VII.1

Objectives for today

- Want to define and study
$$\Phi_e : \overline{M}_H \to \Gamma\backslash D^*$$
and use this to gain some understanding of the surfaces that lie over ∂M_H. This has only been partially carried out. What will try to do is

  - explain and discuss the picture
$$\begin{tikzcd}[column sep=small, row sep=small]
 & & \mathrm{II} \arrow[dr, dash] & & \\
\mathrm{O} \arrow[r, dash] & \mathrm{I} \arrow[ur, dash] \arrow[dr, dash] & & \mathrm{IV} \arrow[r, dash] & \mathrm{V} \\
 & & \mathrm{III} \arrow[ur, dash] & &
\end{tikzcd}$$
for the boundary component structure of period domains of weight $n=2$ and with $h^{2,0}=2$. We will label the boundary components of $\Gamma\backslash D$ as $\Gamma_{\mathrm{I}}\backslash D_{\mathrm{I}}, \Gamma_{\mathrm{II}}\backslash D_{\mathrm{II}}, \dots$

$\boxed{\text{VII}}.2$

- show by example that there are boundary components of $\partial \mathcal{M}_H$ that map to the above

(one might say that the Hodge-theoretic boundary stratification is "realized" by the stratification of $\partial \mathcal{M}_H$. This is not clear in general when there is a good moduli space - eg. [redacted] CI's in dimension three. And it is false in general - eg. surfaces $\Gamma \backslash \mathbb{B}$ where $\mathbb{B}$ = unit ball in $\mathbb{C}^2$ and Γ = discrete group with compact quotient are rigid but may have $p_g > 0$.

VII.3

- discuss how one may at least begin to understand the components of $\partial \mathcal{M}_H$ that map to $\Gamma_{\mathrm{I}}\backslash D_{\mathrm{I}}$

__Note__: This will involve some generalities on singularities of algebraic surfaces - a topic of interest in its own right.

- discuss the other extreme of I and H-surfaces that map to $\Gamma_{\mathrm{IV}}\backslash D_{\mathrm{IV}}$ and are rigid - these are some analogue of trivalent graphs for curves

- <u>Components of $\partial(\Gamma \backslash D^*)$</u>

These are indexed by all possible PLMHS's (equivalence classes of) arising from $\Phi : \Delta^* \to \{T^m\}\backslash D$ where $\log T = N$ and the PLMHS is

$$(V, Q, W(N), F), \quad F \in \check{D}$$

The conditions are

$$\begin{cases} \cdot \ \exp(zN) \cdot F \in D & \text{for } \operatorname{Im} z \gg 0 \\ \cdot \ NF^p \subseteq F^{p-1} \end{cases}$$

The nilpotent orbit

$$t \to \exp\left(\frac{\log t}{2\pi i} N\right) \cdot F$$

polynomial in $\log t$

flag = filtration on $V_{\mathbb{C}}$ in $\check{D}$

closely approximates $\Phi : \Delta^* \to \{T^m\}\backslash D$

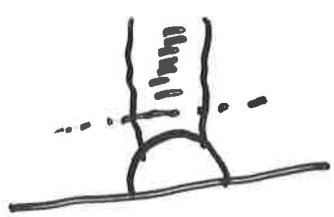
VI.5

[N, F]

- For each equivalence class there is a period domain $D_{[N,F]}$ which corresponds to PHS's on $Gr^{W(N)}(V, Q, N, F)$

Robles has constructed a topology on $\Gamma \backslash D^*$ where $D^* = D \amalg \left(\coprod D_{[N,F]}\text{'s} \right)$

The topology is constructed by

- constructing a Siegel set $\Omega \subset D$ where
$$\Omega = \left\{ \begin{array}{l} \text{approximate fundamental} \\ \text{domain for the action of} \\ \Gamma \text{ on } D \end{array} \right\}$$
- $\{\gamma \in \Gamma : \gamma\Omega \cap \Omega \neq \emptyset$ is finite$\}$
- 
- $\exp(zN) \cdot F \in \Omega$ for $|z| < 1$, $\mathrm{Im}\, z \gg 0$

This topology is Hausdorff (separato) - the analytic structure on $\Gamma \backslash D^*$ has yet to be constructed

• So we need to list the possible LMHS's $(V, Q, W(N), F)$. Recall the "(p,q) decomposition"

- $V_{\mathbb{C}} = \bigoplus_{0 \le p,q \le m} I^{p,q}$

- $W_m = \bigoplus_{p+q \le m} I^{p,q}$

- $\overline{I^{p,q}} \equiv I^{q,p} \mod W_{p+q-2}$

- $N : I^{p,q} \to I^{p-1,q-1}$

We plot the $I^{p,q}$'s by dots in the (p,q) plane where each dot has a number $= \dim I^{p,q}$*. Set

$$\nu^{p,q} = \dim I^{p,q}$$

Then the rules are

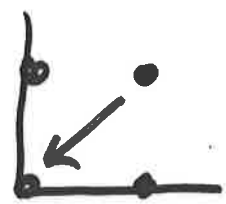
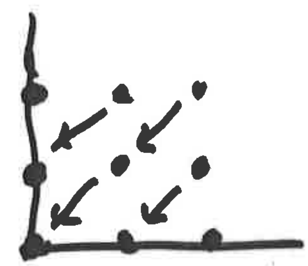
---

* Those are called <u>Hodge diamonds</u>

(i) $\sum_q w^{p,q} = h^{p,m-p}$

(ii) symmetric about $p=q$

(iii) symmetric about $p+q=m$

$m=1$  $m=2$ 

<u>Thm</u>: The boundary components of $\Gamma\backslash D^*$ are indexed by the (equivalence classes of) Hodge diamonds

$m=1$ - Indexed by $g' = w^{0,0}$; then $w^{1,0} = g-g'$

$m=2$ - Indexed by $w^{0,0}, w^{0,1}; w^{1,1}$

There is also a notion of <u>incidence</u>, denoted $\succ$, among boundary components.

Then intuitively

Boundary component A $\succ$ Boundary component B $\iff$ LMHS's corresponding to A can degenerate further to the LMHS's corresponding to B

- for D = Hermitian symmetric domain $\succ$ is a linear order
- for H and I-surfaces, $\succ$ is a non-linear partial order (transitivity)
- in general, $\succ$ is not even a partial order (no transitivity) — ex in CY's with Hodge numbers $(1, h, h, 1)$

<u>Example</u>: $m=1$. Set $D=\mathcal{H}_g$, $\mathcal{A}_g=\Gamma_g\backslash\mathcal{H}_g$

Then

 $\longleftrightarrow \mathcal{A}_{g-g'}$

$$\begin{cases} w^{0,0}=g' \\ w^{0,1}=g-g'=w^{1,0} \end{cases}$$

and

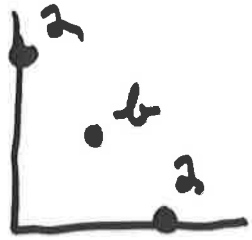
$$\mathcal{A}_g > \mathcal{A}_{g-1} > \cdots > \mathcal{A}_0$$

There is a linear ordering on the principal boundary components of $\mathcal{M}_g$ that maps under Φ_0 to the above.

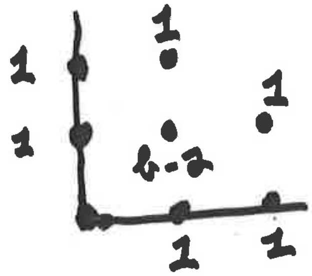
There is a partial ordering on all the boundary components of $\mathcal{M}_g$ that also maps to the linear ordering on the boundary components of $\Gamma_g\backslash\mathcal{H}_g^{\wedge}$

VI.10

Ex: $n=2$, $h^{2,0}=2$, $h^{1,1}=b$



0

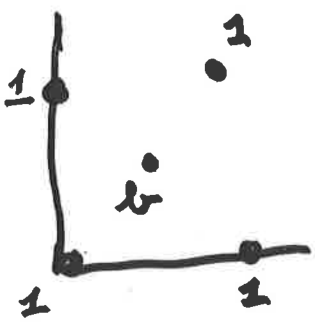


I

$w^{2,0}=1$

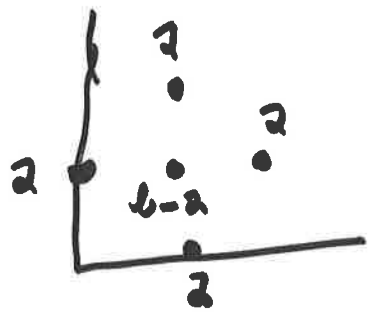
$w^{1,0}=1$

II



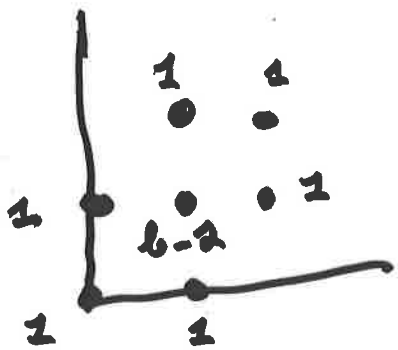
$w^{1,0}=1$

$w^{0,0}=1$



III

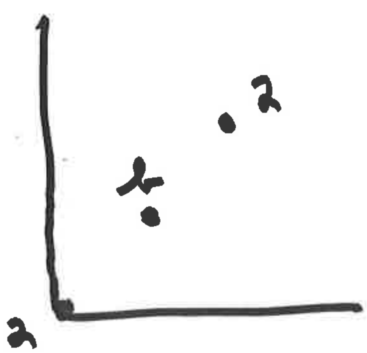
$w^{1,0}=2$



IV

$w^{1,0}=1$

$w^{0,0}=0$



V

$w^{0,0}=2$

The partial (in this case) ordering is

$$\begin{array}{ccccccc} & & & \mathrm{II} & & & \\ & & \succ & & \succ & & \\ 0 & \succ \; \mathrm{I} & & & & \mathrm{IV} \; \succ & \mathrm{V} \\ & & \succ & & \succ & & \\ & & & \mathrm{III} & & & \end{array}$$

Will denote the boundary components by

$D = D_0$, D_{I}, D_{II}, D_{III}, D_{IV}, D_{V}

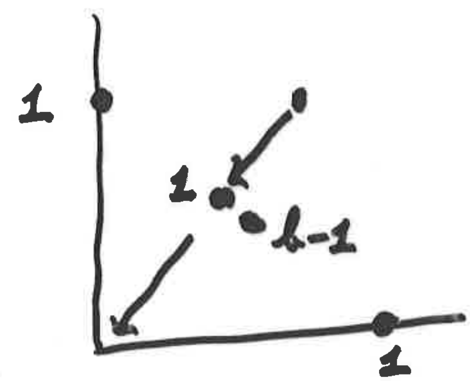
Thus

$$D_{\mathrm{I}} \cong \left\{ \begin{array}{l} \text{period domain} \\ \text{for weight 2} \\ \text{PHS's with } (1, h-2, 1) \\ \text{as Hodge numbers} \end{array} \right\} \times \underbrace{\left\{ \begin{array}{l} \text{period domain} \\ \text{for weight 1} \\ \text{PHS with} \\ (1,1) \text{ a Hodge \#'s} \end{array} \right\}}_{= \; \mathcal{H}_1}$$

$$D_{\mathrm{II}} \cong \left\{ \begin{array}{l} \text{period domain for} \\ \text{weight 2 PHS's} \\ \text{with Hodge \#'s} \\ (1, h, 1) \end{array} \right\} \times \underbrace{\{\text{point}\}}_{= \; \left\{ \begin{array}{l} \text{period domain} \\ \text{for PHS's of weight} \\ 0 \text{ and dimension} = 1 \end{array} \right\}}$$

$\mathbb{Q}$'s $\longrightarrow$

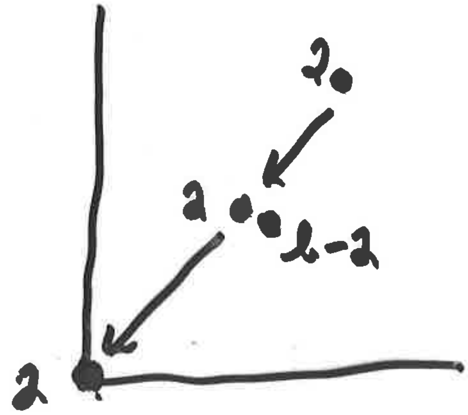
<u>Note</u>: For type II we really should write



because

$$I^{1,1} = N \cdot I^{2,2} \oplus \underset{\substack{\| \\ \ker N}}{I^{1,1}_{\mathrm{prim}}}$$

Similarly, type V is



$\leftarrow \bigoplus_2 \mathbb{Q}(-2)$

$\leftarrow \underbrace{\mathbb{Q}(-1)}_{2} \oplus \underbrace{\mathbb{Q}(-1)}_{b-2}$

The <u>most degenerate</u> boundary components are Hodge-Tate type

<u>Theorem: For H-surfaces all components of $\Gamma\backslash D$ are reached by the image of</u>

$$\Phi_e : \overline{\mathcal{M}}_H \to \Gamma\backslash D^{\sharp},$$

and <u>all incidence relations among the components of $\Gamma\backslash D^{\sharp}$ are also reached by the extended period mapping.</u>

This means: For each LMHS of types I, II, III, IV, V there is a KSBA degeneration $X_t \to X_0$ whose LMHS is of that type. And for any incidence relation such as $\mathrm{I} > \mathrm{II}$, there is a KSBA family

$$\begin{array}{ccccc} X_{s,t} & \to & X_{0,t} & \to & X_{0,0} \\ \downarrow & & \downarrow \Phi_e & & \downarrow \Phi_e \\ \Gamma\backslash D & & \Gamma_{\mathrm{I}}\backslash D_{\mathrm{I}} & & \Gamma_{\mathrm{II}}\backslash D_{\mathrm{I}} \end{array}$$

There are various constructions of the examples, mostly based on using the equation

$$L^2 G = F^2$$

of $\mathbb{X}^b$ for a general H-surface and degenerating this equation

- Heuristic geometric reasoning

Want to construct a degeneration

$$\mathbb{X} \to \mathbb{X}_0$$

under which

- one holomorphic 2-form ω_0 remains holomorphic, $\omega \to \omega_0$
- a general holomorphic 2-form φ becomes singular and contributes to $I^{1,0}$

this is a "thought example"

Suppose that the divisor (ω) becomes a double curve D in the limit. Locally we have something like

$$\varphi = \mathrm{Res}\left(\frac{f(x,y,z)\,dx\wedge dy\wedge dz}{xy-t}\right)$$

$$\downarrow$$

$$\varphi_0 = \mathrm{Res}\left(\frac{f(x,y,z)\,dx\wedge dy\wedge dz}{xy}\right)$$

$$\|$$

$$\frac{f(x,0,z)\,dx\wedge dz}{x} + \frac{f(0,y,z)\,dy\wedge dz}{y}$$

which has a log-pole on the double curve, unless

$$f(x,y,z) = x\,g(x,y,z) + y\,h(x,y,z)$$

vanishes on D.

Conclusion: If we can construct $X \to X_0$ by letting $(\omega) \to$ double curve, then we lose 1 holomorphic form (becomes singular). Moreover, on X_0

$$\mathrm{Res}\,(\mathrm{Res}\,\varphi_0) = f(0,q,z)\,dx$$

is holomorphic and "looks like" a form in $H^{1,0}(\tilde{\mathfrak{D}})$, thus potentially contributing to $I^{1,0}$.

- Recalling the picture

X^b

$t_0 = 0$

$$L_0^2 G = F^2$$
$$L_0 = x t_0^2 \in |\mathfrak{Z}|$$
$$\swarrow \qquad \searrow$$
$$|\mathfrak{Z}-2h| \quad |2h|$$

where $|K_{X^b}|$ = fibres, the above suggests considering

$$X_0^b = \{L_0^2 L_1^2 Q = F^2\}$$

In other words, we put another double curve with pinch points over $t_1 = 0$

The 2-form ω with divisor $xt_1^e|3-h|$ looks as though it should be holomorphic on $\tilde{X}^b$.

More precisely, let

$$X_s = \left\{ \begin{array}{l} \text{surface given by the normalization} \\ \text{of the surface } xt_0^2(sG - xt_1^2 Q) = F^2 \\ \text{and then contracting the } -1 \text{ curves} \end{array} \right\}$$

Here, G and F, Q are general elements of $|3H|, |2H|$

<u>Proposition</u>: The above gives a smooth family $\mathcal{X}^* \to \Delta^*$ of H-surfaces. The extended period map $\Phi_e : \Delta \to \Gamma \backslash D^*$ maps the origin to a type I degeneration. The normalization of X_0 is a K3 surface

Also, $\mathcal{K} \to \Delta$ is a KSBA degeneration.

Sketch of proof

- $x^2 t_0^2 t_1^2 Q = F^2$ is the equation of $\mathbb{X}_0^b \subset \mathbb{P}E$
- $\mathbb{X}_0^b$ has double curve

$$\left(\{x=0\} \cup \{t_0=0\} \cup \{t_1=0\}\right) \cap \{F=0\}$$

- the divisor $S=(x)$ of $x \in |\xi - 2h|$ is $\mathbb{P}^1 \times \mathbb{P}^2$ which under $|\xi| : \mathbb{P}E \to \mathbb{P}^4$ maps to the singular line of

$$Q_0 = \{x_0 x_2 = x_1^2\} \subset \mathbb{P}^4$$

Aside: It is easier to see this picture one dimension down - take $F = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ to get $\mathbb{P}F \longrightarrow \mathbb{P}^1$. Then

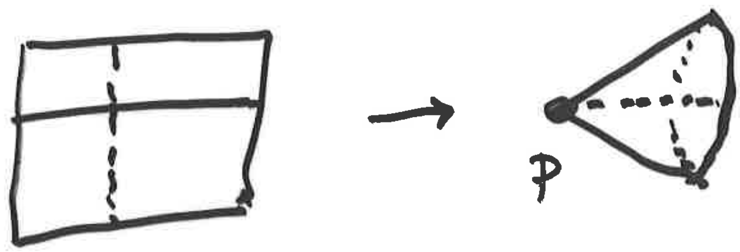
$H^0(\mathbb{P}F, \xi)$ has basis $t_0^2, t_0 t_1, t_1^2, x_3$

and the map $\mathbb{P}F \to \mathbb{P}^3$

$$x_0 = t_0^2,\ x_1 = t_0 t_1,\ x_2 = t_1^2,\ x_3 = x_3$$

and this gives a map $\mathbb{P}F \to Q_0' \subset \mathbb{P}^3$

$H^0(\xi - 2h) = \mathbb{C}x$ where $(x) = \mathbb{P}^1$ $\qquad Q_0' = \{x_0 x_2 = x_1^2\}$

(x) 

The divisor $(x) = \mathbb{P}^1$ with $(x)^2 = -2$ and contracts to the double point $P \in Q_0'$

The map $\mathbb{P}E \longrightarrow G_0 \subset \mathbb{P}^4$ is like this with $P \leftrightarrow$ singular line of Q_0 and $(x) \cong \mathbb{P}^1 \times \mathbb{P}^1$ maps to $Q_{0,\mathrm{sing}}$

$-F|_S \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,2)|$; thus

- $S \cap X^b$ is a $\mathbb{P}^1 \amalg \mathbb{P}^1$ corresponding to the images of E_1, E_2 in $\hat{X}$ = blow up of X at base points of $|K_X|$
- $S \cap X_0^b$ is $2(\mathbb{P}^1 \amalg \mathbb{P}^1)$ corresponding to the x^2 in the equation of X_0^b

This double curve has no pinch points

- the other double curves on $\mathbb{X}_0^6$ are on $t_0 = 0$ and $t_1 = 0$ with pinch points given respectively by

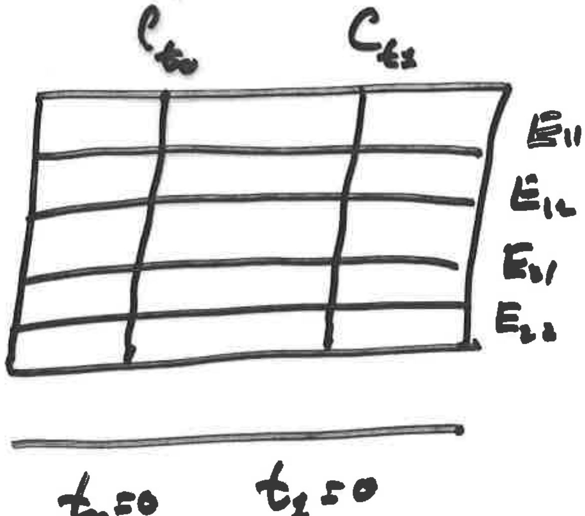
$$\{t_0 = 0\} \cap \{G = F = 0\}$$
$$\{t_1 = 0\} \cap \{Q = F = 0\}$$

More precisely

- on $X \neq 0$, $t_0 = 0$ they are given by $F = G = 0$ on $\mathbb{P}^2_{t_0} \setminus \mathbb{P}^=$
$$\mathbb{P}^= = \mathbb{P}^2_{t_0} \cap \{X = 0\}$$
which is 4 points

- on $X = 0$ and $t_0 = 0$ the equation is locally of the form
$$u^2 v^2 - w^2 = 0$$
$$= (uv - w)(uv + w)$$
which is a double curve without pinch points - similar on $t_1 = 0$

– The normalization $\tilde{X}_0^b$ of X_0^b has the picture



where C_{t_0}, C_{t_1} are elliptic curves branched over $\mathbb{P}^1$ at the 4 pinch points. The E_{ij} are the $\mathbb{P}^1$'s that arise from the normalization of the $2(\mathbb{P}^1 \amalg \mathbb{P}^1)$ above. The involution of $C_{t_0} \to \mathbb{P}^1$ interchanges $E_{11} \cap C_{t_0}$ and $E_{12} \cap C_{t_0}$, etc

– Issue is that $\tilde{X}_0^b$ is <u>not</u> the normalization of $\tilde{X}_0$ of the KSBA limit $X \longrightarrow X_0$. The reason is that the double curve $t_0 = 0$ on X^b is a singularity of $\varphi_{2K_X}(X)$ and not a singularity of X. The correct interpretation is that

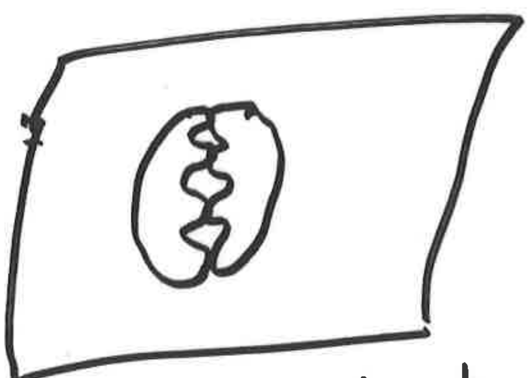
> The limit of C_{t_0} as $X \to X_0$ is a $\mathbb{P}^1$ on $\tilde{X}_0$, while the limit of C_{t_1} on $\tilde{X}_0$ is an elliptic curve with $C_{t_1} \to \mathbb{P}^1$ branched at the 4 pinch points

- The normalization = desingularization $\tilde{X}_0$ of X_0 is a K3 surface with $\omega_0 \in H^0(K_{\tilde{X}_0})$ non-vanishing

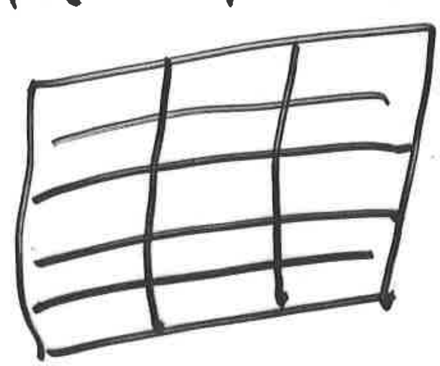
<u>Reason</u>: The divisor (ω) on $\hat{X}$ is $2(E_1 + E_2) + C_{t_1}$. On X the E_1, E_2 are contracted. Moreover

as $X \to X_0$, the ~~div~~ divisor C_{t_1} of ω tends to a double curve on X_0 along which ω_0 is regular and $\neq 0$. Thus on $\tilde{X}_0$, ω_0 gives $K_{\tilde{X}_0} \cong \mathcal{O}_{\tilde{X}_0}$.

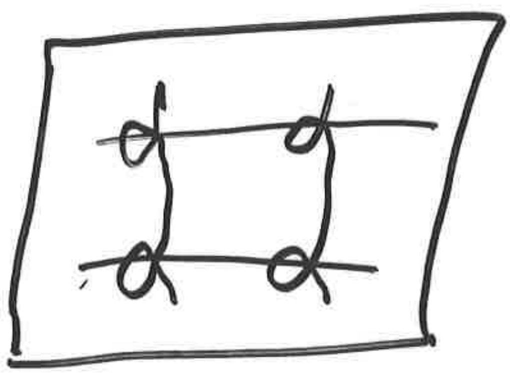
<u>Note</u>: The construction may be reversed as follows: Start with a K3 having 2 elliptic curves C_0, C_1 meeting in 4 points

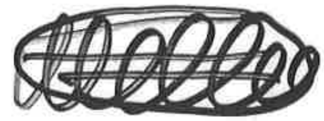


Blow up the 4 points to obtain



- Assume now that each of C_0, C_1 has an involution, and that $C_0 \cap C_1$ are 2 pairs where the 2 points in each pair are interchanged by the involution. Then glue the two pairs of horizontal $\mathbb{P}^1$'s together to obtain a surface with 2 double curves



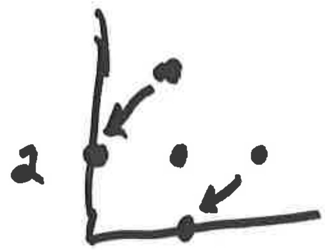


The singular curves are

Now contract C_0 to a $\mathbb{P}^1$ using the involution. This is X_0

VII.25

- We next turn to degenerations of type III where the Hodge diamond of the LMHS is



Geometrically, as $X \to X_0$ we want to lose both generators of $H^0(\Omega^2_X)$ to become singular 2-forms whose residues along the singular curve will give the $I^{1,0}$-part of the LMHS. The above discussion suggests that we let <u>two</u> curves in $|K_X|$ become double curves — e.g. take

$$L_0^2 L_1^2 L_2^2 H = F^2$$

$F \in |2K|$

where $L_i = x t_i^2$ for $i = 0, 1$, $L_2 = x(t_0+t_1)^2$, $H \in |K|$

as the equation of X_0. This is a little subtle as because of the x^3 in the equation the singular curves over $t_1 = 0$, $t_2 + t_1 = 0$ will have an equation

$$y^2 = (x-a_1)^3 (x-a_2)^3 (x-a_3)(x-a_4)$$

Using the birational transformation

$$\begin{cases} u = x \\ v = y/(x-a_1)(x-a_2) \end{cases}$$

on $\mathbb{P}^2$, with inverse

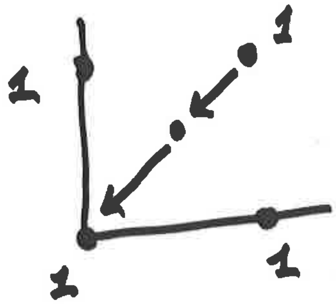
$$\begin{cases} x = u \\ y = v(u-a_1)(u-a_2) \end{cases}$$

the above equation becomes one of the form

$$u^2 = (v-b_1)(v-b_2)(v-b_3)(v-b_4)$$

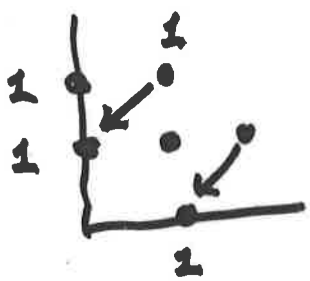
Going through an analysis as before we find on X_0 $\underline{3}$ elliptic curves that give rise to the $I^{2,0}$ in the LMHS.

- For a type II degeneration with Hodge diamond



$$\begin{cases} \dim I^{2,0} = 1 \\ \dim I^{0,0} = 1 \end{cases}$$

what is suggested is that we take a type I degeneration with Hodge diamond



$$\begin{cases} \dim I^{2,0} = 1 \\ \dim I^{1,0} = 1 \end{cases}$$

and degenerate the $I^{1,0}$ to $I^{0,0}$.

For the equation of a type I degeneration

$$x t_0^2 \, x t_1^2 \, Q = F$$

where Q, F are general, what is suggested is that we let the conics $Q=0$, $F=0$ in the $\mathbb{P}^2_{t_1}$ given by $t_1 = 0$ become special - e.g. if we take them to be

then if the curve C_{t_2} for a type I degeneration has the equation

$$y^2 = (x-a_1)(x-a_2)(x-a_3)(x-a_4)$$

then the degenerate equation is

$$y^2 = (x-b_1)^2(x-b_2)^2$$

which is a reducible curve consisting of a pair of $\mathbb{P}^1$'s meeting in 2-points

The holomorphic 2-form with a log-pole on C_{t_2} in the type I degeneration then becomes one whose residue is the unique 1-form on the above curve with log-poles at the nodes - This is the $I^{0,0}$ □

<u>Critique</u>: The above shows that a set is non-empty - it doesn't describe it at all (the above examples may be quite non-generic in components in $\partial\mathcal{M}_H$).

The final topics for these lectures will be

- discuss a method that could lead to descriptions of components - analogous to the basic ones of $\partial\mathcal{M}_g$ - of $\partial\mathcal{M}_I$ and $\partial\mathcal{M}_H$

(the method is of independent interest)

- using this method describe what are the rigid type V degenerations of I and H-surfaces

<u>Central problem</u>: For each Hodge-theoretic boundary component, classify the KSBA boundary components for H and I-surfaces that lie over the Hodge-theoretic one via the extended period mapping, and then describe a general (singular) surface X corresponding to the KSBA boundary component

---

This is probably too ambitious at this point. As for curves it may be better to seperate the question into first describing the "principal" boundary components when X is irreducible, and then deal with may be a combinatorial problem to describe how the general X may be built up from irreducible ones

We will proceed in several steps

(A) Make precise what KSBA singularities are (semi-log-canonical, or slc, models)

(B) Classification of slc singularities for surfaces (Kollár's list)

(C) Theorem: In the list of slc singularities only the

simple elliptic
cusp
(3.3.2)

} isolated

and those in (3.3.4) contribute to the LMHS.

(D) Theorem: Description of the irreducible H and I-surfaces whose LMHS is of Hodge type I

Clarifications: In order to define K_X, note that by (2), ω_X is locally free outside a codimension 2 subset of X, hence it corresponds to a linear equivalence class K_X of Weil divisors which are Cartier outside a codimension 2 subset of X.

One has to be a little careful with E because of the nodes on X. Denote the nodal divisor by $D \subset X$. If we normalize a node, its preimage is 2 points. Corresponding to D, on X' there is a unique divisor D' that is a double cover of D. In (3), E has an f-exceptional part but it also has to contain this divisor D' with coefficient -1.

This definition combines a global condition (4) with purely local conditions (1–3). Singularities satisfying (1–3) are called *semi-log-canonical* or *slc*.

For slc models it is usually better to use *semi-resolutions*, that is, a proper birational morphism $g : X^s \to X$ such that X^s has only double normal crossing points $(xy = 0) \subset \mathbb{C}^{n+1}$ and pinch points $(x^2 = y^2 z) \subset \mathbb{C}^{n+1}$ and g maps the double locus of X^s birationally on the double locus of X; see [26] for details. Let E denote the (reduced) exceptional divisor of a semi-resolution g. Then the canonical ring of X

$$R(X, K_X) := \sum_{m \geqslant 0} H^0(X, \mathcal{O}_X(mK_X))$$

is isomorphic to the *semi-log-canonical ring* of X^s

$$R(X^s, K_{X^s} + E) := \sum_{m \geqslant 0} H^0(X^s, \mathcal{O}_{X^s}(mK_{X^s} + mE)).$$

This actually creates a lot of problems since semi-log-canonical rings are not always finitely generated [30].

It is a quite subtle theorem that semi-log-canonical models actually satisfy the preliminary definition (1.7.5). This is proved in [36, 17.4] and [19].

To get a feeling for semi-log-canonical, let us review the classification of slc surface singularities.

Singularities of semi-log-canonical surfaces

It is convenient to describe the singularities of log canonical surfaces by the dual graph of their minimal resolution. That is, given a singularity $(s \in S)$ with minimal resolution $g : X \to S$ we draw a graph Γ whose vertices are the g-exceptional curves and two vertices are connected by an edge iff the corresponding curves intersect. We use the number $-(E_i \cdot E_i)$ to represent a vertex. In our examples, save in (3.2.4.a), all the exceptional curves are isomorphic to $\mathbb{P}^1$.

Let $\det(\Gamma)$ denote the determinant of the negative of the intersection matrix of the dual graph. This matrix is positive definite for exceptional curves. For instance, if $\Gamma = \{2 - 2 - 2\}$ then

$$\det(\Gamma) = \det \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = 4.$$

For more details concerning the lists below, see [36, Sec.3] or [35].

3.2 (List of log canonical surface singularities).

Each case includes all previous ones.

(3.2.1) Terminal = smooth.

(3.2.2) Canonical = Du Val (= rational double point).

(3.2.3) Log terminal = quotient of $\mathbb{C}^2$ by a finite subgroup of $GL(2,\mathbb{C})$ that acts freely outside the origin. The order of the group is $\det(\Gamma)$. A more detailed list is the following:

(a) (Cyclic quotient)

$$c_1 - \cdots - c_n$$

(b) (Dihedral quotient) Here $n \geqslant 2$ with dual graph

$$\begin{array}{cc} & 2 \\ & / \\ c_1 - \cdots - c_n & \\ & \backslash \\ & 2 \end{array}$$

(c) (Other quotients) The dual graph has 1 fork (with Γ_i as in (a))

$$\begin{array}{c} \Gamma_1 - c_0 - \Gamma_2 \\ | \\ \Gamma_3 \end{array}$$

with 3 cases for $(\det(\Gamma_1), \det(\Gamma_2), \det(\Gamma_3))$:

(Tetrahedral) (2,3,3)
(Octahedral) (2,3,4)
(Icosahedral) (2,3,5).

(3.2.4) Log canonical

(a) (Simple elliptic) $\Gamma = \{E\}$ has a single vertex which is a smooth elliptic curve with self intersection $\leqslant -1$.

(b) (Cusp) Γ is a circle of smooth rational curves, at least one of them with with $c_i \geqslant 3$. (The cases $n = 1,2$ are somewhat special.)

$$\begin{array}{ccccc} & c_n - \cdots - c_{r+1} & \\ / & & \backslash \\ c_1 & & c_r \\ \backslash & & / \\ & c_2 - \cdots - c_{r-1} & \end{array}$$

(c) ($\mathbb{Z}/2$-quotient of a cusp or simple elliptic) Γ has 2 forks.

$$
\begin{array}{ccccc}
2 & & & & 2 \\
 & \diagdown & & \diagup & \\
 & c_1 & - \cdots - & c_n & \\
 & \diagup & & \diagdown & \\
2 & & & & 2
\end{array}
$$

(d) (Other quotients of a simple elliptic) The dual graph is as in (3.2.3.c) with 3 possibilities for $(\det(\Gamma_1), \det(\Gamma_2), \det(\Gamma_3))$:

($\mathbb{Z}/3$-quotient) (3,3,3)
($\mathbb{Z}/4$-quotient) (2,4,4)
($\mathbb{Z}/6$-quotient) (2,3,6).

If X is a non-normal semi-log-canonical surface singularity, then we describe its normalization $\bar{X}$ together with the preimage of the double curve $\bar{B} \subset \bar{X}$.

The *extended dual graph* $(\Gamma, \bar{B})$ has an additional vertex (represented by $\bullet$) for each local branch of $\bar{B}$ connected to C_i if $(\bar{B} \cdot C_i) \neq 0$.

3.3 (List of semi-log-canonical surface singularities). There are 3 irreducible cases. (The number on some edges is the different, which we do not define here [36, Sec.16]. Their role is explained in (3.3.4).

(3.3.1) (Cyclic quotient, one branch of $\bar{B}$)

$$\bullet \overset{1-\frac{1}{\det\Gamma}}{\text{———}} c_1 - \cdots - c_n$$

(3.3.2) (Cyclic quotient, two branches of $\bar{B}$)

$$\bullet \overset{1}{-} c_1 - \cdots - c_n \overset{1}{-} \bullet$$

(3.3.3) (Dihedral quotient) Here $n \geqslant 2$ with dual graph

$$
\begin{array}{ccc}
 & & 2 \\
 & & \diagup \\
\bullet \overset{1}{-} c_1 - \cdots - c_n & & \\
 & & \diagdown \\
 & & 2
\end{array}
$$

(3.3.4) (Reducible cases) We can take several components as above and glue them together along two local branches of $\bar{B}$. The gluing is allowed only if we see the same numbers on the edges.

Thus we can glue 2 copies as in (3.3.1) as long as both have the same $\det(\Gamma)$ or we can take any number of germs as in (3.3.2), make a chain out of them and then either turn the chain into a circle or end it with copies of (3.3.3). To end a chain, we are also allowed to glue a local branch of $\bar{B}$ to itself by an involution. For instance, $\bullet - 1$ glued to itself gives the pinch point $(x^2 = y^2 z) \subset \mathbb{A}^3$.

(E) <u>**Theorem: Description of the rigid H and I-surfaces whose LMHS is of Hodge type V**</u>

These are the analogue of trivalent graphs in the curve case.

(A) We want to explain where slc singularities come from. A complication, caused by the presence of quotient singularities, is that for the varieties we shall consider the "canonical bundle" may not be a line bundle but a Weil divisor. For $\mathcal{X}$ normal we let $\mathcal{X}_{reg} = \mathcal{X} \setminus \mathcal{X}_{sing}$ be the smooth points. For any divisor Z in $K_{\mathcal{X}_{reg}}$ we close it up to get a divisor on $\mathcal{X}$. This is $K_{\mathcal{X}}$. We make the assumption

(1) $m_0 K_{\mathcal{X}}$ is Cartier - i.e. is a line bundle - for some $m_0 \in \mathbb{Z}^+$

We then consider families

$$\mathcal{X} \xrightarrow{\pi} B$$

where B is a smooth pointed curve, and we assume that

(2) $m_0 K_{\mathcal{X}}$ is π-ample

(3) a general fibre X_b is a canonical model (i.e. biregularly equivalent to $\varphi_{mK_{X_b}}(X_b)$ for $m \gg 0$). We also that

(4) for any resolution of singularities $\mathcal{Y} \xrightarrow{p} \mathcal{X}$, we have

$$m_0 K_{\mathcal{Y}} = p^*(m_0 K_{\mathcal{X}}) + E, \quad E \geqq 0$$

(this makes sense since $m_0 K_{\mathcal{X}}$ is a line bundle)

Finally we assume

(5) the assumption (4) holds for <u>any</u> base change

$$\begin{array}{ccc} \mathcal{X}' & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

<u>Defn</u>: When (2) - (5) are satisfied we say that <u>the special fibre X_{b_0} has slc singularities</u>.

<u>Example</u>: This is local with $\mathcal{X}$ the smooth surface given by

$$f(x,y,t) = x^3+y^3+t = 0$$

(we can use m=3 and any smooth cubic in place of x^3+y^3). As usual, Poincaré residues guide us in what canonical bundles should be. Here $K_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}$ via Poincaré residues

$$\mathrm{Res}\left(\frac{g(x,y,t)\,dx\wedge dy\wedge dt}{f(x,y,t)}\right) = g(x,y,-x^m-y^m)\,dx\wedge dy$$

For the base change $t=s^m$, $\mathcal{X}'$ is given by $x^m+y^m+s^m=0$. Desingularizing by blowing up setting $x = \lambda u$, $y = \lambda v$, $s = \lambda$ the Poincaré residue becomes (ignoring constants)

$$\mathrm{Res}\left(\frac{g(\lambda u, \lambda v, \lambda^m)\, du \wedge dv \wedge d\lambda}{\lambda^{m-2}\,(u^m + v^m + t)} \right)$$

For $m \geqq 3$ this 2-form on $\mathcal{Y}'$ has a pole on $\lambda = 0$; i.e.

$$K_{\mathcal{Y}'} = p^* K_{\mathcal{X}'} + E, \quad E < 0$$

Thus (4) picks out nodal curves.

- The next issue is how to recognize slc singularities purely in terms of the singular fibre. Let X be the singular fibre. Let $X' \xrightarrow{\pi} X$ a minimal normal and resolution of singularities with exceptional curves $E_1, \dots, E_\ell$ that are contracted by π to the singular point on X. Then we claim that we have

$$K_{X'} \underset{\mathbb{Q}}{\sim} \tau^* K_X + \sum a_i E_i, \quad a_i \geq -1$$

Here we are following the usual custom of using $\mathbb{Q}$-line bundles to eliminate the the m_0 above. When this is done the $a_i \in \mathbb{Q}$. The point is that for slc singularities we have $a_i \geq -1$.

To explain - not prove (although this argument can be made into a proof) - we drop the $\mathbb{Q}$'s and assume that after SSR we have

and base change

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{p} & \mathcal{X} \\ \downarrow & & \downarrow \\ S & = & S \end{array}$$

where the fibre

$$p^{-1}(X_{b_0}) = \hat{X}_{b_0} + Z$$

a desingularization

, $\hat{X}_{b_0} \to X_{b_0}$ is in $\mathcal{Y}$ over b_0 is a reduced normal crossing divisor

Then

- $K_{\mathcal{Y}} = p^* K_{\mathcal{X}} + \mathcal{E}$, $\mathcal{E} \geq 0$

- $K_{\mathcal{Y}} + \hat{X}_{b_0} = p^*(K_{\mathcal{X}} + X_{b_0}) + \mathcal{E} - Z$

- $K_{\mathcal{Y}} + \hat{X}_{b_0}\Big|_{\hat{X}_{b_0}} = p^*(K_{\mathcal{X}} + X_{b_0})_{\hat{X}_{b_0}} + \mathcal{E} - Z\Big|_{\hat{X}_{b_0}}$

- $K_{\hat{X}_{b_0}} = p^*(K_{X_0}) + (\mathcal{E} - Z)\Big|_{\hat{X}_{b_0}}$,

and all the components of the last term have multiplicity ≥ -1 since $Z, \hat{X}_{b_0}$ are smooth and meet transversely

- For X normal with slc singularities and $f : X' \longrightarrow X$ and

$$K_{X'} \sim f^*(K_X) + \sum q_i E_i , \quad q_i \geq -1$$

as above, we have the following classification of slc singularities

- terminal $a_i > 0$
- canonical $a_i \geq 0$
- log terminal $a_i > -1$
- log canonical $a_i \geq -1$

The "semi" in semi-log-terminal means that we do not require X to be normal, but to have singularities that satisfy

- Serre's condition S_2
- X has double points in codimension 1

Then it will follow that for slc singularities

- X has a double curve with pinch points for its non-normal locus

- In the case of an isolated normal singularity
  - one assumes that none of the E_i are -1 curves (minimality)
  - the matrix $\|(E_i \cdot E_j\|$ is negative definite (Mumford)
  - every coefficient in the inverse matrix is negative (Alexeev)

Using the relations

$$\begin{cases} (K_{X'} + E_j)\cdot E_j = 2g(E_j) - 2 \\ f^* K_X \cdot E_j = 0 \Rightarrow (K_{X'} - \sum_i a_i E_i) E_j = 0 \end{cases}$$

one may solve for the a_i. When this is done there are the following conclusions concerning an isolated slc singularity

- In the terminal case all $a_i = 0$ and X is smooth
- In the canonical case, all E_i are $\mathbb{P}^1$'s with $E_i^2 = -2$ (rational double points)

Then it follows that all $a_i = 0$ and

$$K_{X'} = f^* K_X$$

Thus canonical singularities do not effect the the plurigenera in a KSBA degeneration

- In log terminal case ((3.2.7) in Kollár) the singularities are quotients of $\mathbb{C}^2$ by a finite group and have $\lfloor a_i \rfloor = 0$, where $\lfloor a \rfloor =$ greatest integer $\leq a$

Proof of Theorem C:

Step one: We will deal with the isolated singularities in Kollár's list. Setting

$$L = \sum (1 - \lfloor a_i \rfloor) E_i \geq 0$$

since $a_i \geq -1$, we have

$$K_{\tilde{X}'} + \sum \lfloor -a_i \rfloor E_i + L = K_{\tilde{X}'} + \sum E_i$$

which gives

$$H^0(K_{\tilde{X}'} + \sum \lfloor -a_i \rfloor E_i) \hookrightarrow H^0(K_{\tilde{X}'} + \sum E_i).$$

On a configuration $\bigcup E_i$ of curves meeting transversely we define the logarithmic 1-forms to be the 1-forms on the E_i with log poles and opposite residues at the intersections $E_i \cap E_j$.

We then have an exact sequence

$$H^0(K_{X'}) \to H^0(K_{X'} + \Sigma E_i) \to \begin{pmatrix} \text{logarithmic} \\ \text{1-forms} \\ \text{on } \cup E_i \end{pmatrix}$$

Thus

- If there are no logarithmic 1-forms on $\cup E_i$, then

$$H^0(K_{X'}) \simeq H^0(K_X)$$

In this case under a KSBA degeneration $X_\eta \to X$ no holomorphic 2-forms acquire singularities and

$$\dim I^{2,0} = \dim H^0(\Omega^2_{X_\eta})$$

<u>Step two</u>: When are there no logarithmic 1-forms on $\cup E_i$?

- If all the E_i are $\mathbb{P}^1$'s in a tree ((3.2.3) (a), (b), (c) and (3.3.1), (3.3.3)) there are no logarithmic 1-forms
- In the cusp case (3.2.4)(b) there is 1 logarithmic 1-form (up to a constant)
- In the case (3.3.2) there is at most 1 logarithmic 1-form, depending on how the residues of the 2 end $\mathbb{P}^1$'s interact with the logarithmic 1-forms on the double curve (and this is not an isolated singularity).
- In the cusp case, any 2-form

in $H^0(\Omega^2_{X'}(E_1+\dots+E_\ell))$ will have equal ~~[illegible]~~ to 1 double residues at the $E_i \cap E_{i+1}$. Thus we get a contribution to $I^{0,0}$ in the LMHS

- In the simple elliptic case the residue of forms in $H^0(\Omega^2_{X'}(E))$ gives a form in $H^0(\Omega^1_E)$, which if $\neq 0$ contributes to $I^{1,0}$ in the LMHS

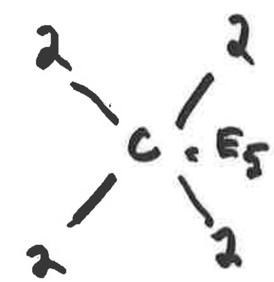
- The question of whether we get $\neq 0$ contributions in the simple elliptic and cusp cases is whether the isolated singular point is the limit of a base point of the canonical series $|K_{X_\eta}|$.

Details/comments/example

- We may solve for the a_i in terms of the $g(E_i)$. When all the E_i are rational and E_i meets $E_{j_1}, \ldots, E_{j_r}$ we have

$$1 + a_i = \frac{2 + (a_{j_1} + \cdots + a_{j_r})}{-E_i^2}$$

- A typical example is (3.3.4)(c)



$$\| E_i \cdot E_j \| = \begin{pmatrix} -2 & & & & 1 \\ & -2 & & & 1 \\ & & -2 & & 1 \\ & & & -2 & 1 \\ 1 & 1 & 1 & 1 & -c \end{pmatrix}, \quad c \geq 3$$

$\Rightarrow a_5 = -1$, all other $a_i > -1$

- For Mumford's theorem, using
$$f : X' \to X \subset \mathbb{P}^N$$
and since $c_1(f^*\mathcal{O}_X(1))^2 > 0$ and
$$E_i \cdot c_1(f^*\mathcal{O}_X(1))^2 = 0$$
the Hodge index theorem gives
$$\| E_i \cdot E_j \| \leqq 0.$$
Since the $[E_i]$ are independent in $H^2(X', \mathbb{Q})$ we actually have $\| E_i \cdot E_j \| < 0$.

- One also has
$$0 \geqq a_i \geqq -1$$

- Finally, for X_η we are using the following

<u>**Heuristic:**</u> $I^{2,0}$ is given by the space of $\omega_\eta \in H^0(\Omega^2_{X_\eta})$ that remain holomorphic in the limit to give $\omega \in H^0(\Omega^2_{X'})$.

- $I^{2,0}$ is given by the residues of the $\lim \omega_\eta = \omega \in H^0(\Omega^2_{X'}(\sum \log E_i))$ that have no residues for $\operatorname{Res}\omega$ on the E_i
- $I^{0,0}$ is given by the double residues of the ω's

If we know $\dim I^{p,0}$ for $p = 0, 1, 2$ then we know all the $I^{p,q}$ for the LMHS. Moreover, in the above we see that as a corollary of the descriptions

$$I^{p,0}(LMHS) = I^{p,0}(MHS \text{ on } H^2(X))$$

for $0 \leqq p \leqq 2$ (Shah's theorem in this case).

<u>Step 3</u>: We have now completed the proof theorem (C) except for the last item. Following is an informal discussion of that case; we shall use the results of this discussion in the analysis of degenerate H and I - surfaces.

Thus let $X_\eta \to X$ be a KSBA degeneration where X has a double curve D with pinch points. Denote by $\tilde{X} \xrightarrow{\pi} X$ the normalization, and by $\tau: \tilde{D} \to \tilde{D}$ the involution on $\tilde{D} = \pi^{-1}(D)$ with fixed points the pinch points on D. Under the specialization $X_\eta \to X$ we have

$$(*) \qquad H^0(\Omega^2_{\mathfrak{X}_\eta}) \longrightarrow H^0(\Omega^2_{\tilde{\mathfrak{X}}}(\tilde{D}))$$

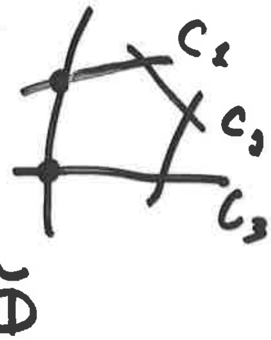
The residue map gives

$$H^0(\Omega^2_{\tilde{\mathfrak{X}}}(\tilde{D})) \to H^0(\Omega^1_{\tilde{D}}),$$

and the image of $(*)$ goes to the (-1)-eigenspace $H^0(\Omega^1_{\tilde{D}})^-$ under the action of τ. Thus the specialization $\mathfrak{X}_\eta \to \mathfrak{X}$ contributes

- $H^0(\Omega^2_{\mathfrak{X}})$ to $I^{2,0}$ in the LMHS
- at most $H^0(\Omega^1_{\tilde{D}})^-$ to $I^{2,0}$ in the LMHS
- to get a contribution to $I^{0,0}$ in the LMHS we need to have either a singularity of type (3.3.x) from Kollár's list

which may pictured ~~up~~ up on $\tilde{X}$ as



where $\omega \in H^0(\Omega^2_{\tilde{D}}(\Sigma C_i))$