Moduli (informal discussion—see Kodaira et al.)

- specify a type
  - of variety
  - want $\mathcal{M}_V = \text{quasi-proj.}$
    - variety whose points are equivalence classes of $V$'s
    - smooth curves

- classical examples
  - polarized abelian varieties
  - marked $H^3$'s

(pseudo domain $D = \text{Hermitian symmetric domain}$)

- in these lectures will take $\mathcal{M}$ corresponding
to smooth, minimal algebraic surfaces $X$
of general type and with fixed Hilbert polynomial $R(X) = \bigoplus mK_X$

- would also like to have a universal family $X \to \mathcal{M}_V$
(any family $x^0_B \to B$ of varieties of type V is induced by a unique map $B \to \mathcal{M}_V$.)

this is too much due to $Aut(X)$; will have to settle for a versal family.

Prop. $X_1 \to B$, $X_2 \to B$ two families of smooth surfaces such that the canonical bundles of the fibres are ample. Then any isomorphism of the generic fibres

$$X_1, \eta \simeq X_2, \eta$$

is induced by an isomorphism $X_1 \simeq X_2$

$$\begin{array}{c}
X_1 \downarrow \phi \downarrow X_2 \\
B \times \phi \\
\end{array}$$

(ultimately reflects the birational invariance of the $H^0(mK_{X_2})$'s)
also want canonical completions $\overline{M}_Y$ - one reason is to be able to do enumerative problems (counting - # of $X$'s with a fixed property $1 \times$) - think Bogomolov and $C^3 \rightarrow P^2$

need to uniquely define limits

$\lim_{b \rightarrow 0} X_b = X_0$

so given $X^r \rightarrow A^r$ want to have

$X \rightarrow A$

in practice need to specify which singularities of $X_0$ are allowed (local) "minimality" (global condition)

Semi-stable reduction (SSR)

- normal crossing variety $X \sim X_1 \cdots X_2 \vee 0$
- reduced NCV - locally uniruled hypersurfaces in $\mathbb{P}^{n+1}$
- LNCV and GNCV (local and global)
  (nodes)
  $\times \times \times$
- given $X^* \to \Delta^*$ there is a base change ($t' = t''$) and completion $X \to \Delta$
  where $\overline{X}_0 = $ reduced NCV

- curves - local condition is nodes, global condition is $\omega_{X_0}^n$ ample

  $\omega_{\overline{X}/\Delta}$ relatively ample

  (using Proj $\mathcal{O}_{X_0}$, contracts the $(\mathbb{P}^2; s_0, s_1)$'s)

- surfaces - right local condition is semi-log-canonical (slc) singularities
  (Kollár, Shepherd-Barron, Alexeev)
• for us the main singularities will be
  - double curve with pinch points
    \[ x^3 z = y^2 \]
  - simple elliptic
    \[ O^{\omega} (d) \mapsto \emptyset \]
  where blow down
  (or contrast) a section with
  \[ H^0 (\omega (d)) \quad (\dim = d \omega) \]
  which gives elliptic normal curve in \( \mathbb{P}^{7-d} \)
  - cusp - \( E \) is cycle of smooth
    rational curves

where all \( C_i \) have \( C_i^3 \Xi - 3 \)
finite quotients of those

reducible cases; glue the above together (more on this later)

global condition is

\( K_X \) is ample

\[- \text{ normal} \Rightarrow K_X = \text{det}(K_{X_{	ext{reg}}})\]

sections are holomorphic \( n \)-forms on \( X_{\text{reg}} \) that are in \( L^2 \)

double curved with pinch points

\[\text{Res} \left( \frac{f(x,y,z) \, dx \wedge dy \wedge dz}{x^2 + z^2 - y^2} \right)\]

where \( f = 0 \) on \( D = \{ x = 0, y = 0 \} \)

for us the singularities will be hypersurface singularities - then there is a list
Local moduli space

- $X = \text{compact, complex manifold. Then there exists a versal family}$

$$\mathcal{X} = \left\{ \begin{array}{c}
\text{proper holomorphic map} \\
\text{with smooth fibres}
\end{array} \right\}$$

$B \xrightarrow{\pi} \mathcal{X}$

such that for any family $X' \to B'$ with $X'_{b'} = X$ we have

$$X' \to X$$

$B' \xrightarrow{\pi'} \mathcal{X}$

- $B \subset H^2(\mathcal{O}_X)$ and $B = H^2(\mathcal{O}_X)$ if $H^2(\mathcal{O}_X) = 0$

$(\mathcal{O}_X = T_X = \text{deal of } \Omega^1_X)$

- For $X$ any analytic variety (compact on a germ) there is some result with $H^2(\mathcal{O}_X)$ replaced by $\text{Ext}^2(\Omega^1_X, \mathcal{O}_X)$
- local to global spectral sequence gives

\[ \cdots \to H^2(\mathcal{E}^0(\Omega^2_X, \mathcal{O}_X)) \to \mathcal{E}^2(\Omega^2_X, \mathcal{O}_X) \to H^0(\mathcal{E}^2(\Omega^2_X, \mathcal{O}_X)) \to \cdots \]

\[ \to T_X \text{Def}^0(X) \to \bigoplus_{x \in X} T_x \text{Def}(X) \]

- **General properties of \( \mathcal{M}'s \)**

  - Murphy's law - aside from curves, abelian varieties, \( K3's \) and \( CI's \), anything that can go wrong will do so (Vakil)

  \[ \exists \, X \text{ with } Y_o \cong \mathbb{P}^2 \times \mathbb{P}^2 \text{ for } t \neq 0 \]

  \[ \downarrow \quad \text{but} \quad Y_0 = \mathbb{P}(\omega_{\mathbb{P}^2}(1) \otimes \mathcal{O}(3)) \not\cong \mathbb{P}_2 \times \mathbb{P}^2 \]

  (jumping of structure at \( t = 0 \); moduli space is 0-dimensional but

  \[ \mathbb{P}^2 \times \mathbb{P}^2 \quad \mathcal{F}_3 \quad \mathcal{F}_4 \quad \cdots \]
Ex (fake quintics): $\mathcal{M}$ for smooth surfaces (minimal, general type) with $K^2 = 5$, $p_g(X) = 4$, $q(X) = 0$ is

\[ m_2 \times m_3 \]

\[ m_2^* = \text{smooth quintics } \mathbb{X} \rightarrow \mathbb{P}^3 \]

\[ m_3 \leftrightarrow \phi \in \mathbb{P} : \mathbb{X} \rightarrow \mathbb{X}' \]

- for $H, I$ surfaces, none of pathologies occur - maybe they should be added to the list with curves...

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- What is the right generalization of smooth curves of genus $g \geq 2$?

Ex can have $\square \times \square \times \square$ with

- $X_t \cong X'_t$ for all $t$
- $X \not\cong X'$
\[ \Psi \rightarrow (\mathcal{A}) \text{-curves. Consider } \{x^2 + y^2 + z^2 = 0\} \subset X_0. \]

- Resolution is blow up with \( \tilde{X}_0 \rightarrow X_0 \)
  where \( E^2 = -2 \)

- \((\mu, \lambda_1, \lambda_2) \rightarrow (\mu, \mu \lambda_1, \mu \lambda_2)\)

\[ \text{Res} \left( f \frac{dx dy dz}{x^2 + y^2 + z^2} \right) = \text{Res} \left( g (\mu \lambda_1 \lambda_2) d\lambda_1 d\lambda_2 \right) \]

\[ \Phi \text{ contracts } E \text{ for all } m \]

- For smooth surface of general type
  this is all that can happen

Take family which locally looks like
\[ x^2 + y^2 + z^2 = \epsilon^2 \] - blow up origin get exceptional \( \tilde{E} \cong \mathbb{P}^2 \times \mathbb{P}^2 \) and blow down \( \tilde{E} \) to \( \mathbb{P}^2 \times \mathbb{P}^2 \) and \( \mathbb{P}_1 \times \mathbb{P}^2 \) - got above phenomenon where fibres over origin have \(-2\) curves
Right generalization of smooth curves with $q \geq 2$ are canonical models (RDP singularities)

- What singularities should be allowed?

- for curves the conditions are

  \[ \begin{align*}
  (i) & \quad X_0 \text{ is nodal} \\
  (ii) & \quad W_{X_0} \text{ is ample}
  \end{align*} \]

(i) arises from semi-stable-reduction (SSR)

This allows \( \exists \) when we want \( \exists \). To resolve this we require \( W_{X_0} \) ample. Rephrased we have \( \Xi \rightarrow B \) where \( \Xi \) is smooth, \( W_{\Xi} \) is s-ample. What about (ii). Here there is a subtlety.
Require $X$ normal with desingularization
$Y \overset{f}{\to} X$ and

$$K_Y = \pi^* K_X + E, \quad E \geq 0$$

Moreover, this should hold for any base change. For curves this implies $X_0$ is nodal.

For surfaces we have $X \overset{f}{\to} B$ where $X$ is normal and $B$ is a smooth curve.

(i) $K_X$ is f-ample

\begin{align*}
(ii) & \quad K_Y = \sigma^* K_X + E, \quad E \geq 0 \\
(iii) & \quad \text{this holds for any base change}
\end{align*}

These are (in $2^{nd}$ approximation) the conditions for surfaces - (iii) implies for $X_0 \overset{f}{\to} X$, any desingularization

$$K_{X_0} = \pi^* K_{X_0} + \sum q_i E_i, \quad q_i \geq -1$$

Then $X_0$ has semi-log-canonical (or slc) singularities.
Theorem (KSBA): (i) There exists a moduli space for surfaces $X$ satisfying

- $X$ is minimal and has canonical singularities
- $X$ is of general type
- $\Theta_X(mK_X)$ is given

(ii) $\tilde{M}$ has a canonical completion $\overline{\tilde{M}}$ and the surfaces $X_0$ corresponding to points of $\tilde{M}$ satisfy

- $X$ is minimal and has $slc$ singularities
- $X$ of general type
- Same Hilbert polynomial

Moreover, $\overline{\tilde{M}}$ is projective.

Issue: No example of description of global structure of surfaces corresponding to $\tilde{M}$; no example of stratification of $\overline{\tilde{M}}$. 
Moduli of smooth $H$-surfaces

- Recall equation of $X \subset PE$

$$x \cdot z^a G = F^a$$

where $x \in 12\mathbb{A}_1$, $t_0 \in 12\mathbb{A}_1$, $G \in 13\mathbb{A}_1$

Abusing notation put $x, G$ together and call it $G$ to have

$$L^2 G = F^a \leq \frac{L \in 12\mathbb{A}_1}{G \in 14\mathbb{A}_1} \leq F \in 12\mathbb{A}_1$$

- Locally this is

$$x z^a = y^a$$

and Jacobian ideal is $\{ x^a, xz, y^a \}$, which globally gives

$$\mathfrak{g} = \{ L^a, LG, F \} \subset \mathcal{O}_E$$

- Away from pinch points, $\mathfrak{g}$ = ideal of double curve - additional generator (skyscraper sheaf) appears at pinch points
Theorem: \( T_X M H \cong H^a(C_X) \cong T_X^b \left( \frac{H^0(PE, \mathcal{O}_X)}{\text{Aut}(PE)} \right) \)

Corollary: \( M H \) is smooth, connected of dimension \( h^a(C_X) = 26 \)

Recall basic diagram:

\[
\begin{array}{c}
\hat{X} \\
\downarrow \phi \\
X
\end{array} \rightarrow \begin{array}{c}
\hat{X} \\
\downarrow \phi
\end{array} \rightarrow \begin{array}{c}
X' \\
\downarrow \phi
\end{array}
\]

\( f = [u^* (x)] \in \text{PE} \)

Will show:

\[
T_X \text{Def}(X) \cong T_{\hat{X}} \text{Def}(\hat{X}) \cong T_X^{eq} \text{Def}^{eq}(X^b)
\]

\[
\cong T_X^{eq} \left( \frac{\text{equation of } X^b}{\text{Aut's}} \right)
\]
Diagram is intrinsic to $X$, so $\approx$'s make sense

$- T_X \text{Def}(X) \cong T_X \text{Def}(\hat{X})$

If $E$ exceptional curve

$e = e_\omega([E]) \in H^2(\Omega^1_X)$

$\theta \in H^2(\Omega^1_X)$

$\omega \in H^0(\Omega^2_X) \cong H^0(\Omega^2_X), \omega|_E = 0$

$\Rightarrow \langle \theta e, \omega \rangle = \langle e, \theta \cdot \omega \rangle = \theta \cdot \omega|_E = 0$

$\Rightarrow [E]$ deforms (to 1st order)

$0 \to K_X^-(\xi) \to K_X^0 \to \Omega^1_X(-2) \to 0$

$\Rightarrow H^2(K_X^-(\xi)) = 0$

$\Rightarrow H^2([E]) = 0$

$\Rightarrow E$ deforms (to 1st order)

$\Rightarrow \text{fog} = \varphi_{2k_X^2}$

$g = \text{normalization}$

$f = \text{contract} \quad g(E_1), g(E_2) = -2 \text{sum}$
\[\text{III - 10 -}\]

- \(\text{Aut} (X) = H^0(\mathcal{O}_X) = 0 = H^0(\mathcal{O}_X^* )\)

- \(\text{Aut} (X^b) = \{ \text{tangent space to equisingular} \} \)
  \(\text{deformations of } X^b \subset PE \)

  \(= 0\)

- **Conclusion**: Will suffice to compute tangent space to the deformations of the equations

  \[R = L^3 G - F^3 = 0\]

- **Crucial computation**: \(h^0(PE,\mathcal{I}^9) = 40\)

- \(\{ LG, L^2, F \} \) not a regular sequence but does have relatively simple free resolution given by the table

  \[
  \begin{array}{ccc}
  \text{LG} & L^2 & F \\
  \text{rel}_2 & L & -G & 0 \\
  \text{rel}_2 & F & 0 & -LG \\
  \text{rel}_3 & 0 & F & -L^3 \\
  \end{array}
  \]
where each row is relation among the generators at the top - single generating

\[ F \times \mathbb{Z}_2 - L \times \mathbb{Z}_2 + G \times \mathbb{Z}_3 = 0 \]

This gives

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
0 \to F^4 \to G^4 \to L^4 \to F^4 \to G^4 \to L^4 \to G^5 \to 0
\end{array}
\end{array}
\end{array}
\]

- tensor (\( \otimes \)) with \( F^4 \), hypercohomology
- spectral sequence will abut to \( H^0(3 \otimes f) \). - lucky because using
  - \( h^0(a^3+bh) \neq 0 \iff a \leq 0, b \leq -2a \)
  - \( h^1(a^3+bh) \neq 0 \iff a \leq 0, b \leq -2a \)
  - \( h^2(a^3+bh) \neq 0 \iff a \leq -3, b \leq 2a+8 \)
  - \( h^3(a^3+bh) \neq 0 \iff a \leq -3, b \leq -2a-6 \)
the spectral sequence degenerates and
\[ h^0(\xi^*) = h^0(\mathbb{P}^2) + h^0(\xi^* \mathbb{P}^2) + h^0(\xi^* \mathbb{P}^2) - h^0(\mathbb{P}^2) - h^0(\xi^*) \]
\[ = 14 + 30 + 2 - 5 - 1 = 40 \]

Note: Using
\[ R^0(\xi^* \mathbb{P}^2) = (R^0 \xi^*) \mathbb{P}^2 \]
\[ = \text{Sym}^\alpha E(-b) \]
\[ E = \mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(2) \]

\[ \left\{ \begin{array}{c}
 h^0(\mathcal{O}_{\mathbb{P}^2}(c)) \neq 0 \iff c \geq 0 \\
 h^2(\mathcal{O}_{\mathbb{P}^2}(c)) \neq 0 \iff c \leq -2 
\end{array} \right. \]

\[ \dim \text{aut}(\mathbb{P}^2) = 15 \]

Proof: \( E \rightarrow \mathbb{P}^2 \) is homogeneous vector bundle so have
\[ 0 \rightarrow \text{Aut} E \rightarrow \text{Aut}(\mathbb{P}^2) \rightarrow \text{Aut}(\mathbb{P}^2) \rightarrow 0 \]

\( \uparrow \)

Fibre preserving automorphisms
need to compute over \( \mathbb{P}^2 \)

\[
h^0(\text{End } E) = h^0(\text{End } E^*)
\]

\[
h^0(\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^2} & \mathcal{O}_{\mathbb{P}^2} & \mathcal{O}_{\mathbb{P}^2}(2) \\ \mathcal{O}_{\mathbb{P}^2} & \mathcal{O}_{\mathbb{P}^2} & \mathcal{O}_{\mathbb{P}^2}(2) \\ \mathcal{O}_{\mathbb{P}^2}(2) & \mathcal{O}_{\mathbb{P}^2}(2) & \mathcal{O}_{\mathbb{P}^2} \end{array})
\]

. **Note.** Have

\[
H^0(\text{End}(E)) \cong \left( \begin{array}{c} \mathfrak{gl}(2, \mathbb{C}) \\ \mathfrak{0} \end{array} \right) U
\]

where \( U = \) unipotent radical of the Lie algebra \( H^0(\text{End}(E)) \)

. \( \text{Aut}(\text{PE}) \) acts on \( \text{PE} \) with two orbits
  - closed orbit \( S \) \( \leftrightarrow \) \( G_0, \text{sing} \)
  - open orbit \( \text{PE} \setminus S \) \( \leftrightarrow \) \( G_0, \text{reg} \)
Alternate computation of $h^2(\mathcal{O}_X)$

$$h^2(\mathcal{O}_X) = -\chi(\mathcal{O}_X) + h^2(\mathcal{O}_X) = 2g + h^2(\mathcal{O}_X)$$

which gives

$$h^2(\mathcal{O}_X) \leq 2g$$

if we know that

$$T(\text{Def}^e_s(X_b^t)) \to T(\text{Def}(X))$$

is injective (almost evident geometrically) then, we get

$$h^2(\mathcal{O}_X) = 2g$$

and $\text{Def}(X)$ is smooth (and unobstructed)

$\mathcal{M}_H$ irreducible needs the analysis of smooth

but not general $X$'s

argument that the space of equations is connected