III.1

Moduli (informal discussion - see Kollár etc)

- $$\left\{\begin{array}{l}\text{specify a type}\\ \text{of variety } V\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{want } \mathcal{M}_V = \text{quasi-proj.}\\ \text{variety whose points are}\\ \text{equivalence classes of } V\text{'s}\end{array}\right\}$$

- classical examples
  - smooth curves
  - polarized abelian varieties
  - marked K3's

(period domain $\mathcal{D}$ = Hermitian symmetric domain)

- in these lectures will take $\mathcal{M}$ corresponding to smooth, minimal algebraic surfaces X of general type and with fixed Hilbert polynomial $R(X) = \bigoplus_{m\geq 0} \chi(mK_X)$

- would also like to have a universal family

$$\mathcal{X} \to \mathcal{M}_V$$

(any family $\mathcal{X}_B \to B$ of varieties of type V is induced by a unique map $B \to \mathcal{M}_V$) - this is too much due to Aut(X); will have to settle for a versal family

- **Prop**: $\mathcal{X}_1 \to B$, $\mathcal{X}_2 \to B$ two families of smooth surfaces such that the canonical bundles of the fibres are ample. Then any isomorphism of the generic fibres

$$X_{1,\eta} \xrightarrow{\sim} X_{2,\eta}$$

is induced by an isomorphism

$$\begin{tikzcd}
\mathcal{X}_1 \arrow[r, "\sim"] \arrow[d] & \mathcal{X}_2 \arrow[d] \\
B \arrow[r, "\sim"] & B
\end{tikzcd}$$

(ultimately reflects the <u>birational</u> invariance of the $H^0(mK_{X_i})$'s)

- also want canonical completions $\overline{\mathcal{M}}_V$ - one reason is to be able to do enumerative problems (counting - # of X's with a fixed property is x) - think Bezout and $\mathbb{C}^2 \hookrightarrow \mathbb{P}^2$

- need to uniquely define limits

$$\lim_{t \to 0} X_t = X_0 \quad ;$$

i.e. given $\mathcal{X}^* \to \Delta^*$ want to have

$$\mathcal{X} \to \Delta$$

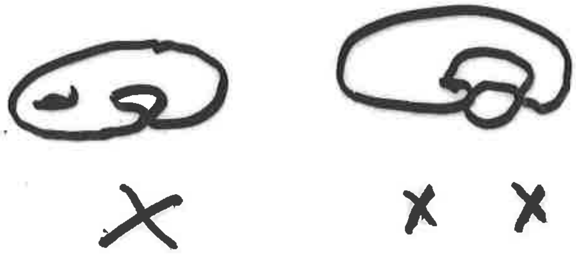
- in practice need to
  - specify which singularities of X_0 are allowed (local)
  - "minimality" (global condition)

Semi-stable reduction (SSR)

- normal crossing variety $x_1^{k_1} \cdots x_\ell^{k_\ell} = 0$

- reduced NCV - locally union of hyperplanes in $\mathbb{C}^{n+1}$

- LNCV and GNCV (local and global)



- given $\mathcal{X}^* \to \Delta^*$ there is a base change ($t' = t^m$) and completion $\mathcal{X} \to \Delta$ where $X_0 =$ [redacted] reduced NCV

- *curves* - local condition is nodes, global condition is ω_{X_0} ample

$$\Updownarrow$$

$\omega_{\mathcal{X}/\Delta}$ relatively ample

(using Proj ω_{X_0} contracts the $(\mathbb{P}^1; \{0,\infty\})'s$)

- *Surfaces* - right local condition is *semi-log-canonical* (slc) singularities (Kollár, Shepherd-Barron, Alexeev)

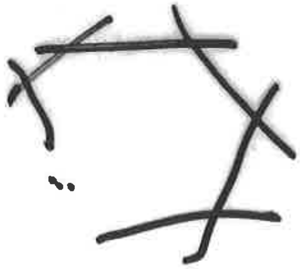
- for us the main singularities will be
  - double curve with pinch points

    $$x^2 z = y^2$$

  - simple elliptic 

    $\mathcal{O}_E(-d) \longrightarrow X$ where blow down (or contract) 0-section with $H^0(\mathcal{O}_E(d))$ (dim $= d \neq 0$, which gives elliptic normal curve in $\mathbb{P}^{d-1}$

  - cusp - E is cycle of smooth rational curves



where all C_i have $\begin{cases} C_i^2 \leq -2 \\ \text{one has } C_i^2 \leq -3 \end{cases}$

- finite quotients of those
- reducible cases; glue the above together (more on this later)

- global condition is

$$\underline{K_X \text{ is ample}}$$

  - X normal $\Rightarrow K_X = j_*(K_{X_{reg}})$

    sections are holomorphic n-forms on X_{reg} that are in L^2

  - double curve D with pinch points

$$\mathrm{Res}\left(\frac{f(x,y,z)\,dx\wedge dy\wedge dz}{x^2z-y^2}\right)$$

    where $f=0$ on $D=\{x=0, y=0\}$

- for us the singularities will be hypersurface singularities - then there is a list

- **Local moduli space**

- X = compact, complex manifold. Then there exists a *versal* family

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] \\ B \end{tikzcd} = \left\{ \begin{array}{l} \text{proper holomorphic map} \\ \text{with smooth fibres} \\ \text{and } X_{b_0} = X \end{array} \right\}$$

such that for any family $\mathcal{X}' \xrightarrow{\pi'} B'$ with $X'_{b_0'} = X$ we have

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d, "\pi'"'] & \mathcal{X} \arrow[d, "\pi"] \\ B' \arrow[r, "f"] & B \end{tikzcd} \qquad \left\{ \begin{array}{l} \mathcal{X}' = \mathcal{X} \times_f B' \\ f(b_0') = b_0 \text{ and} \\ F : X_{b_0'} \cong X_{b_0} \end{array} \right\}$$

- $B \subset H^1(\Theta_X)$ and $B = H^1(\Theta_X)$ if $H^2(\Theta_X) = 0$.
  ($\Theta_X = T_X$ = dual of Ω^1_X)

- For X any analytic variety (compact or a germ) there is same result with $H^1(\Theta_X)$ replaced by $\mathbb{E}xt^1(\Omega^1_X, \mathcal{O}_X)$

III.6.b

- local to global spectral sequence gives

$$\begin{tikzcd}
\arrow[r] & H^1(\mathcal{E}xt^0(\Omega^1_X, \mathcal{O}_X)) \arrow[r] \arrow[d, leftrightarrow] & \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \arrow[r] \arrow[d, leftrightarrow] & H^0(\mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X)) \arrow[r] \arrow[d, leftrightarrow] & {} \\
\arrow[r] & T_X \mathrm{Def}^{es}(X) \arrow[r] & T_X \mathrm{Def}(X) \arrow[r] & \bigoplus_{x \in X} T_{X_x} \mathrm{Def}(X_x) \arrow[r] & {}
\end{tikzcd}$$

- <u>General properties of $\mathcal{M}$'s</u>

- "Murphey's law" - aside from curves, abelian varieties, K3's + CY's, anything that can go wrong will do so (Vakil)

<u>Ex</u> $\mathcal{X} \to \Delta$ with $X_t \cong \mathbb{P}^1 \times \mathbb{P}^1$ for $t \neq 0$

but $X_0 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \not\cong \mathbb{P}^1 \times \mathbb{P}^1$

(jumping of structure at $t = 0$; moduli space is 0-dimensional but

$$\begin{tikzcd}
\underset{\mathbb{P}^1 \times \mathbb{P}^1}{\bullet} & \underset{\mathbb{F}_2}{\bullet} \arrow[l] & \underset{\mathbb{F}_4}{\bullet} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

III.6.c

Ex (fake quintics): $\mathcal{M}$ for smooth surfaces (minimal, general type) with $K_X^2 = 5$, $p_g(X) = 4$, $q(X) = 0$ is

$$\left\{ \begin{array}{l} \mathcal{M}_1^{\#} = \text{smooth quintics } X \hookrightarrow \mathbb{P}^3 \\ \mathcal{M}_2 \leftrightarrow \varphi_{K_X} : X \xrightarrow{2:1} X' \end{array} \right.$$

- for H, I surfaces none of pathologies occur - maybe they should be added to the list with curves...

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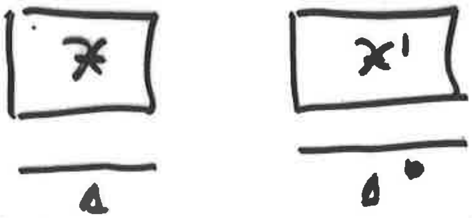
- <u>What is the right generalization of smooth curves of genus $g \geq 2$</u> ?

Ex can have



with

- $X_t \cong X'_t$ for all t
- $\mathcal{X} \ncong \mathcal{X}'$

III.6.d

$\rightsquigarrow$ (-2)-curves. Consider $\{x^2+y^2+z^2=0\} = X_0$

- Resolution is blow up with

$$\begin{tikzcd}\tilde{X}_0 \arrow[r] & X_0 \\ E \arrow[u, phantom, sloped, "\subset"] \arrow[r] & \{0\} \arrow[u, phantom, sloped, "\in"]\end{tikzcd}$$

where $E^2 = -2$

- $(u, \lambda_1, \lambda_2) \to (u, u\lambda_1, u\lambda_2)$

$$\operatorname{Res}\left(g \frac{dx \wedge dy \wedge dz}{x^2+y^2+z^2} \right) = \operatorname{Res}\left(g \frac{(u, u\lambda_1, u\lambda_2)\, du \wedge d\lambda_1 \wedge d\lambda_2}{1+\lambda_1^2+\lambda_2^2} \right)$$

$$\Downarrow$$

$\varphi_{mK_{\tilde{X}_0}}$ contracts E for all m

- for smooth surface of general type this is all that can happen

$$\overbrace{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx}}^{\leadsto}$$

~~Take family which locally looks like $x^2+y^2+z^2=t^2$, smooth total space but singular X_0 blow up origin and~~

- take family that locally looks like $x^2+y^2+z^2=t^2$ - blow up origin to get exceptional $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ and blow down Q to $\mathbb{P}^1 \times \{p_2\}$ and $\{p_1\} \times \mathbb{P}^1$ - get above phenomenon where fibres over origin have -2 curves

⇝ Right generalization of smooth curves with $g \geq 2$ are canonical models (RDP singularities)

- What singularities should be allowed?

- for curves the conditions are

  (i) X_0 is nodal

  (ii) ω_{X_0} is ample

(i) arises from semi-stable-reduction (SSR)

this allows  when we want . To resolve this we require ω_{X_0} ample. Rephrased we have $X \xrightarrow{f} B$ where X is smooth, ω_X is f-ample. What about (ii). Here there is a subtlety.

IV. 6 f

Require X normal with desingularization $Y \xrightarrow{\pi} X$ and

(*) $$K_Y = \pi^* K_X + E, \qquad E \geq 0$$

Moreover this should hold for any base change. For curves this implies X_0 is nodal

For surfaces we have $X \xrightarrow{f} B$ where X is normal and B is a smooth curve

$$\begin{cases} \text{(i) } K_X \text{ is } f\text{-ample} \\ \text{(ii) } K_Y = \pi^* K_X + E, \quad E \geq 0 \\ \text{(iii) this holds for any base change} \end{cases}$$

These are (in 1st approximation) the conditions for surfaces - (iii) implies for $X_0' \xrightarrow{\pi} X_0$ any desingularization

$$K_{X_0'} = \pi^* K_{X_0} + \sum a_i E_i, \qquad a_i \geq -1$$

Then X_0 has <u>semi-log-canonical</u> (or slc) singularities

Theorem (KSBA): (i) There exists a moduli space for surfaces X satisfying

- X is minimal and has canonical singularities
- X is of general type
- $\chi(mK_X)$ is given

(ii) $\mathcal{M}$ has a canonical completion $\overline{\mathcal{M}}$ and the surfaces X_0 corresponding to points of $\partial\mathcal{M}$ satisfy

- X is minimal and has slc singularities
- X general type
- same Hilbert polynomial

Moreover, $\overline{\mathcal{M}}$ is projective.

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Issue: No example of description of global structure of surfaces corresponding to $\partial\mathcal{M}$; no example of stratification of $\overline{\mathcal{M}}$

Moduli of smooth H-surfaces

- Recall equation of $\mathbb{X}^6 \subset \mathbb{P}E$

$$x t_0^2 G = F^2$$

where $x \in |\xi - 2h|$, $t_0 \in |h|$, $G \in |3\xi|$

Abusing notation put x, G together and call it G to have

$$L^2 G = F^2$$

$L \in |h|$
$G \in |4\xi - 2h|$
$F \in |2\xi|$

- Locally this is

$$x^2 z = y^2$$

and Jacobian ideal is $\{x^2, xz, y\}$, which globally gives

$$\mathcal{J} = \{L^2, LG, F\} \subset \mathcal{O}_{\mathbb{P}E}$$

- Away from pinch points, $\mathcal{J}$ = ideal of double curve - additiona generator (skyscraper sheaf) appears at pinch points

- **Theorem**: $T_X \mathcal{M}_H \cong H^1(\Theta_X) \cong T_{X^b}\left(\dfrac{H^0(\mathbb{P}E, \mathbb{P}^4_{\circ} g)}{\mathrm{Aut}(\mathbb{P}E)}\right)$

Corollary: $\mathcal{M}_H$ is smooth, connected of dimension $h^1(\Theta_X) = 26$

- Recall basic diagram

$$\begin{tikzcd}
 & \mathbb{P}E \arrow[r] & \mathbb{P}^4 \\
\hat{X} \arrow[r, "g"] \arrow[d] & X^b \arrow[r, "f"] \arrow[u, phantom, "\cup"] & X' \arrow[u, phantom, "\cup"] \\
X \arrow[urr, "\varphi_{2K_X}"'] & &
\end{tikzcd}$$

, $f = |\mathcal{O}_{\mathbb{P}E}(3)| = |3|$

- Will show

$$T_X \mathrm{Def}(X) \cong T_{\hat{X}} \mathrm{Def}(\hat{X}) \cong T_{X^b} \mathrm{Def}^{es}(X^b)$$

$$\cong T_{X^b}\left(\frac{\text{equation of } X^b}{\mathrm{Aut}'s}\right)$$

- Diagram is intrinsic to X, so $\cong$'s make sense
  - $T_X \mathrm{Def}(X) \cong T_{\hat{X}}(\mathrm{Def}(\hat{X}))$

pf E exceptional curve

$$\begin{cases} e = c_1([E]) \in H^1(\Omega^1_{\hat{X}}) \\ \theta \in H^1(\Theta_{\hat{X}}) \\ \omega \in H^0(\Omega^2_{\hat{X}}) \cong H^0(\Omega^2_X), \quad \omega|_E = 0 \end{cases}$$

$\Rightarrow \langle \theta e, \omega \rangle = \langle e, \theta \cdot \omega \rangle = \theta \cdot \omega|_E = 0$

$\Rightarrow$ $[E]$ deforms (to 1st order)

$$0 \to K_{\hat{X}}(-E) \to K_{\hat{X}} \to \mathcal{O}_E(-1) \to 0$$

$\Rightarrow H^1(K_{\hat{X}}(-E)) = 0$

$\Rightarrow H^1([E]) = 0$

$\Rightarrow$ E deforms (to 1st order)

- $f \circ g = \varphi_{2K_{\hat{X}}}$, g = normalization

  f = contract $g(E_1), g(E_2) = -2$ curves

- $\mathrm{Aut}(\mathcal{X})^{0} = H^{0}(\Theta_{\mathcal{X}}) = 0 = H^{0}(\Theta_{\hat{\mathcal{X}}})$

  $$\mathrm{Aut}(\mathcal{X}^{b})^{0} = \left\{ \begin{array}{l} \text{tangent space to equisingular} \\ \text{deformations of } \mathcal{X}^{b} \subset \mathbb{P}E \end{array} \right\}$$

  $$= 0$$

- Conclusion: Will suffice to compute tangent space to the deformations of the equation

$$R = L^{2}G - F^{2} = 0$$

- Crucial computation:

$$h^{0}(\mathbb{P}E, \mathcal{I}^{q} \otimes \mathcal{J}) = 40$$

– $\{LG, L^{2}, F\}$ not a regular sequence but does have relatively simple free resolution given by the table

	LG	L^2	F
rel_1	L	$-G$	0
rel_2	F	0	$-LG$
rel_3	0	F	$-L^2$

where each row is relation among the generators at the top - single generating syzygy

$$F\,\mathrm{rel}_1 - L\,\mathrm{rel}_2 + G\,\mathrm{rel}_3 = 0$$

- this gives

$$\begin{tikzcd} 0 \arrow[r] & \xi^{-6} \arrow[r, "\begin{pmatrix} F \\ -L \\ G \end{pmatrix}"] & \begin{matrix} \xi^{-4} \\ \oplus \\ \xi^{-6}\otimes\eta \\ \oplus \\ \xi^{-2}\otimes\eta^{-2} \end{matrix} \arrow[r, "\begin{pmatrix} L & F & 0 \\ -G & 0 & F \\ 0 & -LG & -L^2 \end{pmatrix}"] & \begin{matrix} \xi^{-4}\otimes\eta \\ \oplus \\ \eta^{-2} \\ \oplus \\ \xi^{-2} \end{matrix} \arrow[r, "{(LG, L^2, F)}"] & g \arrow[r] & 0 \end{tikzcd}$$

- tensor $(*)$ with ξ^4 - hypercohomology spectral sequence will abut to $H^0(\xi^4 \otimes g)$. - lucky because using

  - $h^0(a\xi + b\eta) \neq 0 \iff a \geq 0, b \geq -2a$
  - $h^1(a\xi + b\eta) \neq 0 \iff a \geq 0, b \leq -2$
  - $h^2(a\xi + b\eta) \neq 0 \iff a \leq -3, b \geq 2a + 8$
  - $h^3(a\xi + b\eta) \neq 0 \iff a \leq -3, b \leq -2a - 6$

the spectral sequence degenerates and

$$h^0(\xi^4 \otimes g) = h^0(\xi^2) + h^0(\xi^4 \otimes \lambda^{-2}) + h^0(\lambda) - h^0(\xi^2 \otimes \lambda^{-2}) - h^0(\mathcal{O}_{\mathbb{P}E})$$

$$= 14 + 30 + 2 - 5 - 1 = 40$$

<u>Note</u>: Using

- $R^0_\pi(\xi^a \otimes \lambda^b) = (R^0_\pi \xi^a)(-b)$

  $= \mathrm{Sym}^a E(-b)$

- $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$

- $\begin{cases} h^0(\mathcal{O}_{\mathbb{P}^1}(c)) \neq 0 \iff c \geqq 0 \\ h^1(\mathcal{O}_{\mathbb{P}^1}(c)) \neq 0 \iff c \leqq -2 \end{cases}$

- $\dim \mathrm{aut}(\mathbb{P}E) = 1$[illegible]

<u>Proof</u>: $E \to \mathbb{P}^1$ is homogeneous vector bundle so have

$$0 \to \underset{\underset{\boxed{\text{fibre preserving automorphisms}}}{\uparrow}}{\mathrm{Aut}\, E} \to \mathrm{Aut}(\mathbb{P}E) \to \mathrm{Aut}(\mathbb{P}^1) \to 0$$

need to compute over $\mathbb{P}^1$

$$h^0(\mathrm{End}\, E) = h^0(E \otimes E^*)$$

$$\| $$

$$h^0 \begin{pmatrix} \mathcal{O}_{\mathbb{P}^1} & \mathcal{O}_{\mathbb{P}^1} & \mathcal{O}_{\mathbb{P}^1}(2) \\ \mathcal{O}_{\mathbb{P}^1} & \mathcal{O}_{\mathbb{P}^1} & \mathcal{O}_{\mathbb{P}^1}(2) \\ \mathcal{O}_{\mathbb{P}^1}(-2) & \mathcal{O}_{\mathbb{P}^1}(-2) & \mathcal{O}_{\mathbb{P}^1} \end{pmatrix}$$

$$\|$$

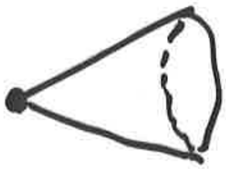
$$11$$

- <u>Note</u>. Have

$$H^0(\mathrm{End}(E)) \cong \begin{pmatrix} \mathfrak{gl}(2,\mathbb{C}) & \mathcal{U} \\ 0 & \mathfrak{gl}(1,\mathbb{C}) \end{pmatrix}$$

where $\mathcal{U}$ = unipotent radical of the Lie algebra $H^0(\mathrm{End}(E))$

- $\mathrm{Aut}(\mathbb{P}E)$ acts on $\mathbb{P}E$ with two orbits
  - closed orbit S $\longleftrightarrow$ $Q_{0,\mathrm{sing}}$
  - open orbit $\mathbb{P}E \setminus S$ $\longleftrightarrow$ $Q_{0,\mathrm{reg}}$



- Alternate computation of $h^1(\Theta_X)$

- $h^1(\Theta_X) = -\chi(\Theta_X) + h^2(\Theta_X)$

$$= 26 + h^2(\Theta_X)$$

which gives

$$h^1(\Theta_X) \leqq 26$$

- if we know that

$$T_{X^b}\Big(\underset{\substack{\| \\ \boxed{\text{equisingular deformations of } X^b \subset \mathbb{P}E}}}{\mathrm{Def}^{es}(X^b)}\Big) \longrightarrow T_X(\mathrm{Def}(X))$$

is injective (almost evident geometrically) then, we get $h^1(\Theta_X) = 26$ and $\mathrm{Def}(X)$ is smooth (and unobstructed)

- $\mathcal{M}_H$ irreducible needs
  - analysis of smooth but not general X's
  - argument that the space of equations is connected