcal{O}_\mathcal{X}^d) \cong H^2(\pi^* \mathcal{O}_\mathbb{P}^2) = H^2(\pi^* \mathcal{O}_\mathbb{P}^d) \)
we have \(-d = -2\) or
\[d = 2\]
This gives
\[1 = \frac{K_X}{\Delta} \cdot E = \sum (m_k - 2) E \cdot F_k\]
from which it follows that
- There is one double fibre \(F_2\)
- \(\deg (\pi|_E) = 2\)
- \(E \cdot F_2 = \frac{1}{2} (E \cdot (\text{general fibre})) = \frac{1}{2} \deg (\pi|_E)\)

**Summarizing:**
- \(X\) is an elliptic surface
- \(p_g(X) = 1\), so \(\deg L = 2\)
- \(\deg (\pi|_E) = 2\)
- \(K_X = F_2\), where \(2F_2\) is the unique multiple fibre
Does this situation occur? At this point the answer to this question is not known. It seems that one needs to determine what at least the biophysical map will look like.
Further example of non-classical phenomena

. extension data analysis gives that there is an $H$-surface at $\bullet$ that is not the further degeneration of an $H$-surface where $LMHS$ is of type $I$ - this doesn't happen classically.

Can Hodge theory be used in other ways than the $LMHS$ to detect/analyze singularities?
For example suppose the monodromy is trivial. For curves there are the compact degenerations

\[ \begin{array}{c}
\bullet & \rightarrow & \bullet \\
\circ & \rightarrow & \circ
\end{array} \]

These are detected by the PHS becoming a direct sum \((\mathbb{C}/\mathbb{Z})\). Radu has found examples where

\[ \text{H-surface} \rightarrow \begin{array}{c}
\bullet \\
\circ
\end{array} \]

One of the Dolgachev singularities on the curve double cover.

What might one be able to do if enough were understood about the connection between HT and moduli?
Recall that Bob Friedman proved the Torelli theorem for $K3$'s knowing local Torelli together with an analysis of the $AT$ at a generic boundary point in $\mathcal{M}(D)$. Here $K3$'s (polarized) are lurking in type I degenerations and so one might be able to bootstrap on the Torelli for $K3$'s using a Friedman type argument?