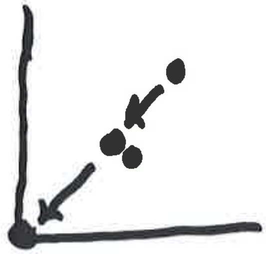
- The most degenerate H and I-surfaces have LMHS of type V



$h^{2,0}=2$

which means they are of <u>Hodge-Tate type</u> - only $I^{p,p}\neq 0$. In this case it is

(*) $$\overset{2}{\oplus}(\mathbb{Q}(-2)) \xrightarrow{N} \overset{2}{\oplus}(\mathbb{Q}(-1)) \xrightarrow{N} \overset{2}{\oplus}\mathbb{Q}$$

$$H^0 \qquad h^0 = h^{1,1}_{\text{prim}} - 2$$

The $Gr(\text{LMHS})$ is rigid; the corresponding period domain is just points (0-dimensional). However (*) has parameters given by the <u>extension data</u>. Recall that the form Q induces a polarization on the LMHS (*). Using the polarization we have a duality between the first and second pieces of ($*$). Hence the 1st order parameters are in

$$\mathrm{Ext}^1_{MHS}\left(\overset{2}{\oplus}\,\mathbb{Q},\ \overset{2}{\oplus}\,\mathbb{Q}(-2)\right)$$

The 2nd order parameters are given by the "extension upon extension" data in ($*$); this is only well-defined when the 1st order parameters vanish.

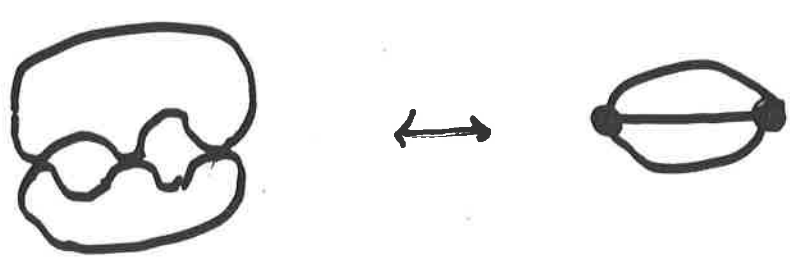
- As a general principle, on a boundary component in moduli, a fibre of the extended period map to the period domains for Gr (LMHS's) maps to extension data. Geometrically, if we have an equi-singular deformation such that the map to Gr(MHS's) is constant, then there is an induced map to the extension data in the MHS's. The simplest example of this is furnished by the moduli of the degenerate genus 2 curve

with normalization  given by a $\mathbb{P}^1$ with 4 marked points $\{p_1, q_1; p_2, q_2\}$. The moduli of the equisingular deformations is given by the cross-ratio of the 4 points. If ω_{p_i, q_i} is the logarithmic differential on $\mathbb{P}^1$ with (opposite) residues at p_i, q_i, then the "period matrix" of the LMHS is given by

$$\int_{\gamma_1} \omega_{p_2, q_2} = \pm \int_{\gamma_2} \omega_{p_1, q_1}$$

where $\partial \gamma_i = p_i - q_i$.

- Of particular interest are the 0-dimensional components of moduli, or rigid degenerate varieties. For curves these correspond to trivalent graphs, the simplest one of which corresponds to the curve



The "building blocks" of the curves corresponding to trivalent graphs are the rigid variety $(\mathbb{P}^1 \setminus \{0,1,\infty\})$. It is of interest to find analogues of this for at least some surfaces. Here we shall carry this for I and H-surfaces (don't know the analogue of

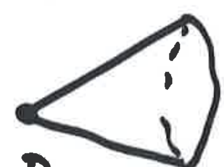
for K3 surfaces). With the notations to be explained below we shall prove the

<u>Theorem</u>: <u>There exists a (unique?) rigid I-surface X_0 whose LMHS is of type V. Its normalization $(\tilde{X}, \tilde{D}) = (\mathbb{P}^2; L_1, L_2, L_3, L_4)$</u>

Here we recall that a smooth I-surface X is a minimal surface with

- $p_g(X) = 2$, $q(X) = 0$
- $K_X^2 = 1$
- X is of general type

Then

$$\varphi_{2K_X} : X \xrightarrow{2:1} Q_0 =$$  $\subset \mathbb{P}^3$

P

with branch locus $P + V \cap Q_0$ where $V \in |\mathcal{O}_{\mathbb{P}^3}(5)|$ is a general quintic. The vertex P is the image of the base point of $|K_X|$, and the curves $C \in |K_X|$ map via φ_{K_C} as double covers of the rulings of Q_0 branched at P + (intersections with V)

The idea is to construct X_0 via the operation: Given 2 disjoint curves C_1, C_2 on a smooth surface Y and an isomorphism $C_1 \xrightarrow{f} C_2$, we define

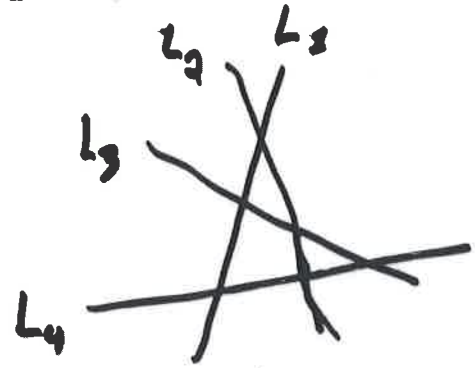
$$Y/f = \left\{ \begin{array}{l} \text{surface with a double curve} \\ \text{obtained by identifying } C_1 \text{ and } C_2 \end{array} \right\}$$

If E is a smooth curve meeting C_1, C_2 and with $f(E \cap C_1) = E \cap C_2$, then E induces a curve on Y/f. If ω is a form in $H^0(\Omega^2_Y(C_1 + C_2 + E))$, then ω induces a form on Y/f iff

$$\operatorname{Res}_{C_1} \omega = -f^*(\operatorname{Res}_{C_2} \omega)$$

and if the double residues of ω at $C_2 \cap E$ are $-$ those at $C_1 \cap E$.

Following Liu-Rollenske we start with $\mathbb{P}^2$ and 4 lines in general position



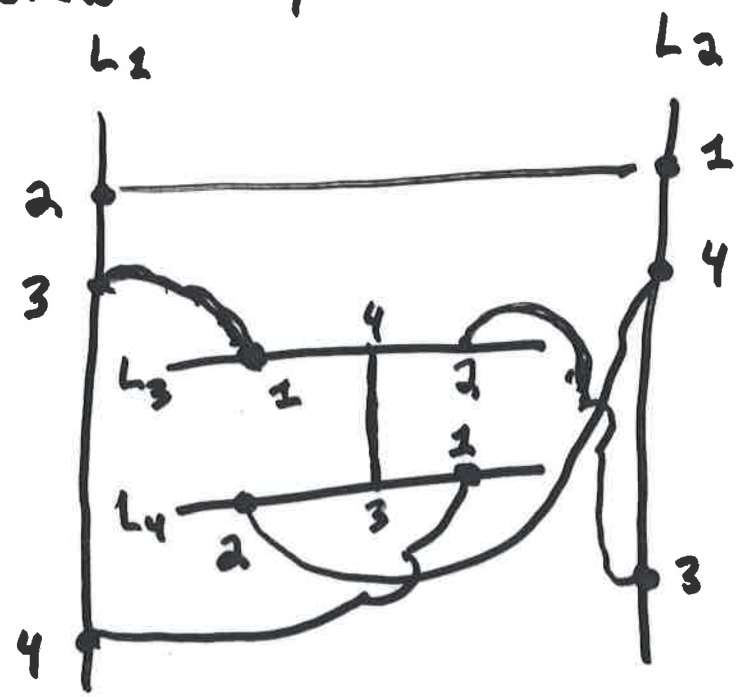
Then $H^0(\Omega^2_{\mathbb{P}^2}(L_1+\dots+L_4)) \cong \mathbb{C}^3$, we want to construct a quotient surface by identifying $L_3 \leftrightarrow L_4$, $L_1 \leftrightarrow L_2$. There are two issues in trying to do this:

(i) the lines are not disjoint, so we have to blow up the

$$P_{ij} = L_i \cap L_j$$

(ii) the identifications $(\mathbb{P}^1, 3 \text{ points}) \cong (\mathbb{P}^1, 3 \text{ points})$ require labelling [redacted] the identifications of the 3 points

First we blow up $\mathbb{P}^2$ at the P_{ij} to obtain exceptional curves E_{ij}. For the next step we draw the picture

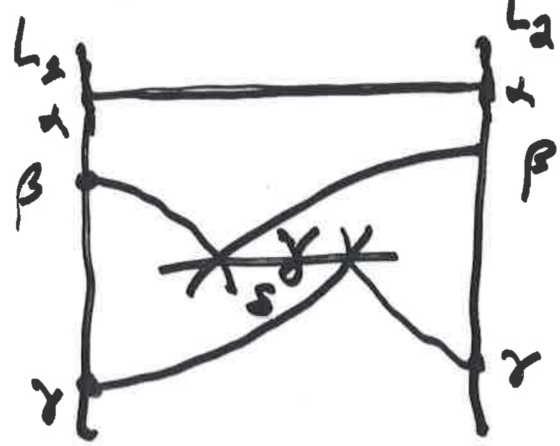


Here "2" on L_1 means P_{12}, "1" on L_2 means P_{21} and the line 2 •——• 1 means $E_{12} = E_{21}$. The identifications are

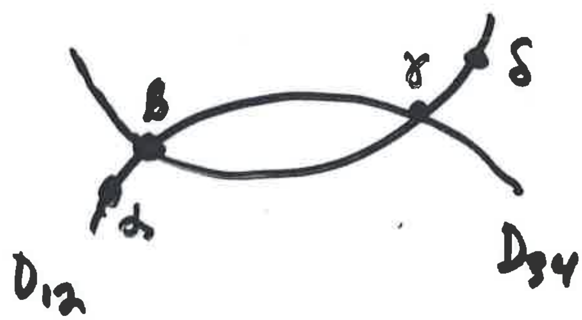
L_3		L_4
1	$\longleftrightarrow$	2
2	$\longleftrightarrow$	1
4	$\longleftrightarrow$	3

L_1		L_2
2	$\longleftrightarrow$	1
3	$\longleftrightarrow$	4
4	$\longleftrightarrow$	3

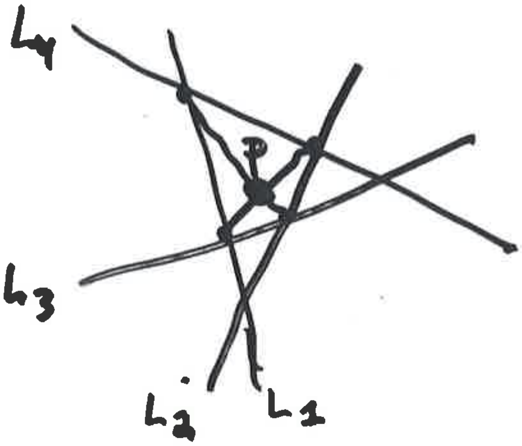
When L_3 is identified with L_4 we obtain



and then identifying L_1 and L_2 gives the double curve



One may identify $|K_{X_0}|$ with the pencil of lines thru p in the figure



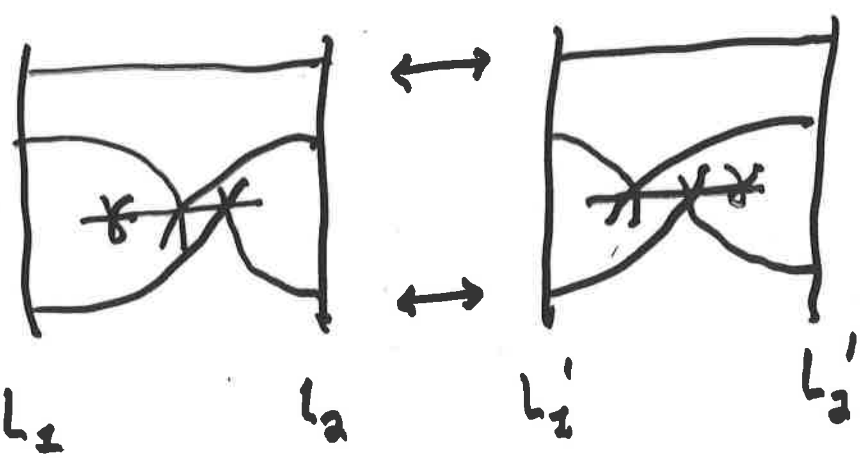
Thus $K_{X_0}^2 = 1$

VIII. 11

- Any construction using <u>tiles</u>, i.e. collections of ($\mathbb{P}^2$, 4 lines) glued together, has

$$K_{Z_0}^2 = \# \text{ of tiles}$$

For the H-surface there are 2 constructions, one of which is



- It is clear that these constructions are rigid. One may suspect that

(*) $\quad - \ T_{X_0} \mathrm{Def}^{sa}(X_0) = 0 \qquad$ (like $H^1(\Theta_X) = 0$)

- for the I-surface, <u>any</u> degeneration where the LMHS has type V and where (*) is satisfied must be the one given above (and maybe the analogous result for H-surfaces)

Regarding the first point (*), the bi-canonical model of X_0 maps to $\mathbb{P}(1,1,2) \underset{2}{\cong} Q_0 \subset \mathbb{P}^3$ with branch locus $P = [0,0,1]$ and the quintic

$$(y=0) + (y(x_0-y)^2) + ((x_1-y)^2)$$

This quintic is rigid (any ~~...~~ equisingular deformation is induced by $\mathrm{Aut}(\mathbb{P}(1,2,2))$,

so one has that

<u>if an equisingular deformation of X_0 induces one of the branch locus of its bi-canonical model, then (*) holds</u>

$\langle o \rangle$

Returning to the central problem (cf. page VII.30), the story is in its earliest stage. We consider a KSBA degeneration

$$X_y \to X$$

and make the following assumptions

(i) X is irreducible and the desingularization $\tilde{X}$ is regular

(ii) the singularities of X are only those that affect the LMHS

(iii) the LMHS is of type I

(iv) X_η is either an H or I-surface

<u>Theorem: The only singularities that can appear are</u>

(a) <u>simple elliptic</u>

(b) <u>those of the form $(\tilde{X}, \tilde{D}, \Sigma)$ where $\tilde{D}$ is a smooth, irreducible curve of genus ≥ 2</u>

(c) <u>two disjoint curves C_1, C_2 with $g(C_i) \geq 2$ and $C_1 \cong C_2$ joined to for a double curve on X</u>

<u>For X an I-surface only the first can occur. For X an H-surface all the possibilities can and do occur</u>

<u>Discussion</u>: • the main assumption is (ii).

• One might conceptually separate the classification question into 3 parts

(a) analyze the classification when the singularities do <u>not</u> effect the LMHS-eq, assume that the monodromy in $\mathcal{X}^* \to \Delta^*$ is a finite group, or even trivial

(b) make the assumption (ii)

(c) combine (a) and (b)

• The assumption that $\tilde{X}$ is regular can most likely be removed

• As will be seen from what follows, the main missing ingredient is to use the hypotheses to analyze the full pluri-canonical ring; here one needs the usual tools of a vanishing theorem and (relatedly) the Riemann-Roch theorem

- The above result is of course far from definitive; it is offered as a step towards what one would like to have

<u>Proof (Step 1)</u>: As background, not presented here, is an analysis of the semi-stable-reduction for the singularities in Kollár's list, as well as for the singularities that arise from the LR-type construction above (and these are related in that SSR reduction for degenerations of type (3.3.4) in Kollár's list may be treated by the method used for the LR constructions).

With this in mind, the contributions to $I^{1,0} \cong \mathbb{C}$ in the LMHS are

- 1 for each simple elliptic singularity (which is not the limit of a base point of $|K_{X_\eta}|$, a case we will ignore)
- $g(\tilde{D}) - g(\tilde{D}/\tau)$
- $g(C_1)$

Step 2: If $C_1, \dots, C_n$ are the disjoint curves that arise in the desingularization $\tilde{X}$ of X so that

$$K_X = K_{\tilde{X}} + C_1 + \dots + C_n ,$$

then

$$K_X^2 = (K_{\tilde{X}} + C_1 + \dots + C_n)^2 = K_{\tilde{X}}^2 + \sum_i (2K_{\tilde{X}} \cdot C_i + C_i^2)$$

so that

$$\boxed{2K_{\tilde{X}} \cdot C_i + C_i^2 = \text{contribution of } C_i \text{ to } K_X^2 - K_{\tilde{X}}^2}$$

The contributions are

$-K_{\tilde{X}} \cdot E = -E^2 > 0$ (using adjunction)

$-2K_{\tilde{X}} \cdot \tilde{D} + \tilde{D}^2 = K_{\tilde{X}} \cdot \tilde{D} + 2g(\tilde{D}) - 2$

$-2K_{\tilde{X}} \cdot (C_1 + C_2) + C_1^2 + C_2^2 + 2C_1 \cdot C_2 =$

$K_{\tilde{X}}(C_1 + C_2) + 4g(C_1) - 4 + 2C_1 \cdot C_2$

Step 3: For any smooth curve $C \subset \tilde{X}$, since $\tilde{X}$ is assumed regular

$$0 \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{\tilde{X}}(C) \to \mathcal{O}_C(C) \to 0$$

gives

$$h^0(\mathcal{O}_{\tilde{X}}(C)) = h^0(\mathcal{O}_C(C)) + 1$$

If $h^0(\mathcal{O}_C(C)) > 0$ this implies that C moves in a pencil. Using $p_g(\tilde{X}) = 1$ we have

$$K_{\tilde{X}} \cdot C \geq 0$$

If

$K_{\tilde{X}} \cdot \tilde{D} < 0$, then $\tilde{D}^2 > 2g(\tilde{D}) - 2$, which implies that $h^0(\mathcal{O}_{\tilde{D}}(\tilde{D})) > 0$ and hence

$$K_{\tilde{X}} \cdot \tilde{D} \geq 0$$

Similarly, $K_{\tilde{X}} \cdot C_1 < 0$ gives

$$C_1^2 > 2g(C_1) - 2 \Rightarrow h^0(\mathcal{O}_{C_1}(C_1)) > 0 \Rightarrow K_{\tilde{X}} \cdot C_1 \geq 0$$

Thus we have for cases (b), (c)

$$K_{\tilde{X}} \cdot \tilde{D} \geq 0, \quad K_{\tilde{X}} \cdot C_1 \geq 0.$$

<u>Step 4</u>: We now restrict to the case $K_{X_{\eta}}^2 = 1$ of I-surfaces. Referring to Step 2, the contribution of $\tilde{D}$ to $K_X^2 - K_{\tilde{X}}^2$ is $K_{\tilde{X}} \cdot \tilde{D} + 2g(\tilde{D}) - 2 \geq 2g(\tilde{D}) - 2 \geq 2$. Since $\tilde{X}$ is assumed to be regular, by the classification of surfaces $K_{\tilde{X}}^2 \geq 0$. From this we have

$K_X^2 \geq 2$, so that the possibility of $(\tilde{X}, \tilde{D}, \tau)$ is ruled out. For (c) we have (using $g(C_1) \geq 2$)

$$2C_1 \cdot C_2 + K_{\tilde{X}} \cdot (C_1 + C_2) + 4g(C_1) - 4 > 1$$

Thus $C_1 \longleftrightarrow C_2$ is ruled out (for an I-surface).

<u>Step 5</u>: We are reduced to the case of one simple elliptic singularity. Then

$$K_X^2 = K_{\tilde{X}}^2 + K_{\tilde{X}} \cdot E, \quad K_{\tilde{X}} \cdot E = -E^2 > 0$$

which gives

$$1 = K_X^2 \Rightarrow K_{\tilde{X}}^2 = 0 \text{ and } K_{\tilde{X}} \cdot E = 1.$$

By the classification of surfaces, the conditions

$$\tilde{X} \text{ regular}, \ K_{\tilde{X}}^2 = 0, \ K_{\tilde{X}} \cdot E = 1, \ p_g(\tilde{X}) = 1$$
$$\Downarrow$$
<u>$\tilde{X}$ is an elliptic surface</u>

(the possibility $mK_{\tilde{X}} = \mathcal{O}_{\tilde{X}}$ for some $m > 1$ is ruled out by $K_{\tilde{X}} \cdot E > 0$).

<u>Step 6</u>: We have $\tilde{X} \xrightarrow{\pi} \mathbb{P}^1$ with

$$R^1_\pi \mathcal{O}_{\tilde{X}} \overset{\text{defn}}{=} L^{-1}, \quad L = \text{line bundle on } \mathbb{P}^1, \text{ say } \deg L = d$$

This gives

$$K_{\tilde{X}} = \pi^*(K_{\mathbb{P}^1} \otimes L) + \sum_i (m_i - 1) F_i$$
$$= \pi^*(h^{d-2}) + \sum_i (m_i - 1) F_i$$

where the $m_i F_i$ are the multiple fibres.

Moreover the restriction of π to E

$$E \xrightarrow{\pi} \mathbb{P}^1$$

is a branched cover. Using

- $1 = K_{\tilde{X}} \cdot E = (d-2) \deg(\pi|_E) + \sum_i (m_i - 1) E \cdot F_i$
- $\mathbb{C} \cong H^2(\mathcal{O}_{\tilde{X}}) \cong H^1(R^1_\pi \mathcal{O}_{\tilde{X}}) = H^1(\pi^* h^{-d})$

we have $-d = -2$ or

$$d = 2$$

This gives

$$1 = K_{\tilde{X}} \cdot E = \sum (m_i - 1)\, E \cdot F_i$$

from which it follows that

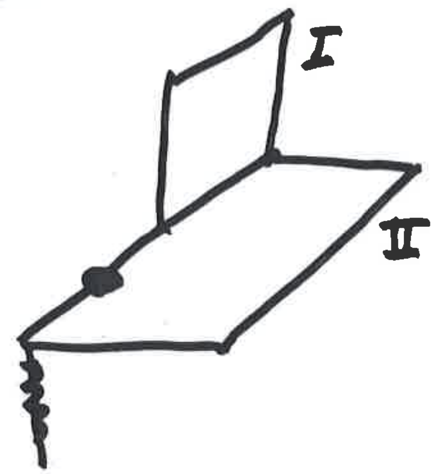
- there is one double fibre F_1
- $\deg(\pi|_E) = 2$
- $E \cdot F_1 = \frac{1}{2}(E \cdot (\text{general fibre})) = \frac{1}{2} \deg(\pi|_E)$

Summarizing:

- $\tilde{X}$ is an elliptic surface
- (4) $p_g(\tilde{X}) = 1$, so $\deg L = 2$
- $\deg(\pi|_E) = 2$
- $K_{\tilde{X}} = F_1$, where $2F_1$ is the unique multiple fibre

Does this situation occur? At this point the answer to this question is not known. It seems that one needs to determine what at least the bicanonical map will look like

Further example of non-classical phenomena

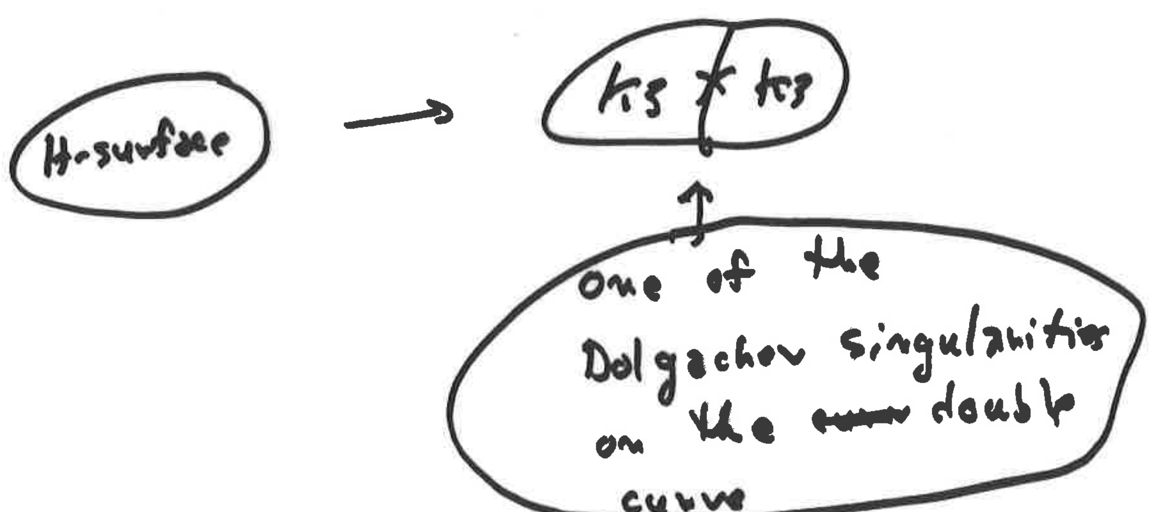


- extension data analysis gives that there is an H-surface at ● that is not the further degeneration of an H-surface whose LMHS is of type I. this doesn't happen classically

Can Hodge theory be used in other ways than the LMHS to detect/analyze singularities?

For example suppose the monodromy is trivial. For curves there are the compact degenerations

These are detected by the PHS becoming a direct sum ($/\mathbb{Z}$). Radu has found examples where



What might one be able to do if enough were understood about the connection between HT and moduli?

Recall that Bob Friedman proved the Torelli theorem for K3's knowing local Torelli together with an analysis of the HT at a generic boundary point in $\partial(\Gamma\backslash D)$. Here K3's (polarized) are lurking in type I degenerations and so one might be able to bootstrap on the Torelli for K3's using a Friedman type argument?