Completion of $\Gamma\backslash D$ to $\Gamma\backslash D^*$

Objectives:

- Given $\Phi : \Delta^* \to \Gamma\backslash D$, $\Gamma = \{T^m\}$ where $T = \exp N$ is unipotent, define

$$\lim_{t \to 0} \Phi(t) = \text{LMHS}$$

- Given $\mathcal{X}^* \to \Delta^*$ discuss how to compute the LMHS when we have $\mathcal{X} \to \Delta$ with
  - $X_0 = \text{NCD}$ (semi-stable reduction)
  - $X_0 = \text{KSBA}$ limit (unique)
- Define D^* as a set, and describe $\partial(\Gamma\backslash D^*)$ for H-surfaces and relate this to $\partial\mathcal{M}_H$ via

$$\Phi_\bullet : \overline{\mathcal{M}}_H \to \Gamma\backslash D^*$$

- Recall that MHS (V, W, F) is defined by the conditions that the induced

$$F^p(Gr^W_k) = \frac{F^p \cap W_k + W_{k-1}}{W_{k-1}}$$

define a Hodge structure on $Gr^W V$

MHS's form an abelian category

- Can one define a $\oplus$ (p,q) decomposition for MHS's? Deligne showed there exists a unique, functorial

  - $V_{\mathbb{C}} = \oplus I^{p,q}$ such that
  - $I^{q,p} \equiv \bar{I}^{p,q} \bmod W_{p+q-2}$

~~$I^{p,q} = F^p \cap$ [illegible] $\bar{F}^q \cap W_{p+q}$ $\bar{F}^{q-1} \cap W_{p+q-2}$ $+ \cdots$~~

$$I^{p,q} = (F^p \cap W_{p+q}) \cap (\bar{F}^q \cap W_{p+q} + \overline{F^{q-1}} \cap W_{p+q-2} + \cdots)$$

- the MHS is $\mathbb{R}$-split if

$$\bar{I}^{p,q} = I^{q,p}$$

$$\Downarrow$$

over the MHS is a $\oplus$ of pure HS's

By Deligne again, to each MHS there is canonically associated an $\mathbb{R}$-split MHS $(V, W, \tilde{F})$ with

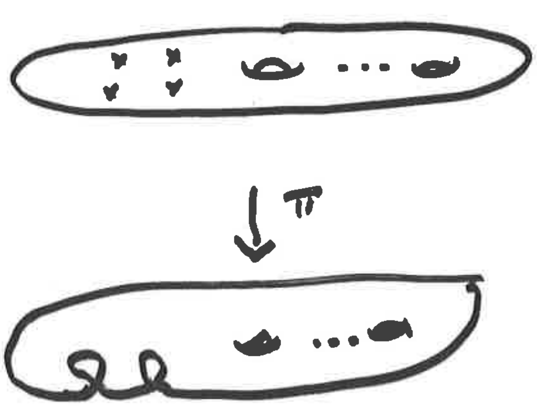
$$Gr^W_F \cong Gr^W_{\tilde{F}}$$

- the conditions to be $\mathbb{R}$-split and the construction of $(V, W, \check{F})$ are linear algebraic - do not have algebro-geometric meaning

- $$\mathrm{Ext}^1_{MHS}(C, A) \cong \frac{\mathrm{Hom}_{\mathbb{C}}(C, A)}{F^0\mathrm{Hom}_{\mathbb{C}}(C, A) + \mathrm{Hom}_{\mathbb{Z}}(C, A)}$$

classifies $0 \to A \to B \to C \to 0$'s.

Example



where $\pi^{-1}(D) = \{p_1, q_1 ; p_2, q_2\}$. Then

- $W_0 = H^0(D)$, D = double locus
- $W_1 = H^1(C)$
- $W_1 / W_0 \cong H^1(\tilde{C})$

The extension data is in

$$J(\tilde{C}) = \frac{H^1(\tilde{C}, \mathbb{C})}{H^{1,0}(\tilde{C}) \oplus H^1(\tilde{C}, \mathbb{Z})} \cong \frac{H^1(\mathcal{O}_{\tilde{C}})}{H^1(\tilde{C}, \mathbb{Z})}$$

arising from

$$0 \to \mathbb{Z} \to \mathcal{O}_{\tilde{C}} \xrightarrow{\exp} \mathcal{O}^*_{\tilde{C}} \to 1$$

the extension class is

$$AJ_{\tilde{C}}(p_1 - q_1 + p_2 - q_2) \in J(\tilde{C})$$

- Recall next that a *limiting mixed Hodge structure* (LMHS) is given by $(V, W(N), F)$ where
  - $W(N)$ is the weight filtration associate to a nilpotent $N \in \mathrm{End}_{\mathbb{Q}}(V)$
  - $N \in F^{-1}\mathrm{End}(V)$ (i.e. $NF^{p} \subseteq F^{p-1}$)

- Given $\Phi : \Delta^{*} \to \{T^{\mathbb{Z}}\}\backslash D$ we have

$$\tilde{\Phi} : \mathcal{H} \to D, \quad \tilde{\Phi}(w+1) = \exp N \cdot \tilde{\Phi}(w)$$

where $w \in \mathcal{H}$, $e^{2\pi i w} \in \Delta^{*}$. Then

$$\Psi(t) = \exp(-wN)\,\tilde{\Phi}(w), \quad t \in \Delta^{*}$$

gives a map

$$\Psi : \Delta^{*} \to \check{D}$$

Theorem (Schmid): Ψ extends to give

$$F_{\lim} = \Psi(0) \in \check{D}$$

and $(V, W(N), F_{\lim})$ is a LMHS

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Note: F_{lim} depends on the choice of coordinate t. Scaling $t \to e^{\lambda} t$ gives

$$F_{lim} \longrightarrow \exp(\lambda N) \cdot F_{lim} = F'_{lim}$$

Since $N \in F^{-1} \operatorname{End}(V)$ we have

$$Gr^{W(N)}_{F_{lim}} = Gr^{W(N)}_{F'_{lim}}$$

- <u>Preliminary construction of D^*</u>

Consider all N's as above. For each N we have well-defined period domains arising from Gr{LMHS}

$$D_{Gr\, W(N)} = \prod \left\{ \begin{array}{l} \text{period domains associated} \\ \text{to the PHS's of the type} \\ \text{appearing in } Gr\,(V, W(N), F_{lim}) \end{array} \right\}$$

Then as a set

$$D^{*} = D \bigcup_{[N]'s} \left(D_{Gr\, W(N)}'s \right)$$

Intuitively, we add to D all the associated graded PHS's to the potential LMHS's

<u>Example:</u> $n=1$. Then $N^2=0$ and the weight filtration is

$$\begin{array}{ccccc} W_0 & \subset & W_1 & \subset & W_2 \\ \parallel & & \parallel & & \parallel \\ \operatorname{Im} N & & \operatorname{Ker} N & & V \end{array}$$

We may then choose an adapted symplectic basis so that

$$Q = \begin{pmatrix} & & +I_{g'} \\ & G_0 & \\ -I_{g'} & & \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & 0 & A \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = {}^{t}A > 0$$

Here recall that

$$N : W_2/W_1 \xrightarrow{\simeq} W_0$$

and Hodge-Riemann I, II give the above.

Then

$$Gr_1^{W(N)}(V, W(N), F_{\lim}) = \left\{ \begin{array}{l} \text{PHS of weight} \\ \text{one with} \\ \dim V'' = g - g' \end{array} \right\}$$

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· Clemens-Schmid

How is the LMHS used to study the degeneration of HS of a family of varieties? The usual way is

- start with $\mathcal{X}^* \to \Delta^*$

- using semi-stable reduction (SSR), consisting of base change plus blowing up we (non-uniquely) arrive at

$$\mathcal{X} \to \Delta$$

locally in $\mathcal{X}$ given by $x_1 \cdots x_k = t$

- the central fibre X_0 is a (global) normal crossing variety (NCV)

$$\left\{ \begin{array}{l} X_0 = \bigcup X_i \ , \quad X_i \text{ smooth} \\ \text{and meet transversely} \end{array} \right.$$

- there is a spectral sequence that

- converges to $H^*(X_0)$
- has $E_1^{p,q} = H^q(X^{[p]})$, $X^{[p]} = \amalg \begin{pmatrix} \text{p-fold} \\ \text{intersections} \\ \text{of the } X_i \end{pmatrix}$
- degenerates at E_2

$$\rightsquigarrow$$

in practice enables us to (usually) compute $H^*(X_0)$ with its MHS

- there is the Clemens retraction mapping

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "r"] & X_0 \\
\cup & \\
X_0 \arrow[uur, "id"'] &
\end{tikzcd}$$

where r is a homotopy equivalence

<u>Clemens - Schmid</u>: There is an exact sequence of MHS's ($n = \dim X_0$)

$$\boxed{H_{2n+2-m}(X_0) \to H^m(X_0) \to H^m_{lim} \xrightarrow{N} H^m_{lim} \to H_{2n-m}(X_0) \to H^{m+1}(X_0)}$$

- <u>Corollary</u> (local invariant cycle theorem)

$$\begin{tikzcd} H^m(X_t)^{inv} \arrow[r, two heads] \arrow[d, phantom, sloped, "="] & H^m(X_0) \arrow[d, phantom, sloped, "\cong"] \\ \ker N & H^m(\mathcal{X}) \end{tikzcd}$$

  - standard that

$$H^m(X_t)^{inv} \twoheadrightarrow H^m(\mathcal{X}^*)$$

  (this is a topological result). Content is that the invariant cycles extend across $t=0$ - this is a <u>Hodge-theoretic</u> result; counterexamples to the topological result

- <u>MHS on $H^m(X_0)$</u>

$$W_k = \bigoplus_{q \le k} E_0^{*,q}$$

Induces weight filtration on

$$E_1^{p,q} = H^q(X^{[p]})$$

The differential d_I is induced from the maps on cohomology arising from the inclusions

$$X^{[p+1]} \hookrightarrow X^{[p]}$$

$$\Downarrow$$

- morphisms of HS's
- preserves weight filtration

$$\Downarrow$$

$$E_2^{p,q} \text{ has a HS}$$

$$\|$$

degeneration of spectral sequence $\longrightarrow$ $E_\infty^{p,q}$ " " "

$$\Downarrow$$

$$Gr^W H^m(X_0) \text{ has a HS}$$

This gives the MHS on $H^m(X_0)$

<u>Corollary (of proof)</u>:

- $N^2 = 0$ if X_0 has at most double loci ($X^{[3]} = \phi$)
- $N^3 = 0$ " " " " " triple loci ($X^{[4]} = \phi$)

- **Dual graph Γ associated to X_0**

  - vertex P_i for each component X_i
  - 1-simplex P_iP_j if $X_i \cap X_j \neq \phi$
  - 2-simplex $P_iP_jP_k$ if $X_i \cap X_j \cap X_k \neq \phi$

$$\vdots$$

$$\Downarrow$$

$E_1^{p,0} = H^0(X^{[p]})$ is Čech complex for $|\Gamma|$

$$\Downarrow$$

$$\boxed{W_0 H^m(X_0) = H^m(|\Gamma|)}$$

Application: $N^m = 0 \Leftrightarrow H^m(|\Gamma|) = 0$

(recall that $N^{m+1} = 0$)

Picture of local invariant cycle theorem

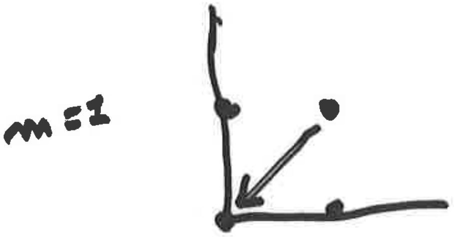
- In (p,q) draw •'s for the $I^{p,q}$ for the LMHS $H^m_{\lim}(X_t)$. Then

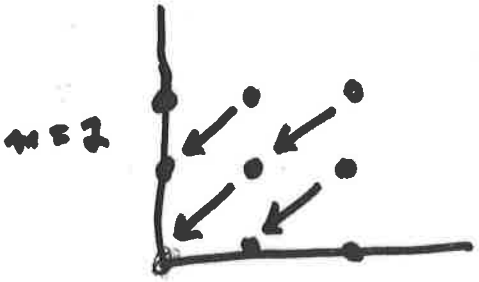
$$0 \leq p,q \leq m$$

(like the $H^{p,q}(X)$ for $H^*(X)$, $\dim X = m$)

Draw ↙ for the action of N.

Then we have



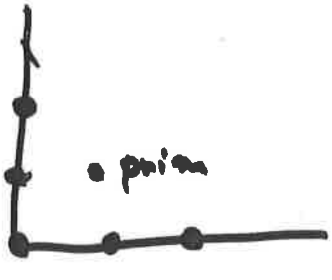


$$\sum_q \dim I^{p,q} = h^{p,0}(X_t)$$

Ker N =

$\cong$

$H^1(X_0)$



Example:

$\{\delta_i\} = \delta_1,\dots,\delta_{g'}$ $\{\delta_2\} = \delta_{g'+1},\dots,\delta_g$

$\{\gamma_i\}$ $\gamma_1,\dots,\gamma_{g'}$ $\{\gamma_2\} = \dots$

$$Q = \begin{pmatrix} & & I_{g'} \\ & Q_0 & \\ -I_g & & \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

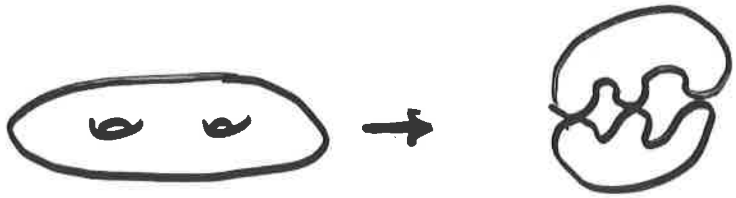
$T\gamma_i = \gamma_i + \delta_i$ (Picard-Lefschetz)

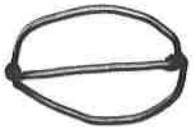
$$\underbrace{\underbrace{\gamma_1,\dots,\gamma_{g'}}_{W_0}, \{\delta_2, \gamma_2\}}_{W_1}, \delta_1,\dots,\delta_{g'}$$

To get SSR we need for X_0

$g' = 2$ $\Gamma =$  $\Rightarrow H^1(|\Gamma|) = \mathbb{Z}^2$

Example



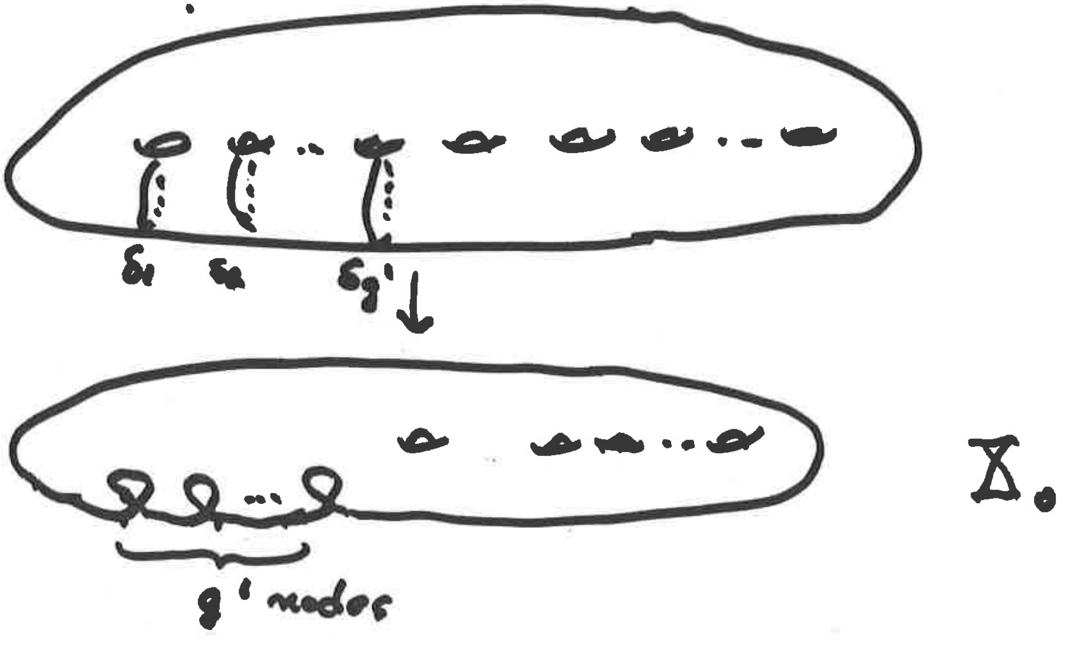
$\Gamma =$ 

$$N = \begin{pmatrix} 0 & \begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix} \\ 0 & 0 \end{pmatrix}$$

$\rightsquigarrow$

$\overset{W}{Gr}$(LMHS's) in these two examples are same

As far as the associated graded's to the LMHS's go, we may restrict to irreducible, stable limit curves



Monodromy is

$$\begin{pmatrix} 0 & 0 & I_{g'} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Setting

$$\mathcal{H}_m = \left\{ \begin{array}{l} \text{period domain for PHS's} \\ \text{of weight 1 with } \dim V = 2m \end{array} \right\}$$

we have

$$\mathcal{H}_g^* = \mathcal{H}_g \cup \mathcal{H}_{g-1} \cup \cdots \cup \mathcal{H}_0 .$$

The extended period mapping

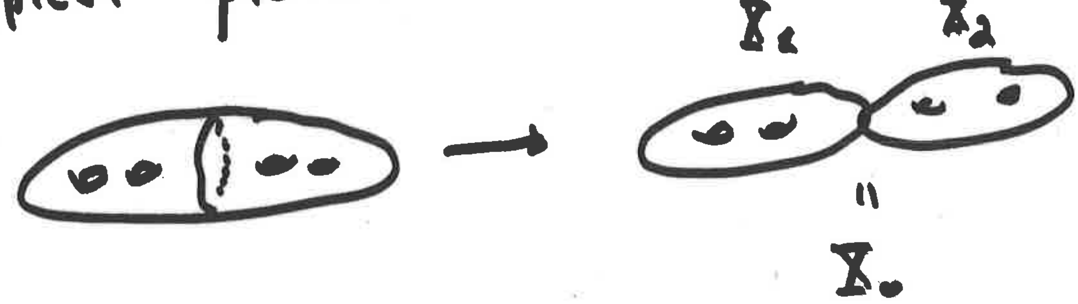
$$\Phi_e : \overline{\mathcal{M}}_g \longrightarrow \Gamma_g \backslash \mathcal{H}_g^*$$

maps the component $\mathcal{M}_{g,g'}$ of the curves above to $\Gamma_{g-g'} \backslash \mathcal{H}_{g-g'} \subset \partial(\Gamma_g \backslash \mathcal{H}_g^*)$ by

$$\Phi_e(X_0) = \Phi(\tilde{X}_0) \in \Gamma_{g-g'} \backslash \mathcal{H}_{g\cdot g'}$$

In general, for a stable nodal curve X_0 the point $\Phi_e(X_0) \in \partial(\Gamma_g \backslash \mathcal{H}_g)$ is defined by the same equation.

Typical picture is



where in this case $\Phi_e(X_0) = \Phi(X_1) \oplus \Phi(X_2)$

Conclusions:

(i) The irreducible, stable curves form the basic "buiding blocks" of $\overline{\mathcal{M}}_g$, in the sense that every $X_0 \in \overline{\mathcal{M}}_g$ is obtained by attaching irreducible stable curves

(ii) We may initially organize $\overline{\mathcal{M}}_g$ by using

$$\Phi_e : \overline{\mathcal{M}}_g \to \Gamma_g \backslash \mathcal{H}_g^*$$

The stratification

$$\Gamma_g\backslash\mathcal{H}_g^* = (\Gamma_g\backslash\mathcal{H}_g)\cup(\Gamma_{g-1}\backslash\mathcal{H}_{g-2})\cup\cdots\cup(\Gamma_0\backslash\mathcal{H}_0)$$

suggests stratifying $\overline{\mathcal{M}}_g$ by pulling back this stratification of $\Gamma_g\backslash\mathcal{H}_g^*$

(iii). In each of these strata of $\overline{\mathcal{M}}_g$ there is an irreducible component $\mathcal{M}_{g,g'}$ consisting of irreducible, stable curves as above. The dual graphs are

(iv) We may then organize the strata in $\overline{\mathcal{M}}_g$ that map to a given Hodge-theoretic stratum by classifying the admissable Γ's with

$$H^1(|\Gamma|) = g'$$

<u>Goal</u>: Use Hodge theory, together with the theory of algebraic surfaces and an understanding of KSBA singularities to obtain a similiar picture for $\overline{\mathcal{M}}_H$ and $\overline{\mathcal{M}}_I$

<u>Next steps</u>: (a) Picture of $\partial(\Gamma\backslash D^*)$ for PHS's of weight $n=2$ with $h^{2,0}=2$

(b) Describe "principal components" of $\partial\mathcal{M}_H$ that map to the strata of $\partial(\Gamma\backslash D^*)$

Addendum: Given a (unique) KSBA degeneration $\mathcal{X} \longrightarrow \Delta$ with T unipotent (no finite group quotient singularities) how much of the LMHS can we compute from X_0?

Answer: There is a natural SSR associated to the KSBA family and from this one has

Theorem: $I^{p,0} \cong I^{p,0}(H^2(X_0))$ for $0 \leq p \leq 2$

That is, the local invariant cycle theorem holds for the part

of the Hodge diamond. Missing is $I^{1,1}_{\text{prim}}$