V. Moduli and Hodge theory

- Review of period domains and their compact duals
- Period mappings
- local Torelli for H and I-surfaces
- global monodromy " " "

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- Given (V, Q, f^p) construct

$$\check{D} = \left\{ F = \{F^p\} \;\middle|\; \begin{array}{l} F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}} \\ Q(F^p, F^{n-p+1}) = 0 \\ \dim F^p = f^p \end{array} \right\}$$

- $\check{D} = G_{\mathbb{C}}/P$ where $G = \mathrm{Aut}_Q(V)$ and $P =$ stabilizer of point in $\check{D}$
- $n = 1$ $\quad F^1 \cong \mathbb{C}^g \subset \mathbb{C}^{2g}$ where $Q = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$

$$G = Sp(2g), \quad P = \begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix}.$$

$$A = I_g \Rightarrow B = {}^tB$$

- $\underline{n=2}$, $F^2 \subset F^{21}$, $Q = \begin{pmatrix} 0 & 0 & I_a \\ 0 & -I_b & 0 \\ I_a & 0 & 0 \end{pmatrix}$

  $f^2 = a$, $f^2 = a+b$

  - $G \cong O(2a, b)$, $P = \begin{pmatrix} A & C & D \\ 0 & B & C' \\ 0 & 0 & {}^tA^{-1} \end{pmatrix}$, $B = {}^tB^{-1}$

  - $A = B = \pm I \Rightarrow C' = {}^tC$

    $A = B = \pm I$, $C = 0 \Rightarrow D = {}^tD$

- $T_E \operatorname{Grass}(k, \mathbb{C}^n) \cong \operatorname{Hom}(E, \mathbb{C}^n/E)$

  - E_t = curve, $e \in E_0$ and $e_t \in E_t$

    with $e_0 = e$

    $$\frac{d}{dt} \longrightarrow \left.\frac{de_t}{dt}\right|_{t=0} \bmod E_0$$

  - write $\dot{E}$ "=" $\left.\frac{dE_t}{dt}\right|_{t=0}$

  - $\check{D} \subset \prod \operatorname{Gross}(f^p, V_{\mathbb{C}})$

    $\Downarrow$

    $T_F\check{D} \subset \bigoplus \operatorname{Hom}(F^p, V_{\mathbb{C}}/F^p)$

- <u>Defn</u>: The <u>infinitesimal period relation</u> (IPR) is the sub bundle $I \subset T\check{D}$ defined by

$$I = \{\dot{F} : \dot{F}^p \subseteq F^{p-1}\}$$

If we write

$$T_F\check{D} = \begin{pmatrix} * & * & * & * \\ A & * & * & * \\ B & C & * & * \\ D & E & F & * \end{pmatrix}, \text{ then}$$

$$I = \begin{pmatrix} * & * & * & * \\ A & * & * & * \\ 0 & C & * & * \\ 0 & 0 & F & * \end{pmatrix}$$

- <u>Examples</u>

$n=1$ - no condition, $\dot{F}^1 \subset F^0 = V_{\mathbb{C}}$

$n=2$ $\left\{ \begin{array}{l} V^{2,0} \xrightarrow{A} V^{1,1} \\ V^{1,1} \xrightarrow{{}^tA} V^{0,2} \\ V^{2,0} \xrightarrow{B=0} V^{0,2} \end{array} \right.$, $B \in \wedge^2 V^{0,2}$

$V^{1,1} = V^{1,1*}$
$V^{0,2} = V^{2,0*}$

- no condition if $h^{2,0} = 1$ (K3 case)
- one condition if $h^{2,0} = 2$ (H+I-surface)

- if $h^{2,0} = a$, $h^{1,1} = b$ then

$$\dim \check{D} = ab + \frac{a(a-1)}{2}$$

- for $a = 2$ this gives

$$\dim \check{D} = 2b + 1$$

<u>Proposition</u>: $\check{D}$ is a <u>contact manifold</u>

Recall that a contact manifold has a 1-form θ with

$$\theta \wedge (d\theta)^b \neq 0$$

This condition is invariant under scaling, so one usually thinks of a line bundle $L \subset T^*\check{D}$ satisfying the above condition.

- By the Darboux theorem there exist local holomorphic coordinates

$$z, x^1, \ldots, x^b, y_1, \ldots, y_b$$

such that

$$\boxed{\theta = c\left(dz - \sum_{1}^{b} dx^\alpha \wedge dy_\alpha\right)}$$

V.3.b

$$\theta = dz + \sum_{i=1}^{b} x^i dy_i$$

$$d\theta = -\sum_{i=1}^{b} dx^i \wedge dy_i$$

- If we have a distribution $I \subset T\check{D}$ then an <u>integral manifold</u> is a submanifold $S \subset \check{D}$ with

$$TS \subset I\big|_S$$

- For a contact system we have
  - $\dim S \leq b$
  - S is given locally by

$$(x^1, \dots, x^d) \to \big(z(x); \underbrace{x^1, \dots, x^d}, 0; \partial_{x^i} z(x), \dots 0\big)$$

$\qquad\qquad ||$

1-jet of an arbitrary function $z(x^1, \dots, x^d)$

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- If all $h^{p,q} \neq 0$, then I is the smallest bracket generating, $G_{\mathbb{C}}$-invariant distribution in $T\check{D}$

- <u>Theorem: Let $\mathcal{X} \to \Delta$ be a family of smooth projective algebraic varieties. Identify all $H^n(X_t, \mathbb{C})$ with $H^n(X_0, \mathbb{C}) = V_{\mathbb{C}}$. Then</u>
  - $F_t^p = H^{n,0}(X_t) + \cdots + H^{p,n-p}(X_t)$ <u>varies holomorphically with t</u>
  - $\dot{F}_t^p \subseteq F_t^{p-1}$ (IPR)

<u>Note</u>. Except in the classical cases a general PHS is <u>not</u> the PHS of an algebraic variety

- <u>Period mapping: $\mathcal{X} \xrightarrow{\pi} S$ with $\mathcal{X} \subset \mathbb{P}^N$</u>

<u>For $\tilde{\mathcal{X}} \to \tilde{S}$ identify all then $H^n(X_{\tilde{s}})_{prim}$ with V. Then we have</u>

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- $\tilde{\Phi} : \tilde{S} \to D \subset \check{D}$

$$\begin{array}{ccl} \tilde{s} & \to & F^p_{\tilde{s}} = \left\{ \begin{array}{l} \text{Hodge filtration} \\ \text{for } H^n(X_{\tilde{s}}) \end{array} \right\} \end{array}$$

- $\tilde{\Phi}_* : T\tilde{S} \to I \subset TD$

- <u>~~mono~~ dromy representation</u>

$$\rho : \pi_1(S, s_0) \to \Gamma \subset \mathrm{Aut}_{\mathbb{Q}}(V_{\mathbb{Z}})$$

- $\tilde{\Phi}(\gamma \tilde{s}) = \rho(\gamma)\, \tilde{\Phi}(\tilde{s})$

- $\Phi : S \to \Gamma \backslash D$ is <u>period mapping</u>

<u>Assuming $\Phi(S)$ is smooth, it is an integral manifold of the IPR</u>

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Will now discuss the local period mapping for H and I - surfaces. Recall that we have

- H-surface: $K_X^2 = 2$, $h^{2,0}(X) = 2$,

  $h^{1,1}_{prim}(X) = 27$

- I-surface: $K_X^2 = 1$, $h^{2,0}(X) = 2$

  $h^{1,1}_{prim}(X) = 28$

<u>Proof that $h^{1,1}_{prim}(X) = 27$ for H-surface:</u>

- $b_0 = b_4 = 1$
- $b_1 = b_3 = 0 \quad (q(X) = h^{1,0}(X) = 0)$
- $b_2 = h^{2,0} + h^{1,1} + h^{0,2}$
  $= 4 + h^{1,1}$

$\Rightarrow \chi_{top}(X) = 6 + h^{1,1}$

<u>Noether</u>: $\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + \chi_{top}(X))$

$$\chi(\mathcal{O}_X) = 1 - q(X) + p_g(X) = 3, \qquad \frac{1}{12}(K_X^2 + \chi_{top}(X)) = \frac{1}{12}(2 + 6 + h^{1,1}(X))$$

$\Rightarrow h^{1,1}(X) = 28$

$\Rightarrow h^{1,1}_{prim}(X) = 27$

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- $\mathcal{M}_H, \mathcal{M}_I \overset{\text{defn}}{=}$ KSBA moduli space for H, I - surfaces

- $\dim \mathcal{M}_H = 26$
  $\dim \mathcal{M}_I = 28$

Theorem (generic local Torelli): For a general point of $\mathcal{M}_H, \mathcal{M}_I$ the differential of the period mapping is injective

- For an H-surface, $\Phi(\mathcal{M}_H)$ has dimension 1 less than maximal -
- For an I-surface, $\Phi(\mathcal{M}_I)$ has maximal dimension

Thus we "know" what the PHS's of the X's locally look like as solutions to a PDE system - Only other know example is CY 3-folds

- How does one prove local Torelli?

For surfaces the differential of the period mapping is

$$T_X \mathrm{Def}(X) \to \mathrm{Hom}(H^{2,0}(X), H^{1,1}_{prim}(X))$$

$$\wr\wr \qquad\qquad\qquad \wr\wr$$

$$H^1(\Theta_X) \to H^0(\Omega^2_X)^* \otimes H^1(\Omega^1_X)_{prim}$$

which gives

$$(*) \qquad H^1(\Theta_X) \otimes H^0(\Omega^2_X) \to H^1(\Omega^1_X)$$

induced by

$$\Theta_X \otimes \Omega^2_X \xrightarrow{\lrcorner} \Omega^1_X .$$

Want to show that

(*) <u>has no left kernel</u>

This is a <u>multiplicative</u> cohomological problem

- One case when this is easy is K3's (and in general for CY's). Then

$$\Omega^2_X \cong \mathcal{O}_X \Rightarrow \Theta_X \cong \Omega^1_X$$

and (*) is

$$H^1(\Theta_X) \otimes \mathbb{C} \xrightarrow{\sim} H^1(\Omega^1_X)$$

- In general the method so far has been
  - realize X as a hypersurface in a "known" variety Y
  - describe $H^0(\Omega^2_X)$ by Poincaré residues of sections of a line bundle on Y
  - describe $T_X \operatorname{Def}(X)$ and $H^1(\Omega^1_X)$ in terms of sections of line bundles on Y

In other words, move the question to a multiplicative problem about sections of line bundles over Y. For

$T_X \mathrm{Def}(X)$ we "vary the equation of X". What to do about $H^1(\Omega^1_X)$?

- Recall $\mathbb{P}E \xrightarrow{\pi} \mathbb{P}^1$, $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$

with line bundles

- $\xi = \mathcal{O}_{\mathbb{P}E}(1)$
- $\eta = \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$

where

$$H^0(k\xi + l\eta) \cong H^0(\mathrm{Sym}^k E(l))$$

$$\| $$

$$\mathrm{Sym}^k E \otimes \mathcal{O}_{\mathbb{P}^1}(l)$$

Ex $H^0(\xi)$ gives $f: \mathbb{P}E \to \mathbb{P}^4$ with image $Q_0 = \{x_0 x_2 = x_1^2\}$

- $X^b \subset \mathbb{P}E$ has equation

$$R = L^2 G - F^2 = 0, \quad \begin{cases} L \in H^0(\xi) \\ G \in H^0(4\xi - 2\eta) \\ F \in H^0(2\xi) \end{cases}$$

Then

$$\dot{R} = 2LG\dot{L} + L^2\dot{G} - 2F\dot{F} \in H^0(4\xi\otimes\mathcal{J})$$

where $\mathcal{J} \subset \mathcal{O}_{\mathbb{P}E}$ is the Jacobian ideal generated by partial derivatives of the defining equation of X^b. Thus

$$T_{X^b}\mathrm{Def}(X^b) \cong H^0(4\xi\otimes\mathcal{J}) / \mathrm{Aut}\,\mathbb{P}E$$

- Relation between X^b and X is

$$\begin{array}{ccc} \mathbb{P}E & \xrightarrow{f} & \mathbb{P}^g \\ \cup & & \\ X^b & & \end{array}$$

and

$$f(X^b) = \varphi_{2K_X}(X)$$

Then

- $X^b = \varphi_{2K_X}(X)$ blown up at base points of $|K_X|$
- $T_{X^b}\mathrm{Def}(X^b) \cong T_X\mathrm{Def}(X)$

- How to compute $H^0(K_X)$ in terms of data on $\mathbb{P}E$? Recall that the adjunction conditions for Poincaré residues on a surface having a double curve with pinch points is vanishing on the double curve.

- Claim: $K_{\mathbb{P}E} = -3\xi = \xi^{-3}$.

Proof: V = vector space of dimension n

$$K_{\mathbb{P}V} = \det(V^* \otimes \mathcal{O}_{\mathbb{P}V}(-1)) \cong \Lambda^n V^* \otimes \mathcal{O}_{\mathbb{P}V}(-n)$$

$$\| $$

$$K_{\mathbb{P}^{n-1}}$$

Then from $0 \to \pi^*\Omega^1_{\mathbb{P}^1} \to \Omega^1_{\mathbb{P}E} \to \Omega^1_{\mathbb{P}E/\mathbb{P}^1} \to 0$

we have

$$K_{\mathbb{P}E} = \pi^* K_{\mathbb{P}^1} \otimes K_{\mathbb{P}E/\mathbb{P}^1} \qquad V = E_t^*$$

$$= -2h - 3\xi + 2h = -3\xi$$

- $R \in |4\xi| \Rightarrow$ take Poincaré residues of $\xi^4 \otimes K_{PE} = \xi$

Adjunction conditions given by vanishing on $L = 0$, $L \in |\eta|$

$$\Downarrow$$

- $K_{X^b} \cong \xi - \eta$

- $H^0(K_{X^b}) = H^0(\xi - \eta)$
$$\cong H^0(E(-1))$$
$$\cong H^0(\mathcal{O}_{P^2}(1))$$

- This leads to the natural map

$$T_{X^b}\,\mathrm{Def}(X^b) \otimes H^0(K_{X^b})$$
$$\|$$
$$\frac{H^0(4\xi \otimes \mathcal{J})}{(\cdots)} \otimes H^0(\xi - \eta) \longrightarrow \frac{H^0((5\xi - \eta) \otimes \mathcal{J})}{(\cdots)}$$

<u>Guess</u>: $H^1(\Omega^1_X)_{prim} \cong \dfrac{H^0((5\xi - \eta) \otimes \mathcal{J})}{(\cdots)}$

- With the notations
  - t_0, t_1 = basis for $H^0(\eta)$
  - x = generator of $H^0(\xi - 2\eta)$
  - $xt_0^2, xt_0t_1, xt_1^2, r_0, r_1$ = basis for $H^0(\xi)$

the "Fermat-like" H-surface is

$$\boxed{x t_0^2 (x^3(t_0^6+t_1^6) + r_0^3 + r_1^3) = (r_0^2+r_1^2)^2}$$

A fair amount of computation (in the typed notes, pages 68-74) it is shown that

- this gives an H-surface
- the differential of the period map is given by the diagram on a previous overhead
- this map has no left kernel

<u>Q: Local Torelli for any smooth H-surface?</u>

- <u>Global monodromy</u>

For the period mapping

$$\Phi : \mathcal{M} \to \Gamma \backslash D, \quad \Gamma \subset \underset{\substack{\| \\ G_{\mathbb{Z}}}}{\mathrm{Aut}_{\mathbb{Q}}(V_{\mathbb{Z}})}$$

one says that Γ is <u>arithmetic</u> if $[\Gamma : G_{\mathbb{Z}}] < \infty$. For a while it was felt that Γ is always arithmetic, but examples of Picard - Deligne - Mostow showed this might not be the case. Now there are flood of examples, called <u>thin matrix groups</u>. In the tensor algebra $T(V) = \bigoplus_{m,n} V^{\otimes m} \otimes V^{* \otimes n}$ one has

$$T(V)^{\Gamma} = T(V)^{G_{\mathbb{Z}}},$$

so the invariants do not distinguish Γ from an arithmetic group.

<u>Theorem: The global monodromy groups of H and I-surfaces are arithmetic groups</u>

Proof: In the notes for H-surfaces

<u>Conclusions</u>: For H and I-surfaces, Hodge theory (via the period mapping) provides a very strong invariant of $\mathcal{M}_H, \mathcal{M}_I$.

<u>Next issue</u>: What can Hodge theory tell us about the surfaces corresponding to $\partial\mathcal{M}_H, \partial\mathcal{M}_I$?

Even for curves, Torelli does not hold for $\partial\mathcal{M}_g$. But it <u>does</u> provide a very good guide to the stratification of $\partial\mathcal{M}_g$ and what the curves are over each stratum.

Next lectures

- a possible completion $\Gamma\backslash D^* \supset \Gamma\backslash D$ and the strata of $\partial(\Gamma\backslash D)$ - the picture is

$$\begin{tikzcd} & & \mathrm{II} \arrow[dr, dash] & & \\ 0 \arrow[r, dash] & \mathrm{I} \arrow[ur, dash] \arrow[dr, dash] & & \mathrm{IV} \arrow[r, dash] & \mathrm{V} \\ & & \mathrm{III} \arrow[ur, dash] & & \end{tikzcd}$$

- description of the surfaces lying over II

- work of Liu-Rollenske and a combinatorial description of the most degenerate I and H-surfaces (seminar talk of Mark Green)