Hodge theory and degenerations of algebraic surfaces

- important question in algebraic geometry is classification of varieties with given properties / invariants
- this leads to moduli spaces $\mathcal{M}$ and their compactifications $\overline{\mathcal{M}}$ - degeneration of smooth varieties to ones with "canonical" singularities
- Hodge theory is an important method to study the geometry / topology of algebraic varieties and their degenerations

- much studied in the classical case (curves, abelian varieties, K3 surfaces) and CY's - not been much explored for surfaces of general type (like $g \geq 2$ for curves)
- this course will discuss how Hodge theory may be used to study the geometry of the degenerations of two examples of algebraic surfaces (the first ones that are encountered in the general classification) - both surfaces are very concrete to describe

- the general plan of the course is
  - review standard stuff from algebraic geometry, moduli, and Hodge theory (establish notations etc)
  - study H- and I-surfaces - give their equations and different ways of looking at them
  - discuss moduli / singularities
  - discuss Hodge theory / degenerations of Hodge structures
  - apply this to the two classes of surfaces

notes posted on

http://math.miami.edu/~pg

Cast of characters - notations

- X = smooth projective variety
- vector bundles $E \to X$

  $\mathcal{O}_X(E)$, $H^q(X, E)$, $h^q(X, E)$
- $K_X = \Lambda^n T^* X$, dim $X = n$
- Ω^p_X
- line bundles $L \to X$, $L = [D]$

  for D = divisor locally given by (f) -

  $K_X = (\omega)$
- on X surface $L \cdot L' = c_1(L) \cdot c_1(L') = D \cdot D'$
- <u>duality</u>: $H^q(E)^* \cong H^{n-q}(E^* \otimes K_X)$
- <u>vanishing</u>: L ample $\Rightarrow H^q(L^*) = 0$, $q < n$
- $H^0(mK_X)$, $R_X = \bigoplus_{m \geq 0} H^0(mK_X)$

  pluricanonical ring, $P_m = h^0(mK_X)$
- $\kappa(X) = \mathrm{Tr\,deg}\, R_X$ - general type $\kappa(X) = n$

• $X = C$ = smooth algebraic curve = compact Riemann surface

$$\chi(\mathcal{O}_X) = 1 - g, \quad g = h^1(\mathcal{O}_X) = h^0(\Omega^1_X)$$

• $E \xrightarrow{\pi} X$, $(\mathbb{P}E)_x = \mathbb{P}E^*_x = E^* \setminus \{0\} / \mathbb{C}^*$

$\mathcal{O}_{\mathbb{P}E}(1) = \xi$ – fibres are 1-dimensional quotients of E

• $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d)) = F_d$, usually $d \geqq 0$

• $H^0(\mathbb{P}E, \xi^m) \cong H^0(\mathbb{P}^1, \mathrm{Sym}\, E)$

• $X = S$ = smooth algebraic surface

$$\chi(\mathcal{O}_X) = 1 - q(X) + pg(X) \quad \begin{array}{l} q(X) = h^0(\Omega^1_X) = h^1(\mathcal{O}_X) \\ pg(X) = h^0(K_X) = h^2(\mathcal{O}_X) \end{array}$$

$$= \frac{1}{12}(K_X^2 + \chi_{top}(X)) \quad \text{(Noether)}$$

• $\chi(L) = \frac{1}{2}(L^2 - L\cdot K_X) + \chi(\mathcal{O}_X)$

(Riemann-Roch for line bundles on surfaces)

- singular varieties
  - normal ; normalization $\tilde{X} \to X$;

    codim $X_{sing} \geq 2$

- curves - X_{sing} = nodes = $\begin{cases} xy = 0, \\ \end{cases}$  

- surfaces
  - $xy = 0$, $x^2 z = y^2$
  - isolated
    - simple elliptic
    - cusp

- $(u, v) \to (u, v^2, uv)$ normalization of pinch point

- $x^3 + y^3 + z^3 = 0$ 

- adjunction conditions - dualizing sheaf

  $f(x_1, \dots, x_n) = 0$, $\omega = g(x) \dfrac{dx_1 \wedge \dots \wedge dx_n}{f(x)}$

  $$\operatorname{Res}_X \omega = g(x) \frac{dx_2 \wedge \dots \wedge dx_n}{f_{x_1}(x)} \Bigg|_X = - g(x) \frac{dx_1 \wedge dx_3 \wedge \dots \wedge dx_n}{f_{x_2}(x)} \Bigg|_X = \dots$$

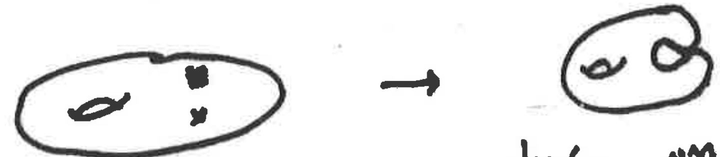
$$\begin{array}{l} - K_X \longleftrightarrow \mathrm{Res}_X\, \omega \text{ holomorphic on } \tilde{X} \to X \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \uparrow \end{array}$$

normalization, or any desingularization

- $\omega_X \longleftrightarrow$ all $\mathrm{Res}_X\, \omega$'s ; duality theorem holds for this ω_X

Ex $\mathrm{Res}\left(\dfrac{g(x,y,z)\, dx\wedge dy\wedge dz}{x^2 z - y^2} \right) = \dfrac{du\wedge dv}{u}$

Ex for



since $\mathrm{Res}\left(\dfrac{dx\wedge dy}{xy}\right) = \begin{cases} dx/x & \text{on } y=0 \\ -dy/y & \text{on } x=0 \end{cases}$

we get for ω_X logarithmic 1-forms with opposite residues

- rational map

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & Y \\ \cup & \nearrow_{\text{holomorphic}} & \\ U & & \end{array}$$

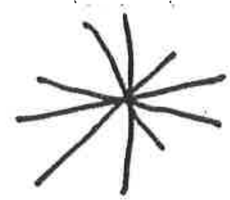
Zariski-open

graph $\Gamma \subset X \times Y$ is an algebraic subvariety

Ex $x/y : \mathbb{C}^2 \dashrightarrow \mathbb{P}^1$

$$\begin{array}{ccl} \mathbb{P}^1 \subset & \Gamma = & \{ x t_1 = y t_0 \text{ where } [t_0, t_1] \in \mathbb{P}^1 \} \\ \downarrow & \downarrow & \\ \{0\} \subset & \mathbb{C}^2 & \end{array}$$

$\Gamma = Bl_{\{0\}} \mathbb{C}^2$



L

Ex: $\varphi_L : X \dashrightarrow H^0(L)^*$

$\varphi_L(x) = [s_0(x), \dots, s_n(x)]$, $s_i \in H^0(L)$ basis

- $\varphi_{K_X} : X \dashrightarrow \mathbb{P} H^n(\mathcal{O}_X)$

$\varphi_{K_C} : C \longrightarrow \mathbb{P}^{g-1}$ canonical curve, $g \geq 2$

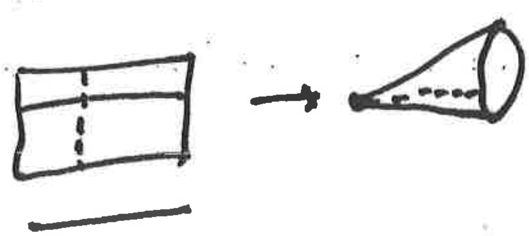
- $\varphi_\xi : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \longrightarrow Q_0 \subset \mathbb{P}^3$

$H^0(E)$ has basis $x, t_0^2, t_0 t_1, t_1^2$

$\cong$

$H^0(\mathbb{P}E, \xi)$

$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}(0)) \longrightarrow p$



- main tool in studying algebraic surfaces is using $C \in |L|$ and cohomology of

$$0 \to \mathcal{F} \to \mathcal{F} \otimes L \to \mathcal{F} \otimes L|_C \to 0$$
$$\mathcal{F} \otimes L|_C = \mathcal{F} \otimes L \otimes \mathcal{O}_C$$

together with

- $K_C = K_X \otimes L|_C$ (adjunction)
- vanishing theorems
- duality

<u>Ex</u> $C \in |K_X|$

$K_C = 2K_X|_C \quad (K_C^{1/2} = K_X|_C)$

$\Rightarrow \varphi_{2K_X}|_C = \varphi_{K_C}$; bicanonical map on surfaces especially important

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<u>Note</u> • X smooth, C irreducible $\Rightarrow \omega_C = K_X \otimes [C]|_C$

Defns: **I-surface** is minimal, regular, general type with

$$\begin{cases} p_g(X) = 2 \\ K_X^2 = 1 \end{cases}$$

H-surface is minimal, regular, general type with

$$\begin{cases} p_g(X) = 2 \\ K_X^2 = 2 \end{cases}$$

Riemann-Roch plus MVT ($h^1(mK_X) = h^1((m-1)K_X^*) = 0$ for $m \geq 2$) gives

$$P_m(X) = h^0(mK_X) = \begin{cases} m(m-1)/2 + 3 \\ m(m-1) + 3 \end{cases}$$

for $m \geq 2$. Here

$$\frac{1}{2}(L^2 - K_X \cdot L) = \frac{1}{2}(m^2 - m)K_X^2 = \left(\frac{m(m-1)}{2}\right)K_X^2.$$

- $\varphi_{2K_X} : X \dashrightarrow \begin{cases} \mathbb{P}^3 & \text{- I-surface} \\ \mathbb{P}^4 & \text{- H-surface} \end{cases}$

- Why these surfaces?

(*) $$p_g(X) \leq \frac{K_X^2}{2} + 2 \qquad \text{(Noether)}$$

$\Downarrow$

I, H surfaces are close to extremal (and have small invariants)

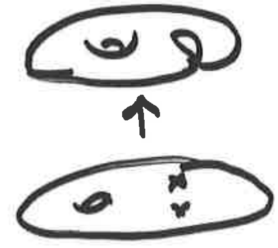
- will see that (*) reflects Clifford's theorem for $C \in |K_X|$
- will also see that I, H surfaces have very nice equations for $\varphi_{2K_X}(X)$:

$\left\{ \begin{array}{l} \text{I-surface: branched cover } \cancel{\text{over}}\ \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)) \\ \text{H-surface: hypersurface in } \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)) \end{array} \right.$

Each of those is a birational model - point is that the surface is described by 1 equation

Moduli

- curves -- stable curve C has
  - ordinary nodes D
  - ω_C ample



- then have $\tau : \tilde{D} \to \tilde{D}$, $D = \tilde{D}/\tau$

where

  - $\tau : \tilde{D} \to \tilde{D}$
  - $K_{\tilde{C}} + \tilde{D}$ ample

$$\left\{ \begin{matrix} \text{stable} \\ \text{curves } C \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} (\tilde{C}, \tilde{D}, \tau) \\ \text{as above} \end{matrix} \right\}$$

- <u>Defn</u>: $\mathcal{M}_g = \left\{ \begin{matrix} \text{set of stable curves } C \\ \text{with } \chi(\mathcal{O}_C) = 1 - g \end{matrix} \right\}$

- $\mathcal{M}_g$ is
  - projective of dimension $3g-3$
  - irreducible
  - almost smooth

- Surfaces - singularities are semi-log-canonical (slc) - up to action of finite groups these are

  - smooth
  - double curve with pinch points
  - isolated
    - canonical (no adjunction conditions)
    - simple elliptic
    - cusp
  - combinations of above

- for us they will all be hypersurface singularities - there is a list (looks worse than it is)

- **stable** X has
  - slc singularities
  - K_X ample

- then have (X, D) and $(\tilde{X}, \tilde{D}, \tau)$ and

$$\left\{ \begin{array}{c} \text{stable surfaces} \\ (X, D) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} (\tilde{X}, \tilde{D}, \tau) \text{ as} \\ \text{above} \end{array} \right\}$$

- Hilbert polynomial $\underset{m \geq 0}{\oplus} \chi(mK_X)$; then $HP(X) = \underset{m \geq 0}{\oplus} \chi(mK_X)$

  - $HP(X)$ constant on components of moduli space

  - component with fixed HP is projective

  - $\mathcal{M}_I, \mathcal{M}_H$ smooth, irreducible with

$$\begin{cases} \dim \mathcal{M}_I = 28 \\ \dim \mathcal{M}_H = 26 \end{cases}$$

- for curves the stratification of $\mathcal{M}_g$ is fairly well understood
  - "principal" strata

  - most degenerate strata; dimension of monodromy cone is maximal
- for surfaces of general type no example and nothing much about the global structure of singular surfaces (except recent work of Liu-Rollenske) nor structure of $\partial\mathcal{M}$ is known

Goal of course: Use Hodge theory as a guide to study $\mathcal{M}_I$ and $\mathcal{M}_H$

More the latter as more interesting surface

<u>Hodge theory</u> - <u>the</u> basic invariant of complex algebraic varieties is its (mixed) Hodge structure on cohomology

- M = manifold, $A^\bullet(M) = \oplus A^q(M) = C^\infty$ differential forms $\sum f_{i_1 \dots i_q}(x)\, dx_{i_1} \wedge \dots \wedge dx_{i_q}$

<u>de Rham:</u> $H^*(M, \mathbb{R}) \cong H^*(A^\bullet(M), d)$

- X = complex manifold

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X), \quad A^{q,p}(X) = \overline{A^{p,q}(X)}$$

$$\varphi = \sum \varphi_{i_1 \dots i_p j_1 \dots j_q} \overbrace{dz_{i_1} \wedge \dots \wedge dz_{i_p}}^{p} \wedge \overbrace{d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}}^{q}$$

Hodge decomposition

- X compact Kähler (projective $\Rightarrow$ Kähler)

$$H^n(X) = \bigoplus_{p+q=n} H^{p,q}(X), \quad H^{q,p}(X) = \overline{H^{p,q}(X)}$$

Hodge filtration

- $$F^p H^n(X) = \bigoplus_{p' \geq p} H^{p',q}(X) \Rightarrow F^p \oplus \overline{F}^{n-p+1} \cong H^n(X)$$

- $$H^{p,q}(X) \cong \left\{ \varphi \in A^{p,q}(X) : d\varphi = 0 \Leftrightarrow \begin{matrix} \partial\varphi = 0 \\ \bar\partial\varphi = 0 \end{matrix} \right\} \Big/ \text{exact}$$

$$\cong H^{p,q}_{\bar\partial}(X) \qquad \text{(Dolbeault)}$$

- $$H^q(X,E) \cong H^{0,q}_{\bar\partial}(X,E)$$

$$H^q(X,\Omega^p_X) \cong H^{p,q}_{\bar\partial}(X)$$

$$\Downarrow$$

$$\boxed{H^{p,q}(X) \cong H^q(X,\Omega^p_X)}$$

- $$H^1(X) \cong H^0(\Omega^1_X) \oplus \underset{\begin{array}{c} \wr\shortparallel \\ H^1(\mathcal{O}_X) \end{array}}{\overline{H^0(\Omega^1_X)}}$$

- $$H^2(X) \cong H^0(\Omega^2_X) \oplus \underset{\begin{array}{c} \uparrow c_1 \\ \mathrm{Pic}(X) \end{array}}{H^1(\Omega^1_X)} \oplus \underset{\begin{array}{c} \wr\shortparallel \\ H^2(\mathcal{O}_X) \end{array}}{\overline{H^0(\Omega^2_X)}}$$

- **Defn:** Hodge structure $(V, F^{\bullet})$ (weight n)
  - $V = \mathbb{Q}$-vector space (usually $V_{\mathbb{Z}}$ with $V_{\mathbb{Z}} \otimes \mathbb{Q} = V$)
  - $F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}}$
  - $F^p \oplus \bar{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}}, \quad 0 \leqq p \leqq n$

- $V^{p,q} = F^p \cap \bar{F}^q$ and $V = \oplus V^{p,q}$, $V^{q,p} = \overline{V^{p,q}}$

  $C = i^{p-q}$ on $V^{p,q}$ is Weil operator

- Polarized Hodge structure (PHS) (V, Q, F)
  - $Q : V \otimes V \to \mathbb{Q}, \quad Q(u, v) = (-1)^n Q(v, u)$
  - $\begin{cases} (HR\ I): Q(F^p, F^{n-p+1}) = 0 \\ (HR\ II): Q(u, C\bar{u}) > 0, \quad u \neq 0 \end{cases}$

curve • $H^1(C) \cong H^0(\Omega^1_C) \oplus \overline{H^0(\Omega^1_C)}$, Q = cup product

surface • $H^2(X) = \underbrace{H^0(\Omega^2_X) \oplus H^1(\Omega^1_X)_{\text{prim}} \oplus \overline{H^0(\Omega^2_X)}}_{V_{\mathbb{C}}} \oplus \mathbb{C}\omega$

Q = cup-product

$\omega = c_1(L)$

$H^*(X)$, X arbitrary

$\Updownarrow$

- Two extensions of HT:
  - <u>mixed Hodge structure</u> (MHS)
  - <u>variation of Hodge structure</u> (VHS)

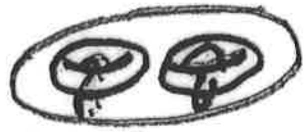
$\Updownarrow$

$H^*(X_t)$

- These come together in <u>limiting mixed HS</u> (LMHS)

$$H^n_{\lim}(X_t) \text{ as } X_t \to X_0$$

- <u>Period domain</u> $D = \left\{ \begin{array}{c} \text{set of PHS's with} \\ \text{given } h^{p,q} = \dim V^{p,q} \end{array} \right\}$



<u>Ex</u> $H^1(C) \cong \mathbb{Z}^{2g}$

$Q = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

$D \cong \mathfrak{H}_g$ via $Z \to (I, Z)$, $F^1 = \{\text{span of rows}\}$
$\qquad\qquad\qquad\qquad\qquad\quad\; \| $
$\qquad\qquad\qquad\qquad\qquad H^0(\Omega^1_C)$

$\int \omega \wedge \omega' = 0$, $(i)\int \omega \wedge \bar{\omega} > 0$

Ex $n=2$, $D \subset \mathrm{Grass}(h^{2,0}, V_{\mathbb{C}})$

$$\overset{\cup}{F}, \begin{cases} Q(F,F)=0 \\ Q(F,\bar{F})>0 \end{cases}$$

$$\begin{cases} V^{2,0} = F \\ V^{2,0} \oplus V^{1,1} = F^{\perp} \end{cases}$$

- Hermitian symmetric domain (HSD) $\Leftrightarrow h^{2,0}=1$
(K3 case)

- <u>non-classical case</u>: $D \neq$ HSD

- equivalence class of PHS's
$\updownarrow$
quotient by action of $G_{\mathbb{Z}} = \mathrm{Aut}(V_{\mathbb{Z}}, Q)$
$\updownarrow$
$\Gamma \backslash D$
where $G_{\mathbb{C}}$ acts on $\mathrm{Grass}(h^{2,0}, V_{\mathbb{C}})$

- <u>Period mapping</u>

$$\updownarrow$$

$$\begin{array}{ccc} \Phi : \mathcal{M} & \longrightarrow & \Gamma\backslash D \\ \cap & & \cap \\ \Phi_e : \overline{\mathcal{M}} & \longrightarrow & \Gamma\backslash D^* \end{array}$$

$$\updownarrow$$

<u>extended period mapping</u>

<u>Strategy</u>: $\Gamma\backslash D^*$ can be understood using Lie theory and geometry of homogeneous complex manifolds - use this to help study $\partial\mathcal{M} = \Phi_e^{-1}(\partial(\Gamma\backslash D^*))$