Adjunction

Q: What is $K_X$ for a singular variety?
Q: What is $\omega_X$ to have duality on a singular variety?

- Issue is local and we use the notation $K_X$ for a line bundle and its sheaf of sections. Also we will only be concerned with case where $X$ is locally a hypersurface

$$f(x_1, \ldots, x_m + z) = 0, \quad f(x) \text{ irreducible}$$

The basic idea is to use residues

* For some singularities only $K^m_X$ is a line bundle for some $m > 0$ - we will only get into this if we have do later
From

\[ 0 = df \bigg|_{\mathcal{X}} = \sum f_{x_i}(x) \, dx_i = 0 \]

we have for any \( i \neq j \)

\[ \left. \frac{dx_{i_1} \cdots dx_{i_k} \cdots dx_{n+1}}{f_{x_i}(x)} \right|_{\mathcal{X}} = \pm \left. \frac{dx_{j_1} \cdots dx_{j_l} \cdots dx_{n+1}}{f_{x_j}(x)} \right|_{\mathcal{X}} \]

(\( f_{x_1} \, dx_1 + f_{x_2} \, dx_2 = 0 \Rightarrow \frac{dx_i}{f_{x_2}} = -\frac{dx_i}{f_{x_2}} \), \ldots )

\[ \Rightarrow \varphi = g(x) \left( -2 \right)^{k} \frac{dx_{i_1} \cdots dx_{i_k} \cdots dx_{n+1}}{f_{x_i}(x)} \left|_{\mathcal{X}} \right. \]

is well-defined

\[ \text{Defn: } \varphi = \text{Res} \left. \frac{g(x) \, dx_{i_1} \cdots dx_{i_k} \cdots dx_{n+1}}{f(x)} \right|_{\mathcal{X}} = \text{Res} \left. \frac{g(x)}{f(x)} \right|_{\mathcal{X}} \]

For \( \tilde{\mathcal{X}} = \text{normalization of } \mathcal{X} \)

\[ \text{Defn: } K_{\tilde{\mathcal{X}}} = \left\{ \varphi : \tilde{\varphi} \text{ holomorphic on } \tilde{\mathcal{X}} \right\} \]

\[ = \left\{ \varphi : \tilde{\varphi} \text{ holomorphic on } \tilde{\mathcal{X}}_{\text{reg}} \right\} \]
Ex \[ xy = 0 \] \[ \frac{g(x,y)\,dx\,dy}{xy} \rightarrow g(0,y)\,dy \text{ on } x = 0 \]

\[ K_{\mathcal{X}} = \text{Res} \left\{ \frac{g(x,y)\,dx\,dy}{xy} \right\} : g(0,0) = 0 \]

Ex \[ x^2z = y^2 \quad \Rightarrow \quad (u, v) \rightarrow (u, uv, v^2) \]

\[ \frac{g(x,y,z)\,dx\,dy\,dz}{x^2z - y^2} \rightarrow 2 g(u, uv, v^2)\,du\,dv \]

\[ K_{\mathcal{X}} = \text{Res} \left\{ \frac{g(x,y,z)\,dx\,dy\,dz}{x^2z - y^2} \right\} , \quad g = 0 \text{ on the double curve } x = 0, y = 0 \]

\[ \mathcal{X} \supset \mathcal{B} = \{ u = 0 \text{ in } (u,v) - \text{plane} \} \]

\[ \downarrow \quad \downarrow \]

\[ \mathcal{X} \supset \mathcal{D} \]

\[ \mathcal{D} \rightarrow \mathcal{D} \text{ is 2:1 with involution } \]

\[ \tau (u,v) = (u,v) \]
Note: \( \text{Res}_{D}(\phi) = \int_{(0,0,u^2)} du \) is holomorphic on \( \tilde{D} \) with \( \tau^* (\cdot) = (\cdot) \)

\( \rightarrow \) Double residue is in \( \Omega^1_{\tilde{D}} \)

**Defn:** \( \omega^{\times} = \text{all } \phi = \text{Res}_{x}(\tilde{E}) \text{ as above} \)

For this \( \omega^{\times} \) we have

\[ H^0(E)^* \cong H^{n-8}(E^0, \omega^{\times}) \]

**Ex:** For algebraic curves

we have 1-forms with log poles upstairs and \( \pm \) residues at identified points.


Nother's inequality

\[ p_2(x) \leq \frac{k_2^2}{2} + 2 \]

\[ h^0(L_2) + h^0(L_1) - 1 \geq h^0(L_2 \otimes L_1) \geq h^0(L_2) + h^0(L_1) - 1 \]

If \( h^0(L_1) = d_1 + 2 \)

- \( q_{L_1} : X \rightarrow \mathbb{P}^{d_1} \), image spans \( \mathbb{P}^{d_1} \)

- \( \mathbb{P}^{d_1} \times \mathbb{P}^{d_2} \rightarrow \mathbb{P}^{d_1 + d_2 - 2} \) via \( V_1 \otimes V_2 \rightarrow V_2 \otimes V_2 \)

and image of independent points on each give spanning set in \( \mathbb{P}^{d_1 + d_2 - 2} \)

Corollary (Clifford): \( \deg L = g - 1, L^3 = K_C \)

\[ \Rightarrow \dim \{ L \} \leq \frac{g - 1}{2} \]

- Suppose \( X \) regular and use (x) plus

\[ 0 \rightarrow K_X \rightarrow 2K_X \rightarrow K_C \rightarrow 0 \]

\[ h^0(K_X) + h^0(K_X) - 1 \leq h^0(2K_X) = 0 \]

\[ 2g - 2 \leq g^2 + K_X^2 \]
Proof that for a general H-surface \( E \) there is a unique hyperelliptic curve \( C_0 \in \ker \chi \).

This will use concepts to be introduced later.

- Hodge bundle, given by \( \det \pi^* \varpi \otimes \tilde{\mathcal{F}} \), which has general fibre \( H^0(\Omega^2_C) \).

- Moduli \( C_0 \in \ker \chi \). These are the singular fibres in \( \ker \chi \) for a general \( E \). The pictures are:

- Reducible fibres

- Difference is \( - \) vanishing cycle \( S \neq 0 \)
- \( - \) vanishing cycle \( S = 0 \)
Estves formula

\[ h = 9 \lambda - \delta_0 - 3 \delta_1 \]

number of hyperelliptic fibres

Will see that \( \lambda = 5 \)

Noether

\[ \chi(X,\mathcal{O}_X) = \frac{1}{12} \left( K_X^2 + \chi_{\text{top}}(X) \right) \]

\[ \frac{n}{3} \quad \frac{2}{2} \]

\[ \Rightarrow \chi_{\text{top}}(X) = 34 \]

\[ \Rightarrow b_2(X) = 32, \quad h^{2,1}(X) = 28 \]

\[ \chi(X,\mathcal{O}_\hat{X}) = \frac{1}{12} \left( K_{\hat{X}}^2 + \chi_{\text{top}}(\hat{X}) \right) \]

\[ \frac{n}{3} \quad \frac{0}{0} \]

\[ \chi_{\text{top}}(\hat{X}) = 36 \]

\[ \delta_2 = 0 \quad (\text{base point at node- but general } X \text{ has a distinct base points}) \]

\[ \chi(\hat{X}) = \chi(\mathbb{P}^1) \chi(C) + \delta_0 \Rightarrow \delta_0 = 44 \]
Use of Bertini for $H$-surfaces

- equation of $\mathfrak{a}^2 \subset PE$ is
  \[ x t_0^2 G = F^2 \]

Consider pencil
  \[ x_0 t_0^2 (\lambda_0 G_0 + \lambda_2 G_2) = (\lambda_0 + \lambda_2) F^2 \]

By usual Bertini, generic member is smooth outside the base locus

\[
\begin{cases}
  x t_0^2 = 0 \\
  F = 0
\end{cases}
\]

These separate into

(i) $t_0^2 = 0 = F$, \( x \neq 0 \)

(ii) $x = 0 = F$

(i) = double conic (pinch points are (i) + $G = 0$)

(iii) = blown up base points of $|K_X|$

Note: This gives another proof of unique $HE C_0 \in |K_X|$
## II.1

### H. and I. Surfaces \(X\)

- \(X\) smooth, irreducible, minimal, general type \((\kappa(X) = 2)\)
- Numerical (Hilbert polynomial):
  \[ K_X^a = 2, 1 \]

- Hodge-theoretic:
  - \(\varphi(X) = 0\) \(\Rightarrow h^2(O_X) = 0\)
  - \(\rho_g(X) = 2\) \(\Rightarrow h^2(O_X^2) = 2\)

**Objective:** Get good understanding of

\[ X \xrightarrow{\text{geometrically}} \text{Specializations} \xrightarrow{\text{equations}} \]

- two pictures \(\Rightarrow\)
  - \(\varphi_K(X) = X' \subset P^4\)
  - \(X^b = \text{hypersurface in } P^4\)
    - where \(E = O_{P^2} \otimes O_{P^2} \otimes O_{P^2}\)
  - picture of \(\kappa_X\)
- ideas \(\Rightarrow\) build up \(\oplus H^0(mK_X)\) inductively
Theorem $\mathcal{X}$ = general $H$-surface

(i) general $C \in |K_{\mathcal{X}}|$ smooth and $g(C) = 3$

(ii) $\varphi_{2k_{\mathcal{X}}}: \mathcal{X} \to \mathcal{X}' \subset \mathbb{P}^4$ birational with image

\[
\begin{cases}
    x_0x_2 = x_2^3 \\
    x_0G(x) = F(x)^2
\end{cases}
\]

F, G = quadric, cubic

(iii) there exists unique hyperelliptic $C_0 \in |K_{\mathcal{X}}|$.

(iv) $\mathcal{X}'$ sing = double conic with 8 pinch points

\[
\varphi_{2k_{\mathcal{X}}}|_{C_0} = \varphi_{K_{C_0}}: C_0 \to \mathbb{P}^1
\]

(i) no fixed component in $|K_{\mathcal{X}}|$.

Bertini - will see two distinct base points where all $C$ smooth.

(ii) idea: use $m \geq 2$

\[
0 \to (m-1)K_{\mathcal{X}} \to mK_{\mathcal{X}} \to K_C \to 0
\]

exact on $H^0(\cdot)$ level and know $h^0(mK_{\mathcal{X}}) = m(m-1) + 3$; also

$h^0(K_C^{m_{12}})$ since $C$ = plane quartic curve.
\[ P_1 = P_9 = 2, \quad P_2 = 5, \quad P_3 = 9, \quad P_4 = 15, \quad P_6 = 33 \]

- \( H^0(K_X) \quad t_0, t_2 \quad \text{weight 1} \)

- \( H^0(\omega_X) \quad t_0^a, t_0^b, t_3, t_2^3, x_3, x_4 \quad \text{weight 2} \)

- \( H^0(3K_X) \quad P_3(t_2, x_1) \), \( \Omega \) of weight 3
  \[
  \dim = 8
  \]
  \[ \exists \{ c \in H^0(K^*_C) \} \neq 0 \]

- \( H^0(4K_X) \quad P_4(t_2, x_1), \quad t_0, t_2, t_4, \gamma \Rightarrow \text{one reaction} \]
  \[
  \dim = 14
  \]
  \[ t_2 \gamma = F_4(t_0, t_0^* t_1, t_2^3, x_3, x_4) \]

- \( H^0(6K_X) \quad P_6(t_2, x_1, \gamma) - \text{then one relation} \)
  \[
  t_0^2, t_0^3 \gamma, t_0 x_1 \gamma, t_0 x_1 \gamma \quad \text{mod} \quad P(t_0, t_1, x_3, x_4) \]

  \[ \exists \gamma^2 = G(t_0, t_0^* t_1, t_2^3, x_3, x_4) \]

  change notation to \( \gamma = F \) and square to have
  \[
  t_0^2 G = F^2, \quad x_0 G = F^2 \quad \text{in} \quad P^4 \]
- remainder of the argument to be completed later - some observations

- $P(1,1,1,1,2) = \mathbb{C}^4 \Rightarrow \mathbb{C}^*$

  where for $\lambda \in \mathbb{C}^*$, $\lambda \cdot (x_0, x_1, x_2, x_3, x_4) = (\lambda x_0, \lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4)$

- $\text{image here is } \mathcal{Q}_0$

- $R(\mathcal{X}) \cong \mathbb{C} [x_0, x_1, x_2, x_3, x_4, \mathcal{Y}] / (\mathcal{X} - F, \mathcal{X}^3 - G)$

- will see that for $F, G$ general

  the resulting $\hat{\mathcal{X}}' \subset \mathbb{P}^4$ has $\hat{\mathcal{X}}' = \mathcal{X}$

  smooth $H$-surface with the stated properties

- pictures of $H$-surface

  $\hat{\mathcal{X}} \subset \mathbb{P}^4$, $\hat{\mathcal{X}} \rightarrow \hat{\mathcal{X}}' \subset \mathbb{P}^4$
Recall: \( E = \mathcal{O}_{\mathbb{P}^2}(\alpha_1 \cdot \alpha_2 \cdot \alpha_3(2)) \), \( \mathcal{PE} \to \mathbb{P}^2 \)

- \( (\mathcal{PE})_t = \mathcal{PE}_t^2 \)

- \( \mathcal{E} = \mathcal{O}_{\mathcal{PE}}(2) \), \( R^1_\mathcal{E} \mathcal{E} = \{ 0 \text{, } m \geq 0 \} \)

- \( \mathcal{H} = \mathcal{O}_{\mathcal{PE}}(2) \), \( \text{Pic} \mathcal{PE} \cong \mathbb{Z} \)

- \( \mathcal{E}^2 = 2 \mathcal{P}^2 \) \( \implies \mathcal{H} \in H^6(\mathcal{PE}, \mathbb{Z}) \)

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**Theorem:** (i) \( \mathcal{PE} \to \mathcal{Q}_x \subseteq \mathbb{P}^4 \) is the natural desingularization of \( \mathcal{P}(x, 1, 2, 2) \)

(ii) We have \( \begin{array}{c}
\begin{array}{c}
\mathcal{E} \xrightarrow{q_{2k+2}} \mathcal{H} \xrightarrow{f} \mathcal{PE} \to \mathbb{P}^4, \ f = 171 \\
\end{array}
\end{array} \)

\( \mathcal{P}^2 = \mathbb{P}^2 \)

- Proof of (i): \( \Pi_x(m \mathcal{H} + \mathcal{E}) \equiv S^m \mathcal{E}(2) \)

\( \to H^0(\mathcal{E} - 2 \mathcal{H}) = \mathbb{C} \cdot x \), \( (x) = S \subseteq \mathcal{P}(\alpha_1 \alpha_2) \)

- \( \Pi_x \mathcal{E} \) has basis \( [t_0, t_1, t_2, t_3, x_0, x_1, x_2, x_3, x_4] \)

\( \begin{array}{c}
\begin{array}{c}
\mathcal{PE} \to \mathbb{P}^4 \text{ with image } \mathcal{Q}_0, \ S \to \mathcal{Q}_0, \text{ sing}
\end{array}
\end{array} \)
- intrinsically, $\xi_\mathcal{E} \in H^0(\mathcal{PE}, \mathcal{E})$ so $x_{i,j} \xi_\mathcal{E} \in H^0(\mathcal{E})$

and map is

\[
\begin{aligned}
    x_0 &= x t_0^2 \\
    x_2 &= x t_0 t_2 \\
    x_3 &= x t_2 \\
    x_4 &= x_3 \\
    x_7 &= x_4
\end{aligned}
\]

- LHS = coordinates in $\mathbb{P}^4$
- RHS = sections of $\mathcal{E}$

- $S = (X) = \mathbb{P}(\mathcal{E}^2 \oplus \mathcal{E}^2) \cong \mathbb{P}^2 \times \mathbb{P}^2$ maps to $L = \mathbb{C}^* \times \mathbb{C}^*$

- Picture

\[
\begin{array}{c}
\mathbb{P}^2 \\
\downarrow \\
S \\
\downarrow \\
\mathbb{P}^2
\end{array}
\]

- Picture

\[
\begin{array}{c}
\text{fibres of } S \rightarrow \mathbb{P}^2 \\
\text{map isomorphically to } L
\end{array}
\]

Proof of (ii): Equation of $f^{-2}(X')$ is $X^2 \mathcal{E}^2$ PE

\[
x t_0 \mathcal{E}^2 G = F^2
\]

$F \in \mathcal{E}^2$, $G \in \mathcal{E}^2$

- for generic $F, G$ this is smooth away from $t_0 = 0$
- writing as $t_0^3(xG) = F^3$ we see double curve $t_0 = 0$, $F = 0$ with pinch points

\[
\begin{aligned}
  x = 0, F = 0 & \text{ points in } t_0 = 0 \\
  G = F = 0 & \text{ is 6 } \quad \quad \quad \\
\end{aligned}
\]

- the first is because $\mathcal{X} \to H^0 (\varpi, 1^{2})$

and $F = 0 |_{t_0 = 0}$ is a conic

- picture of $\mathcal{X}$

- general $C_x$ = smooth plane quartic

- special $C_x$'s = nodal ones

- base points are 2 marked points = bitangent $L$ to all $C_x$

- $C_{x_0} \to \{ F = 0 \} n \{ t_0 = 0 \}$ with $6 + 2 = 8$ branch points
- \( K_{\mathcal{X}} = g^*(\xi - \eta) \)

- \( K_{\mathcal{PE}} = \xi^3 \)

\[ \text{Yamabe: } 0 \rightarrow V \rightarrow T_{\mathcal{PE}} \rightarrow \xi^2 TP^2 \rightarrow 0 \]

- \( \text{Euler: } PV^3 = P^{m-2}, \quad \dim V = n \)

\[ 0 \rightarrow \Omega_{P^{m-2}} \rightarrow V^2(1) \rightarrow TP^{m-2} \rightarrow 0 \]

\[ 1 \rightarrow \Sigma x_i \chi_{x_i} \]

\[ \det T_{\mathcal{PE}} = h^2 \otimes \bigotimes E^0 \otimes \xi^3 = \xi^3 \]

\[ h^{-2} \]

- Adjunction gives result

\[ \mathbb{E}^b \circ 1 \mathcal{E}^{*} \]

\[ K_{\mathcal{X}} = \xi^3 \circ K_{\mathcal{PE}} \otimes [D] \]

\[ = \xi^3 - \eta \]

\[ \text{Note: } H^0(K_{\mathcal{X}}) \cong H^0(\mathcal{X} \otimes \mbox{Ker} \eta^2) \cong H^0(P^2, \mathcal{E}(1)) \]

\[ \cong H^0(\mathcal{O}_{P^2}(1)) \]