Adjunction

Q: What is K_X for a singular variety?

Q what is ω_X to have duality on a singular variety?

- Issue is local and we use* the notation K_X for a line bundle and its sheaf of sections. Also we will only be concerned with case where X is locally a hypersurface

$$f(x_1, .., x_{n+1}) = 0, \quad f(x) \text{ irreducible}$$

The basic idea is to use <u>residues</u>

---

* For some singularities only $K_X^{\otimes m}$ is a line bundle for some $m > 0$ - we will only get into this if we have do later

From

$$0 = df\Big|_X = \sum f_{x_i}(x)\,dx_i = 0$$

we have for any i, j

$$\left.\frac{dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_{n+1}}{f_{x_i}(x)}\right|_X = \pm\, \frac{dx_1\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_{n+1}}{f_{x_j}(x)}\Bigg|$$

$$\left(f_{x_1}dx_1 + f_{x_2}dx_2 = 0 \Rightarrow \frac{dx_1}{f_{x_2}} = -\frac{dx_2}{f_{x_1}}, \dots \right)$$

$$\Rightarrow \quad \varphi = g(x)\left.\frac{(-1)^i\, dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_{n+1}}{f_{x_i}(x)}\right|_X$$

is well-defined

Defn: $\varphi = \operatorname{Res}_X \left\{ g(x)\,\frac{dx_1\wedge\cdots\wedge dx_{n+1}}{f(x)} \right\} = \operatorname{Res}_X \Phi$

For $\tilde{X}$ = normalization of X

Defn $K_X = \{ \varphi : \tilde{\varphi} \text{ holomorphic on } \tilde{X} \}$

$= \{ \varphi : \tilde{\varphi} \text{ holomorphic on } \tilde{X}_{reg} \}$

<u>Ex</u> $xy=0$

$$\frac{g(x,y)\,dx\wedge dy}{xy} \begin{array}{l} \nearrow \ g(x,0)\,\frac{dx}{x} \ \text{ on } y=0 \\ \searrow \ -g(0,y)\,\frac{dy}{y} \ \text{ on } x=0 \end{array}$$

$$K_X = \text{Res}\left\{ \frac{g(x,y)\,dx\wedge dy}{xy} \right\} \ : \ g(0,0)=0$$

<u>Ex</u> $x^2 z = y^2$, $(u,v) \longrightarrow (u, uv, v^2)$

$$\frac{g(x,y,z)\,dx\wedge dy\wedge dz}{x^2z-y^2} \longrightarrow 2\,\frac{g(u,uv,v^2)\,du\wedge dv}{u}$$

$$K_X = \text{Res}\left\{ \frac{g(x,y,z)\,dx\wedge dy\wedge dz}{x^2z-y^2} \right\},$$

$g=0$ on the double curve $x=0,\ y=0$

- $\tilde{X} \supset \tilde{D} = \{u=0 \text{ in } (u,v)\text{-plane}\}$

$$\begin{tikzcd} \tilde{X} \arrow[d] & \supset & \tilde{D} \arrow[d] \\ X & \supset & D \end{tikzcd}$$

- $\tilde{D} \to D$ is 2:1 with involution

$\tau($[illegible]$,v)=(0,-v)$

Note: $\operatorname{Res}_{\tilde{D}}(\tilde{\varphi}) = g(0,0,v^2)\,dv$ is holomorphic on $\tilde{D}$ with $\tau^*(\cdot\cdot) = -(\cdot\cdot)$

$\rightarrow$ Double residue is in $\Omega^{1\,-}_{\tilde{D}}$

Defn: $\omega_X =$ all $\varphi = \operatorname{Res}_X(\Phi)$ as above

For this ω_X we have

$$H^q(E)^* \cong H^{n-q}(E^* \otimes \omega_X)$$

Ex For algebraic curves

we have 1-forms with log poles upstairs and ± residues at identified points

Noether's inequality

$$p_g(X) \leq \frac{K_X^2}{2} + 2$$

$$\boxed{(*) \quad \bullet\ h^0(L_1),\ h^0(L_2) \neq 0 \Rightarrow h^0(L_1 \otimes L_2) \geq h^0(L_1) + h^0(L_2) - 1}$$

Pf $h^0(L_i) = d_i + 1$

- $\varphi_{L_i} : X \to \mathbb{P}^{d_i}$, image spans $\mathbb{P}^{d_i}$
- $\mathbb{P}^{d_1} \times \mathbb{P}^{d_2} \hookrightarrow \mathbb{P}^{d_1+d_2-1}$ via $V_1 \times V_2 \to V_1 \otimes V_2$ and image of independent points on each give spanning set in $\mathbb{P}^{d_1+d_2-1}$

<o>

Corollary (Clifford): $\deg L = g-1$, $L^2 = K_C$

$$\Rightarrow \dim |L| \leq \frac{g-1}{2}$$

- Suppose X regular and use $(*)$ plus

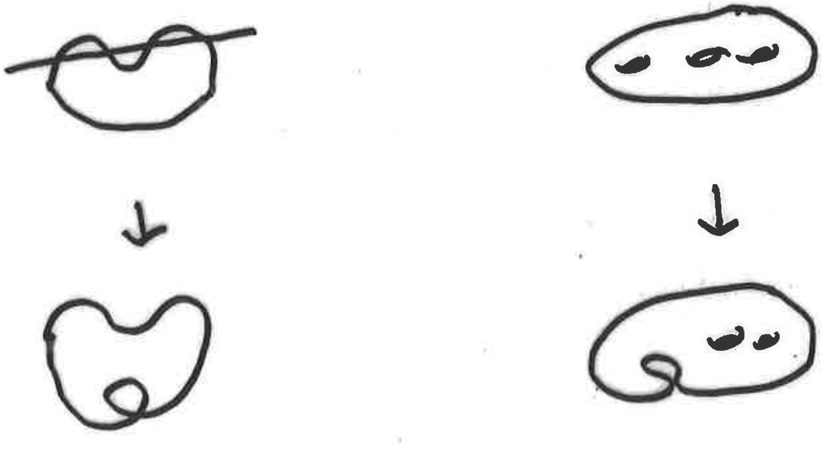
$$0 \to K_X \to 2K_X \to K_C \to 0$$

$$h^0(K_X) + h^0(K_X) - 1 \leq h^0(2K_X) =$$

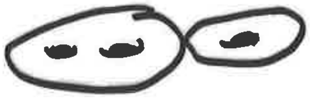
$$\overset{\parallel}{2p_g - 1} \leq 1 + p_g + K_X^2$$

Proof that for a general H-surface X there is a unique hyperelliptic curve $C_0 \in |K_X|$

- this will use concepts to introduced later
  - Hodge bundle, given by $\det R^0_\pi \omega_{\hat{X}/\mathbb{P}^1}$ which has general fibre $H^0(\Omega^1_{C_t})$
  - nodal $C_i \in |K_X|$; these are the singular fibres in $|K_X|$ for a general X - the pictures are



  - reducible fibres



  - difference is
    - vanishing cycle $\delta \neq 0$
    - vanishing cycle $\delta = 0$

$$\left\{\begin{array}{l} \lambda = \deg(\text{Hodge bundle}) \\ \delta_1 = \#\text{ reducible fibres} \\ \delta_0 = \#\text{ irreducible fibres} \end{array}\right.$$

- **Esteves formula**

$$\left.\begin{array}{c}\text{number of} \\ \text{hyperelliptic} \\ \text{fibres}\end{array}\right\} = h = 9\lambda - \delta_0 - 3\delta_1$$

Will see that $\lambda = 5$

- **Noether**

- $$\underset{3}{\underset{\shortparallel}{\chi(\mathcal{O}_X)}} = \frac{1}{12}\left(\underset{2}{\underset{\shortparallel}{K_X^2}} + \chi_{top}(X)\right)$$

$$\Rightarrow \chi_{top}(X) = 34$$

$$\Rightarrow b_2(X) = 32, \quad h^{1,1}(X) = 28$$

- $$\underset{3}{\underset{\shortparallel}{\chi(\mathcal{O}_{\hat{X}})}} = \frac{1}{12}\left(\underset{0}{\underset{\shortparallel}{K_{\hat{X}}^2}} + \chi_{top}(\hat{X})\right)$$

$$\chi_{top}(\hat{X}) = 36$$

- $\delta_1 = 0$ (base point at node - but general X has 2 distinct base points

- $\chi(\hat{X}) = \chi(\mathbb{P}^1)\chi(C) + \delta_0 \Rightarrow \delta_0 = 44$

Use of Bertini for H-surfaces

- equation of $X^b \subset \mathbb{P}E$ is

$$x\,t_0^2\,G = F^2$$

Consider pencil

$$x_0 t_0^2 (\lambda_0 G_0 + \lambda_1 G_1) = (\lambda_0 + \lambda_1) F^2$$

By usual Bertini, general member is smooth outside the base locus

$$\begin{cases} x t_0^2 = 0 \\ F = 0 \end{cases}$$

These separate into

(i) $t_0^2 = 0 = F$, $x \neq 0$

(ii) $x = 0 = F$

(i) = double conic (pinch points are (i) + $G=0$)

(ii) = blown up base points of $|K_X|$

Note: This gives another proof of unique $HE\ C_0 \in |K_X|$

II.1

H. and I. surfaces X

- X smooth, irreducible, minimal, general type ($\kappa(X)=2$)
- Numerical (Hilbert polynomial):

$$K_X^2 = 2, 1$$

- Hodge-theoretic:
  - $q(X)=0$ $\quad (h^1(\mathcal{O}_X)=0)$
  - $p_g(X)=2$ $\quad (h^2(\mathcal{O}_X)=2)$

Objective: Get good understanding of

X
- geometrically
- equations

⇝ Specializations $X \to X_0$

- two pictures
  - $\varphi_{2K_X}(X) = X' \subset \mathbb{P}^4$
  - $X^\flat$ = hypersurface in $\mathbb{P}E$ where $E = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$
- ideas
  - picture of $|K_X|$
  - build up $\oplus H^0(mK_X)$ inductively

<u>Theorem</u> X = general H-surface

(i) general $C \in |K_X|$ smooth and $g(C) = 3$

(ii) $\varphi_{2K_X} : X \longrightarrow X' \subset \mathbb{P}^4$ birational with image

$$\begin{cases} x_0 x_2 = x_1^2 \\ x_0 G(x) = F(x)^2 \end{cases}$$

F, G = quadric, cubic

(iii) there exists unique hyperelliptic $C_0 \in |K_X|$

(iv) X'_{sing} = double conic with 8 pinch points

$$\varphi_{2K_X}\big|_{C_0} = \varphi_{K_{C_0}} : C_0 \xrightarrow{2:1} \mathbb{P}^1$$

(i) no fixed component in $|K_X|$

Bertini - will see two distinct base points where all C smooth

(ii) idea: use

$$- \quad 0 \to (m-1)K_X \to mK_X \to K_C^{m/2} \to 0$$

exact on $H^0(\cdot\cdot)$ level and

know $h^0(mK_X) = m(m-1) + 3$; also

$h^0(K_C^{m/2})$ since C = plane quartic curve

$P_1 = p_g = 2$

$P_2 = 5$, $P_3 = 9$, $P_4 = 15$, $P_6 = 33$

- $H^0(K_X)$ t_0, t_1 weight 1

- $H^0(2K_X)$ $t_0^2, t_0 t_1\ t_1^2, \underbrace{x_3, x_4}$ $\sim$ = weight 2

- $H^0(3K_X)$ $P_3(t_i, x_\alpha)$, Ξ of weight 3

  dim = 8 $\Xi|_C \in H^0(K_C^{3/2})$ it $\neq 0$

- $H^0(4K_X)$ $P_4(t_i; x_\alpha)$, $t_0\Xi, t_1\Xi \Rightarrow$ one reation mod P_4's

  dim = 14 2

  $$t_1\Xi = F_2(t_0^2, t_0 t_1\ t_1^2, x_3, x_4)$$

- $H^0(6K_X)$ $P_6(t_i, x_\alpha, \bar{\Xi})$ - then one relation

  $\Xi^2, t_0^3\Xi, t_0x_3\Xi, t_0x_4\Xi$ mod $P(t_0, t_1, x_3, x_4)$'s

  $$\Downarrow$$

  $$\Xi^2 = G(t_0^2, t_0t_1, t_1^2, x_3, x_4)$$

change notation $t_0\Xi = F$ and square to have

$t_0^2 G = F^2$, $x_0 G = F^2$ in $\mathbb{P}^4$

- remainder of the argument to be completed later - some obervations

  - $\mathbb{P}(1,1,2,2) = \mathbb{C}^4 \setminus \{0\} / \mathbb{C}^*$

    where for $\lambda \in \mathbb{C}^*$, $\lambda(t_0, t_1, x_3, x_4) = (\lambda t_0, \lambda t_1, \lambda^2 x_3, \lambda^2 x_4)$

$$\Downarrow$$

$$\begin{tikzcd} \varphi_{2K_X} : X \arrow[r] & \mathbb{P}^2(1,1,2,2) \arrow[r, hook] & \mathbb{P}^4 \\ & & \boxed{\text{image here is } Q_0} \arrow[u] \end{tikzcd}$$

  - $R(X) \cong \mathbb{C}[\underset{1}{t_0}, \underset{1}{t_1}, \underset{2}{x_3}, \underset{2}{x_4}, \underset{3}{\Phi}] / (t_0 \Phi - F, \Phi^2 - G)$

  - will see that for F, G general the resulting $X' \subset \mathbb{P}^4$ has $\tilde{X}' = X$ smooth H-surface with the stated properties

  - pictures of H-surface

$$X' \subset \mathbb{P}^4, \quad \begin{tikzcd} \hat{X} \arrow[d] \\ \mathbb{P}^2 \end{tikzcd}, \quad \begin{tikzcd} \hat{X} \arrow[r] \arrow[d] & X^6 \arrow[d] & \subset \mathbb{P}E \arrow[d] \\ \mathbb{P}^2 \arrow[r, phantom, "="] & \mathbb{P}^2 \arrow[r, phantom, "="] & \mathbb{P}^2 \end{tikzcd}$$

- **Recall:** $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$, $\mathbb{P}E \xrightarrow{\pi} \mathbb{P}^1$
  - $(\mathbb{P}E)_t = \mathbb{P}E_t^*$
  - $\xi = \mathcal{O}_{\mathbb{P}E}(1)$, $R^q_\pi \xi^m = \begin{cases} 0 & m \geqq 0 \\ S^m E, & m = 0 \end{cases}$
  - $h = \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$, $\operatorname{Pic} \mathbb{P}E \cong \mathbb{Z}\{\xi, h\}$
  - $\xi^2 h = 2\xi^3 = 1 \in H^6(\mathbb{P}E, \mathbb{Z})$
- **Theorem:** (i) $\mathbb{P}E \xrightarrow{|\xi|} Q_0 \subset \mathbb{P}^4$ is the natural desingularization of $\mathbb{P}(1,1,2,2)$

(ii) We have

$$\begin{tikzcd}
X \arrow[rrd, "\varphi_{2K_X}"] & & \\
\hat{X} \arrow[u] \arrow[r, "g"] \arrow[d] & \mathbb{P}E \arrow[r, "f"] \arrow[d] & \mathbb{P}^4, \quad f = |\xi| \\
\mathbb{P}^1 \arrow[r, phantom, "="] & \mathbb{P}^1 &
\end{tikzcd}$$

- **Proof of (i):** $\pi_*(m\xi + lh) \cong S^m E(l)$

$\Rightarrow H^0(\xi - 2h) = \mathbb{C} \cdot x$, $(x) = S \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})$

- $\pi_* \xi$ has basis $x \cdot \underbrace{\{t_0^2, t_0 t_1, t_1^2\}}_{H^0(\mathcal{O}_{\mathbb{P}^1}(2))}, \underbrace{x_3, x_4}_{H^0(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})}$

$\Downarrow$

$\mathbb{P}E \to \mathbb{P}^4$ with image Q_0, $S \to Q_{0,\text{sing}}$

- intrinsically, $t_i \in H^0(\mathbb{P}E, \eta)$ so $x t_i t_j \in H^0(\xi)$ and map is

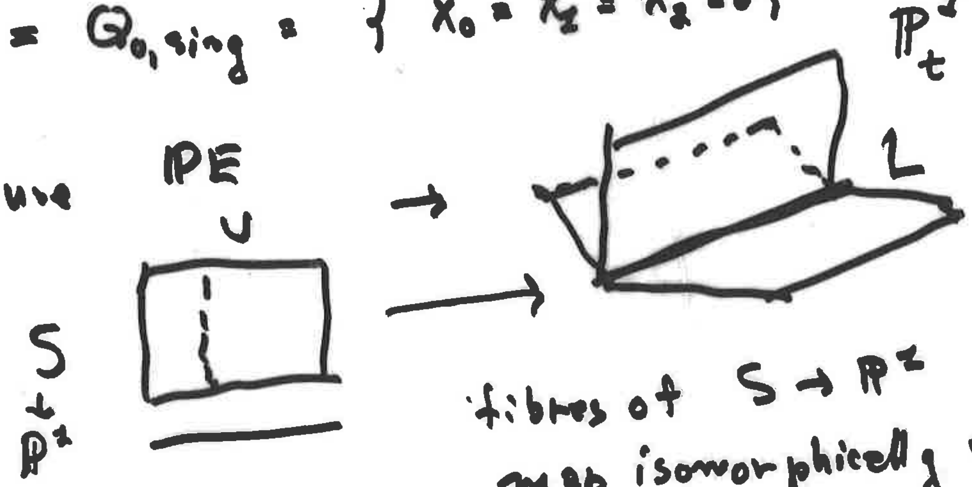
$$\begin{cases} x_0 = x t_0^2 \\ x_1 = x t_0 t_1 \\ x_2 = x t_1^2 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$$

- LHS = coordiates in $\mathbb{P}^4$
- RHS = sections of ξ

- $S = (x) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{P}^1 \times \mathbb{P}^1$ maps to

$L = Q_{0,\text{sing}} = \{ x_0 = x_1 = x_2 = 0 \}$

- Picture



fibres of $S \to \mathbb{P}^1$ map isomorphically to L

<u>Proof of (ii)</u>: Equation of $f^{-1}(\Xi')$ is $\Xi^b \subset \mathbb{P}E$

$$\boxed{x t_0^2 G = F^2}$$

$F \in |2\xi|$, $G \in |3\xi|$

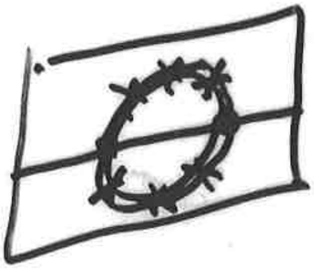
- for generic F, G this is smooth away from $t_0 = 0$

- writing as $t_0^2(xG) = F^2$ we see double curve $t_0 = 0, F = 0$ with pinch points

$$\begin{cases} x = 0, F = 0 & = 2 \text{ points in } t_0 = 0 \\ G = F = 0 & = 6 \quad '' \quad '' \quad '' \end{cases}$$

- the first is because $x\Big|_{t_0=0} \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ and $F=0\Big|_{t_0=0}$ is a conic

- picture of X'



- general C_t = smooth plane quartic
- special C_{t_i}'s = nodal ones
- base points are 2 marked points = bitangent L to all C_t
- $C_{t_0} \xrightarrow{2:1} \{F=0\} \cap \{t_0 = 0\}$ with $6+2 = 8$ branch points

$$\boxed{f \circ g\Big|_{C_t} = \varphi_{K_{C_t}}}$$

- $K_{\hat{\underline{X}}} = g^*(\xi - h)$

- $K_{\mathbb{P}E} = \xi^{-3}$

Proof: $0 \to V \to T\mathbb{P}E \to \pi^* T\mathbb{P}^1 \to 0$

$$\overset{\cong}{h^{2}}$$

Euler: $\mathbb{P}V^* = \mathbb{P}^{n-1}$, $\dim V = n$

$$0 \to \mathcal{O}_{\mathbb{P}^{n-1}} \to V^*(1) \to T\mathbb{P}^{n-1} \to 0$$

$$1 \to \Sigma x_i \partial/\partial x_i$$

$$\det T\mathbb{P}E = h^2 \otimes \underset{h^{-2}}{\underset{\shortparallel}{\det E^*}} \otimes \xi^3 = \xi^3$$

- Adjunction gives result

$$X^b \in |\xi^4|$$

$$K_{X^b} = \xi^4 \otimes K_{\mathbb{P}E} \otimes [-D]$$

$$= \xi - h$$

Note: $H^0(K_{\hat{X}}) \cong H^0(X^b, \xi \otimes h^{-1}) \cong H^0(\mathbb{P}^1, E(-1))$

$$\cong H^0(\mathcal{O}_{\mathbb{P}^1}(1))$$