Next topics

- Hodge theory (HS, MHS, LMHS)

- Moduli (generalities, KSBA singularities)

- Connection between HT and moduli

  - generalities

  - H-surfaces

Have posted in the link pages S-1 through S-6 which give a summary of H- and I- surfaces (general type, $p_g(X) = 2$, $q(X) = 0$, $K_X^2 = 2, 1$)

<u>Hodge theory</u> - <u>the</u> basic invariant of complex algebraic varieties is its (mixed) Hodge structure on cohomology

- M = manifold, $A^\bullet(M) = \oplus A^q(M) = C^\infty$ differential forms $\sum f_{i_1 \dots i_q}(x)\, dx_{i_1} \wedge \dots \wedge dx_{i_q}$

<u>de Rham:</u> $H^\bullet(M, \mathbb{R}) \cong H^\bullet(A^\bullet(M), d)$

- X = complex manifold

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X), \quad A^{q,p}(X) = \overline{A^{p,q}(X)}$$

$$\varphi = \sum \varphi_{i_1 \dots i_p j_1 \dots j_q}\, \underbrace{dz_{i_1} \wedge \dots \wedge dz_{i_p}}_{p} \wedge \underbrace{d\bar z_{j_1} \wedge \dots \wedge d\bar z_{j_q}}_{q}$$

Hodge decomposition

- X compact Kähler (projective $\Rightarrow$ Kähler)

$$H^n(X) = \bigoplus_{p+q=n} H^{p,q}(X), \quad H^{q,p}(X) = \overline{H^{p,q}(X)}$$

Hodge filtration

- $F^p H^n(X) = \bigoplus_{p' \geq p} H^{p',q}(X) \Rightarrow F^p \oplus \overline{F}^{n-p+1} \cong H^n(X)$

- $H^{p,q}(X) \cong \dfrac{\left\{ \varphi \in A^{p,q}(X) : d\varphi = 0 \Leftrightarrow \begin{matrix} \partial\varphi = 0 \\ \bar\partial \varphi = 0 \end{matrix} \right\}}{\text{exact}}$

$$\cong H^{p,q}_{\bar\partial}(X) \qquad \text{(Dolbeault)}$$

- $H^q(X, E) \cong H^{0,q}_{\bar\partial}(X, E)$

$$H^q(X, \Omega^p_X) \cong H^{p,q}_{\bar\partial}(X)$$

$$\Downarrow$$

$$\boxed{H^{p,q}(X) \cong H^q(X, \Omega^p_X)}$$

- $H^1(X) \cong H^0(\Omega^1_X) \oplus \underbrace{\overline{H^0(\Omega^1_X)}}_{\substack{\cong \\ H^1(\mathcal{O}_X)}}$

- $H^2(X) \cong H^0(\Omega^2_X) \oplus \underbrace{H^1(\Omega^1_X)}_{\substack{\uparrow c_1 \\ \mathrm{Pic}(X)}} \oplus \underbrace{\overline{H^0(\Omega^2_X)}}_{\substack{\cong \\ H^2(\mathcal{O}_X)}}$

• <u>Defn</u>: <u>Hodge structure</u> $(V, F^{\bullet})$ (<u>weight n</u>)

– $V = \mathbb{Q}$-vector space (usually $V_{\mathbb{Z}}$ with $V_{\mathbb{Z}} \otimes \mathbb{Q} = V$)

– $F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}}$

– $F^p \oplus \bar{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}}$, $0 \leq p \leq n$

• $V^{p,q} = F^p \cap \bar{F}^q$ and $V = \oplus V^{p,q}$, $V^{q,p} = \overline{V^{p,q}}$

$C = i^{p-q}$ on $V^{p,q}$ is <u>Weil operator</u>

• <u>Polarized Hodge structure</u> (PHS) (V, Q, F)

– $Q : V \otimes V \to \mathbb{Q}$, $Q(u,v) = (-1)^n Q(v,u)$

– $\begin{cases} (\mathrm{HR\ I}) : Q(F^p, F^{n-p+1}) = 0 \\ (\mathrm{HR\ II}) : Q(u, C\bar{u}) > 0, \quad u \neq 0 \end{cases}$

curve • $H^1(C) \cong H^0(\Omega^1_C) \oplus \overline{H^0(\Omega^1_C)}$, Q = cup product

surface • $H^2(X) = \underbrace{H^0(\Omega^2_X) \oplus H^1(\Omega^1_X)_{\mathrm{prim}} \oplus \overline{H^0(\Omega^2_X)}}_{V_{\mathbb{C}}} \oplus \mathbb{C}\omega$

Q = cup-product $\qquad$ $\omega = c_1(L)$

$H^{\cdot}(X)$, X arbitrary

$\Updownarrow$

· Two extensions of HT

- <u>mixed Hodge structure</u> (MHS)
- <u>variation of Hodge structure</u> (VHS)

$\Updownarrow$

$H^{*}(X_t)$

· These come together in <u>limiting mixed HS</u> (LMHS)

$$H^n_{\lim}(X_t) \text{ as } X_t \to X_0$$

· <u>Period domain</u> $D = \left\{ \begin{array}{c} \text{set of PHS's with} \\ \text{given } h^{p,q} = \dim V^{p,q} \end{array} \right\}$

Ex $H^1(C) \cong \mathbb{Z}^{2g}$

$Q = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

$D \cong \mathcal{H}_g$ via $Z \longrightarrow (I, Z)$, $F^1 = \left\{ \begin{array}{c} \text{span of} \\ \text{rows} \end{array} \right\}$

$\int \omega \wedge \omega' = 0$, (i) $\int \omega \wedge \bar{\omega} > 0$ $\qquad H^0(\Omega^1_C)$

Ex $n=2$, $D \subset \mathrm{Grass}(h^{2,0}, V_{\mathbb{C}})$

$$F \in D, \quad \begin{cases} Q(F,F)=0 \\ Q(F,\bar{F})>0 \end{cases}$$

$$\begin{cases} V^{2,0} = F \\ V^{2,0} \oplus V^{1,1} = F^{\perp} \end{cases}$$

- Hermitian symmetric domain (HSD) $\Leftrightarrow h^{2,0}=1$
  (K3 case)

- non-classical case: $D \neq$ HSD

- equivalence class of PHS's
  $\Updownarrow$
  quotient by action of $G_{\mathbb{Z}} = \mathrm{Aut}(V_{\mathbb{Z}}, Q)$
  $\Updownarrow$
  $\Gamma \backslash D$
  where $G_{\mathbb{C}}$ acts on $\mathrm{Grass}(h^{2,0}, V_{\mathbb{C}})$

- Period mapping

$$\updownarrow$$

$$\begin{array}{ccc} \Phi : \mathcal{M} & \longrightarrow & \Gamma\backslash D \\ \cap & & \cap \\ \Phi_e : \overline{\mathcal{M}} & \longrightarrow & \Gamma\backslash D^* \end{array}$$

$$\updownarrow$$

extended period mapping

Strategy: $\Gamma\backslash D^*$ can be understood using Lie theory and geometry of homogeneous complex manifolds — use this to help study $D\mathcal{M} = \Phi_e^{-1}(\partial(\Gamma\backslash D^*))$

IV. 1

- <u>Hodge theory</u>
  - (polarized) Hodge structures
  - mixed Hodge structures
  - variation of Hodge structure
  - limiting mixed Hodge structures

- Hodge structure (V, F) of weight n

$$\begin{cases} F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}} \\ F^p \oplus \bar{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}}, \quad 0 \leq p \leq n \end{cases}$$

- morphism of weight r

$$\varphi : V \longrightarrow V'$$

$$\varphi(F^p) \subseteq F'^{p+r}$$

- $V = \bigoplus_{p+q=n} V^{p,q}, \quad \bar{V}^{p,q} = V^{q,p}$

$$V^{p,q} = F^p \cap \bar{F}^q$$

- $V^{p,q} \longrightarrow V'^{p+r, q+r}$, and commutes with conjugation

- morphisms are <u>strict</u>

$$\varphi(V_{\mathbb{C}}) \cap F'^{\,p+r} = \varphi(F^p)$$

- $n = 2p$, <u>Hodge classes</u>

$$Hg^p = F^p \cap V$$
$$= V^{p,p} \cap V$$

- morphism of weight r

$$\Updownarrow$$

$$\varphi \in Hg^r(V^* \otimes V')$$
$$\|$$
$$Hg^r(\mathrm{Hom}(V, V'))$$

<u>Rule of thumb</u>: best possible linear algebra properties that can be intrinsically formulated are true (this is the case throughout HT)

- Notation: $\mathbb{Q}(-1) =$ HS of weight -2

- **Cohomology of projective varieties**

$-\ (H^n(X, \mathbb{Q}), F) =$ HS of weight n

$-\ H^{p,q}(X) = \dfrac{\{\varphi \in A^{p,q}(X) : d\varphi = 0\}}{\{\varphi = d\psi,\ \psi \in A^{p+q-1}(X)\}}$

$$d\varphi = 0 \Leftrightarrow \begin{cases} \partial\varphi = 0 \\ \bar\partial\varphi = 0 \end{cases}$$

$$\varphi = d\psi \Leftrightarrow \begin{cases} \varphi = \partial\psi' \\ \varphi = \bar\partial\psi'' \end{cases} \Leftrightarrow \varphi = \partial\bar\partial\eta$$

$-\ H^{p,q}(X) \cong H^q(\Omega^p_X)$

- **Hard Lefschetz** : $W = H^*(X, \mathbb{Q})$

$-\ X = c_1(L)$ where $L \to X$ ample, $\dim X = d$

Theorem: $X^k : H^{d-k}(X) \xrightarrow{\sim} H^{d+k}(X)$

$-\ h(\varphi) = \deg\varphi - d \Rightarrow [h, X] = 2X$

$-$ given $X, h \in \mathrm{End}(W)$ as above, there exists a unique y with $\begin{matrix}[h,y] = -2y \\ [X,y] = h\end{matrix}$

$\rightarrow$ $H^*(X, \mathbb{Q})$ is an sl_2-module

- <u>sl_2</u> $\quad V_k = \mathrm{Sym}^k \mathbb{Q}^2 = \mathrm{span}\{u^k, u^{k-1}v, \dots, v^k\}$

$$x = v\,\partial/\partial u, \quad y = u\,\partial/\partial v, \quad h = [x, y]$$

<u>weights</u> of h are $k, k-2, \dots, -k$

$$\bullet \underset{y}{\overset{x}{\rightleftarrows}} \bullet \;\cdots\; \rightleftarrows \bullet$$

- <u>Defn</u>: $H^{d-k}(X)_{prim} = \ker\{x^{k+1} : H^{d-k} \to H^{d+k}\}$

$$\rightarrow \quad H^*(X) = \bigoplus_{\substack{0 \le l \le k \\ 0 \le k \le d}} x^l H^{d-k}(X)_{prim}$$

- additional structure of

$$\tilde{Q} : H^{d-k}(X) \otimes H^{d+k}(X) \to \mathbb{Q}$$

preserved by sl_2-action

<u>Defn</u>: $Q_k : H^{d-k}(X)_{prim} \otimes H^{d-k}(X)_{prim} \to \mathbb{C}$

$$Q_k(\varphi, \psi) = \tilde{Q}(x^k\varphi, \psi) = \int_X x^k \wedge \varphi \wedge \psi$$

$\to$ Q_k satisfies Hodge-Riemann I, II

$$\begin{cases} \text{(I)} & Q_k(F^\ell, F^{d-k-\ell+1}) = 0 \\ \text{(II)} & i^{p-q} Q_k(\varphi, \bar{\varphi}) > 0 \ , \quad p+q = d-k \end{cases}$$

- <u>Lefschetz again</u>: $Y \subset X$ smooth section of $L \to X$

$$\begin{cases} H^q(X, \mathbb{Z}) \xrightarrow{\sim} H^q(Y, \mathbb{Z}) & 0 \leq q \leq d-2 \\ H^{d-1}(X, \mathbb{Z}) \hookrightarrow H^{d-1}(Y, \mathbb{Z}) \end{cases}$$

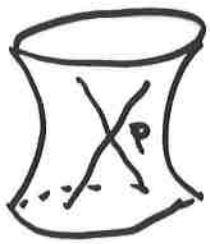
also morphisms of Hodge structures

<u>Ex</u> X = surface, Y = curve
- Y connected
- $H^0(\Omega^1_X) \hookrightarrow H^0(\Omega^1_Y)$

$$H^2(X,\mathbb{Q}) = H^2(X,\mathbb{Q})_{prim} \oplus \mathbb{Q}[Y]$$

$$\| $$

$$\{\varphi : \varphi \cdot [Y] = 0\}$$



$L_1^2 = 0 = L_2^2, \; L_1 \cdot L_2 = 1$

- two rulings ; $L_1 + L_2 = T_p X \cap X$
  $\sim \mathbb{P}^2 \cap X =$ plane conic Y

- $L_1 - L_2$ primitive

• standard morphisms of HS's for $Y \subset X$

$$\begin{cases} \text{Rest}: & H^q(X) \to H^q(Y) \\ \text{Gy}: & H^q(Y)(-1) \to H^{q+2}(X) \end{cases}$$

<o>

• <u>Mixed Hodge structure</u> (V, F, W)

$$\begin{cases} W_0 \subset W_1 \subset \dots \subset W_m = V \\ F^n \subset F^{n-1} \subset \dots \subset F^0 = V_{\mathbb{C}} \end{cases}$$

such that induced F on $\mathrm{Gr}^W_r V$ are HS's wt r

$$F^q(Gr^r_W V) = \frac{F^q \cap W_{r,\mathbb{C}} + W_{r-1,\mathbb{C}}}{W_{r-1,\mathbb{C}}}$$

- morphism of degree r is $\varphi : V \to V'$

$$\begin{cases} W_k \longrightarrow W'_{k+2r} \\ F^p \to F'^{p+r} \end{cases}$$

$\to$ induced maps $Gr^s_W \to Gr^{2r+s}_{W'}$ are morphisms of HS's

- <u>strictness</u>: $\varphi : V \to V'$ as above

$$\begin{cases} \varphi(W_k) = W'_{k+2r} \cap \varphi(V) \\ \varphi(F^p) = F'^{p+r} \cap \varphi(V) \end{cases}$$

<u>Theorem (Deligne)</u>: $\mathcal{U}$ = complex algebraic variety

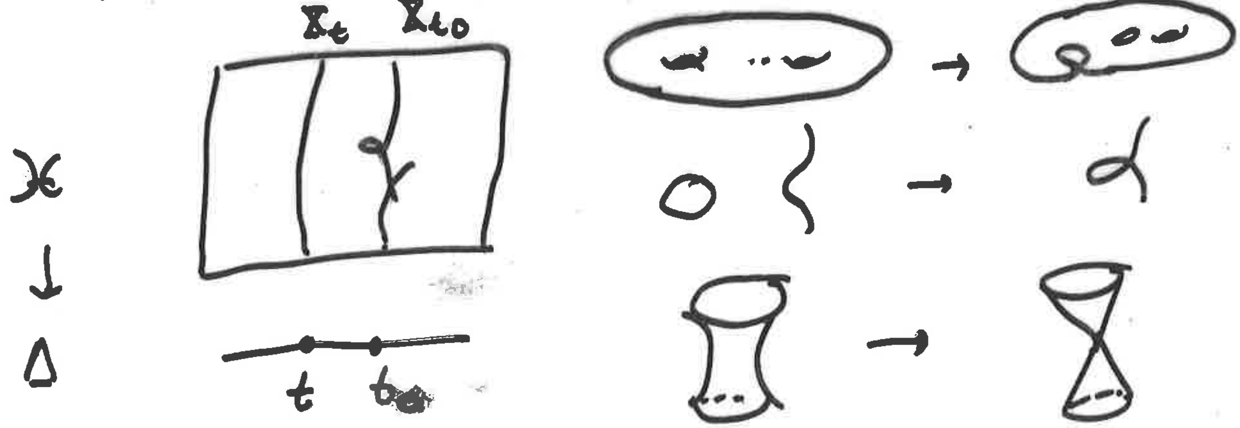
$\to H^n(\mathcal{U})$ has a canonical MHS

- $\mathcal{U}$ complete $\Rightarrow$ weights $0, .., n$
- $\mathcal{U}$ affine $\Rightarrow$ weights $n, .., 2n$

- Because of strictness, MHS's form an abelian category

- <u>Limiting mixed Hodge structures</u>

- typically algebraic varieties come in families (eg., moduli - to be discussed later), and a "general member" will be smooth but there will be some singular, or degenerate members in the family



- $\boxed{\text{What is } \lim\limits_{t \to 0} H^n(X_t) = H^n_{\lim} \text{ ?}}$

- It is <u>not</u> $H^n(X_0)$
  - dimensions don't agree
  - can modify X_0 (eg, blowing up) without changing $H^n(X_t)$ but changing $H^n(X_0)$

- <u>Monodromy</u>

$$\begin{array}{c} \mathcal{X}^* = \{ X_t : t \in \Delta^*, \text{ smooth} \} \\ \downarrow \\ \Delta^* \end{array}$$

is a differentiable fibre bundle.

topology doesn't change when we follow X_t along a path

$$\rightsquigarrow$$

$$T : H^n(X_\eta) \to H^n(X)$$

<u>Monodromy theorem</u>: $(T^m - I)^{n+1} = 0$

T — eigenvalues are roots of unity

T — Jordan blocks have length $\leq n$

- make a base change $t = t'^m$

$$\Delta'^* \to \Delta^*$$

to have $\mathcal{X}'^* \to \Delta'$ and

$$T' = T^m$$

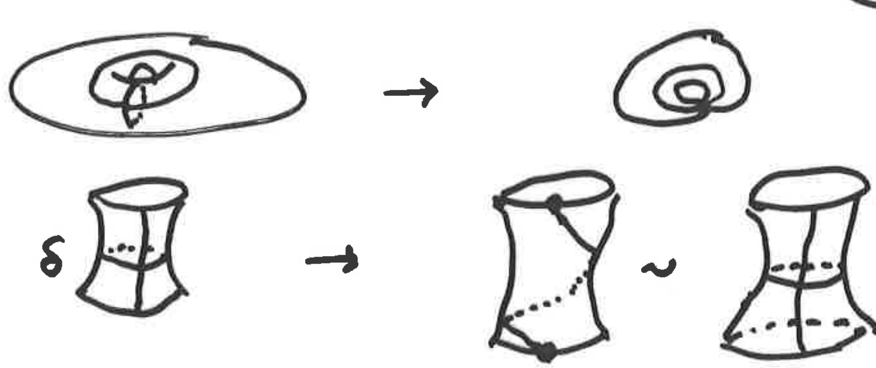
$$\Downarrow$$

assume T <u>unipotent</u> with

$$\begin{cases} N = \log T = (T-I) - \frac{1}{2}(T-I)^2 + \\ N^{n+1} = 0 \end{cases}$$

vanishing cycle

Ex



$$\begin{cases} \delta \to \delta \\ \gamma \to \gamma + \delta \end{cases}$$

(Picard-Lefschetz)

- Given (V, N) with $N^{n+1} = 0$ there exists a unique filtration

$$W_{-n} \subset W_{-n+2} \subset \cdots \subset W_{n-2} \subset W_n$$

such that

$$\begin{cases} N : W_k \to W_{k-2} \\ N^k : W_k \xrightarrow{\sim} W_{-k}, \quad k \geq 0 \end{cases}$$

<u>Notes</u>: There exists an $sl_2 = \{N, Y, N^+\}$

$$\Downarrow$$

$$V = \bigoplus_{k \leq n} m_k V_k$$

V_n: $\quad x^n \quad x^{n-1}y \quad \cdots \quad xy^{n-1} \quad y^n$

$N = y\frac{d}{dx}$ $\quad \cdot \to \cdot \qquad \cdot \to \cdot$

W_{-n}

W_{-n+2}

W_{n-2}

W_n

If we have (V, Q) and $N \in \mathrm{End}_Q(V)$, then we may choose $\mathfrak{sl}_2 \subset \mathrm{End}_Q(V)$

and

$$W_k^{\perp} = W_{2m+k-2}$$

Defn: A limiting mixed Hodge structure (LMHS) is a MHS $(V, W(N), F^{\cdot})$ such that $N \in F^{-1}\,\mathrm{Hom}(V, V)$

(i.e., $N: F^p \subseteq F^{p-1}$)

A polarized limiting mixed Hodge structure (PLMHS) has a Q and

$$\begin{cases} Q_k : Gr_{k,P} \otimes Gr_{k,P} \to \mathbb{Q} \\ Q_k(u, v) = Q(N^k u, v) \end{cases}, \quad Gr_{k,P} = \text{primitive stuff}$$

satisfies HR I, II

$\boxed{\text{Summary of } H \text{ and } I\text{- surfaces}}$

<u>H- surfaces</u>: $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$

- $\mathbb{P}E \xrightarrow{\pi} \mathbb{P}^1$

- $\xi = \mathcal{O}_{\mathbb{P}^1}(1)$ and $h = \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ generate $\mathrm{Pic}(\mathbb{P}E)$

- $$\begin{array}{ccc} H^0(\mathbb{P}E, k\xi + l h) & \cong & H^0(\mathbb{P}^1, \mathrm{Sym}^k E \otimes \mathcal{O}_{\mathbb{P}^1}(l)) \\ \| & & \| \\ H^0(k\xi + lh) & & H^0(S^k E(l)) \end{array}$$

- $H^0(\xi - 2h) \cong \mathbb{C}\cdot x$ where $(x) = S \cong \mathbb{P}^1 \times \mathbb{P}^1$

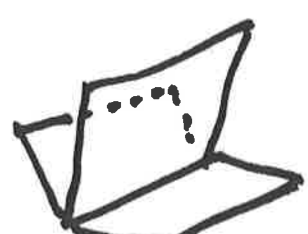
  (S corresponds to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \subset \mathbb{P}E$)

- $H^0(\xi)$ has basis $\underbrace{t_0^2, t_0 t_1, t_1^2}, \underbrace{x_3, x_4}$

  $\cong$

  $H^0(E) = H^0(\mathcal{O}_{\mathbb{P}^1}(2)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})$

- $$\begin{array}{ccc} \mathbb{P}E & \xrightarrow{f} & Q_0 \subset \mathbb{P}^4 \\ \cup & & \cup \\ S & \longrightarrow & Q_{0,\,sing} = L \end{array}$$



- $$\begin{tikzcd} \hat{X} \arrow[r, "g"] \arrow[d, "\pi"] & \mathbb{P}E \arrow[r, "f"] \arrow[d] & Q_0 \\ \mathbb{P}^2 \arrow[r, phantom, "="] & \mathbb{P}^2 & \end{tikzcd}$$, $f(S) = Q_{0,\mathrm{sing}}$

where $\hat{X}$ = blow up of X at base points of $|K_X|$ and $f \circ g = \varphi_{2K_{\hat{X}}}$

<u>note</u>: $H^0(mK_{\hat{X}}) \cong H^0(mK_X)$ and

$$\begin{tikzcd} \hat{X} \arrow[r, "\varphi_{2K_{\hat{X}}}"] \arrow[d] & \mathbb{P}^4 \\ X \arrow[ur, "\varphi_{2K_X}"'] & \end{tikzcd}$$

- equation of $X^b = g(\hat{X})$ is

$$t_0^2 x \cdot G = F^2 \qquad G \in H^0(3\xi),\ F \in H^0(2\xi)$$

for generic G, F a Bertini-type argument gives that X^b is smooth away from $t_0 = 0$

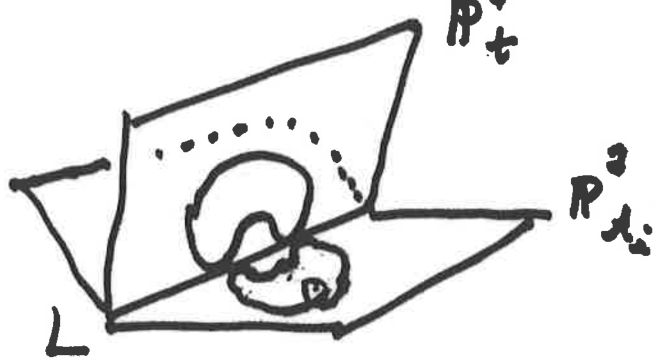
pinch points
- $x = 0$ gives 2 for $F|_{\mathbb{P}^2_x \times \{0\}}$
- $x \neq 0$ gives 6 for $F = G = 0$ in $\mathbb{P}^2$

- fibres of $\hat{X} \to \mathbb{P}^1$ are $C \in |K_X|$

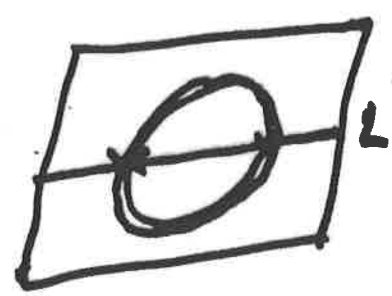
and

$$f \circ g|_C = \varphi_{K_C} \quad (\text{or } \varphi_{\omega_C})$$

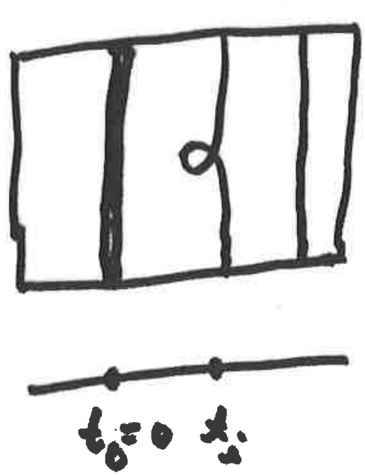
leading to picture



- two points on L are base points of $|K_X|$
- finitely many nodal C_{t_i}'s
- one hyperelliptic C_{t_0}



- picture of $\hat{X}$



Noether

$$\begin{array}{ccc} \chi(\mathcal{O}_{\hat{X}}) & = \frac{1}{12}\big(& K_{\hat{X}}^2 + \chi_{top}(\hat{X})\big) \\ \| & & \| \\ \chi(\mathcal{O}_X) = 3 & & 0 \\ & & \text{because } K_{\hat{X}} = \pi^* K_X + E_1 + E_2 \end{array}$$

- $$\begin{array}{ccccc} \chi_{top}(\hat{X}) = & \chi_{top}(\mathbb{P}^2) & \chi_{top}(C) & + & \left(\begin{array}{c}\#\text{ of nodal} \\ \text{curves } C_{t_0}\end{array}\right) \\ \| & & \| & & \| \\ 36 & & -8 & & 44 \end{array}$$

- $h^{1,1}(X) = h^{1,1}(\hat{X}) - 2 = 28$

Adjunction

- $K_{\mathbb{P}E} = -3\xi$

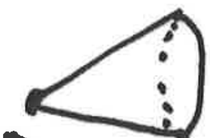
- $X^b \subset |4\xi|$

- $K_{X^b} = -3\xi + 4\xi$ corrected by $-\eta$ for double curve with pinch points

$$\Rightarrow h^0(K_{X^b}) = \underbrace{h^0(\xi - \eta)}_{\text{on } \mathbb{P}E} = \underbrace{h^0(E(-1))}_{\text{on } \mathbb{P}^2} = 2$$

<u>I- surfaces</u> : $F = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$

- $H^0(\xi)$ has basis $t_0^2, t_0 t_1, t_1^2, y$
- $H^0(\xi - 2h) = \mathbb{C}\cdot x$, $(x) =$ $\cong \mathbb{P}^1$
- $\mathbb{P}F \xrightarrow{f'} Q_0' \subset \mathbb{P}^3$, $Q_0' = \{x_0 x_1 = x_2^2\}$

  $\uparrow$
- resolves double point $=$  p
- $$\begin{tikzcd} \hat{X} \arrow[r, "g'"] \arrow[d] & \mathbb{P}F \arrow[r, "f'"] \arrow[d] & Q_0' \\ \mathbb{P}^1 & \mathbb{P}^1 & \end{tikzcd}$$ $f' \circ g' = \varphi_{2K_X}$
- 2- sheeted branced cover branched over $p + V$, $V =$ general quintic in $\mathbb{P}^3$
- fibres of $\hat{X} \to \mathbb{P}^1$ are $g = 2$ hyperelliptic curves and $\varphi_{2K_X}\big|_C = \varphi_{K_C}$ is 2:1 map to rulings of Q_0'

- no special fibres (unlike H-surface)
- $\varphi_{2K_X} : X \longrightarrow \mathbb{P}(1,1,2)$ $(= Q_0')$
- $\varphi_{5K_X} : X \hookrightarrow \mathbb{P}(1,1,2,5)$

$$z^2 = f_{10}(t_0, t_1, y)$$