Topics in the geometric application of Hodge theory

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Introduction
I. Mixed Hodge structures and extension data
II. Limiting mixed Hodge structures and extension data
III. First order variations of mixed Hodge structures and its extension data

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Introduction

The classification of algebraic varieties is a central part of algebraic geometry. Among the common tools that are employed are birational geometry, especially using the analysis of singularities, and Hodge theory. The latter is frequently used as a sort of "black box": the cohomology of an algebraic variety has a functorial mixed Hodge structure and the formal properties of the corresponding category have very strong consequences.

On the other hand, geometric structures arise naturally in Hodge theory and one aspect of this will be the focus of these lectures. This aspect originates from the fact that mixed Hodge structures have extension data expressed by linear algebra.

However *limiting* mixed Hodge structures that arise when singular varieties are smoothed are also constructed by linear algebra but from this construction there is an associated geometric structure, and it is this property and its uses, illustrated by examples, that these lectures will mainly be centered around. The first two lectures will be elementary, mostly consisting of linear algebra constructions; in them we will assume no knowledge of Hodge theory or algebraic geometry, and the examples will be mostly given by pictures.

In the third lecture we will take up a related topic, namely the differential of the period mapping at a singular variety. This discussion will also be formulated by linear algebra but it will use some background from sheaf cohomology as used in the study of Hodge theory of algebraic varieties.

I. Mixed Hodge structures and extension data

- A *Hodge structure of weight n* (V, F) is given by a $\mathbb{Q}$-vector space V and a decreasing *Hodge filtration*

$$F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}}$$

on the complexification that satisfies the conditions

$$F^p \oplus \overline{F}^{n-p} \xrightarrow{\sim} V_{\mathbb{C}} \quad 0 \leqq p \leqq n$$

that F be opposite to $\overline{F}$. Setting

$$V^{p,q} = F^p \cap \overline{F}^q$$

these conditions are equivalent to having a *Hodge decomposition*

$$V_{\mathbb{C}} = \overset{p+q=n}{\oplus} V^{p,q}, \quad \overline{V}^{p,q} = V^{q,p}.$$

The relation between the two is given by

$$F^p = \bigoplus^{p' \geqq p} V^{p',q}.$$

Example: The *Tate Hodge structure* $\mathbb{Q}(-1)$ has $V = \mathbb{Q}$ and is of weight 2 with $V^{1,1} = \mathbb{C}$.

- A *polarized Hodge structure* (V, Q, F) is given by a Hodge structure (V, F) together with a *polarizing form*

$$Q : V \otimes V \to \mathbb{Q}$$

with $Q(u, v) = (-1)^n Q(v, u)$ and satisying the two Hodge-Riemann bilinear relations

$$\begin{aligned} &\text{(HRI)} \quad Q(V^{p,q}, V^{p',q'}) = 0, \quad p' \neq n - p \\ &\text{(HRII)} \quad i^{p-q} Q(V^{p,q}, \overline{V}^{p,q}) > 0. \end{aligned}$$

The first is equivalent to

$$Q(F^p, F^{n-p+1}) = 0;$$

this together with (HRII) gives $F^{n-p+1} = (F^p)^\perp$.

In practice the Hodge structures that arise in geometry are almost always polarizable. For these lectures this will be assumed but the notation will be suppressed.

Example: For X a smooth projective complex algebraic variety the cohomology group $H^n(X, \mathbb{Q}) = V$ has a polarizable Hodge structure.

- Also in practice there will usually be a lattice $V_\mathbb{Z} \subset V$ with $\mathbb{Q} \otimes_\mathbb{Z} V_\mathbb{Z} = V$. This will also be assumed in these lectures.

- A *mixed Hodge structure* (V, W, F) is given by a $\mathbb{Q}$-vector space V with two filtrations

$$\begin{aligned} W_0 \subset W_1 \subset \cdots \subset W_m \qquad & (\textit{weight filtration}) \\ F_n \subset F_{n-1} \subset \cdots \subset F_0 = V_Q \quad & (\textit{Hodge filtration}) \end{aligned}$$

such that on the *associated graded*

$$\mathrm{Gr}_k^W(V) = W_k / W_{k-1}$$

the Hodge filtration induces a Hodge structure of weight k. Specifically

$$F^p \,\mathrm{Gr}_k^W(V) = \frac{F^p \cap W_k}{F^p \cap W_{k-1}}.$$

We really should use $W_{k,\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Q}} W_k$ etc., but to avoid notational clutter we shall not do so.

Example: The cohomology $H^m(X, \mathbb{Q})$ of a complex algebraic variety has a mixed Hodge structure. For X complete the weights are $0 \leqq k \leqq m$; for X affine they are $m \leqq k \leqq 2m$.

Example: For complex algebraic varieties $Y \subset X$ the cohomology groups $H^m(X, Y; \mathbb{Q})$ and $H^m(X \backslash Y; \mathbb{Q})$ have mixed Hodge structures.

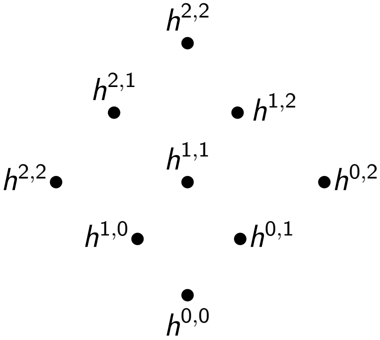
- Mixed Hodge structures don't have Hodge decompositions but they do have canonical Hodge-Deligne decompositions

$$\begin{cases} V_{\mathbb{C}} = \oplus I^{p,q}, & \overline{I}^{p,q} \equiv I^{q,p} \text{ modulo } W_{p+q-2} \\ W_k = \bigoplus\limits_{p+q \leqq k} I^{p,q}, & I^{p,q} \xrightarrow{\sim} \left(\mathrm{Gr}^W_{p+q}(V)\right)^{p,q}. \end{cases}$$

The $I^{p,q}$ may be defined by

$$I^{p,q} = (F^p \cap W_{p+q}) \cap (\overline{F}^q \cap W_{p+q}) + \overline{F}^{q-1} \cap W_{p+q-1} + \overline{F}^{q-2} \cap W_{p+q-2} + \cdots .$$

They are pictured by a Hodge-Deligne diagram, here drawn for $n = 2$ and with $h^{p,q} = \dim I^{p,q}$:



- A *morphism of weight* $2r$ from (V, W, F) to (V', W', F') is given by $\Phi : V \to V'$ satisfying

$$\begin{cases} \Phi(W_k) \subset W'_{k+2r}, \\ \Phi(F^p) \subset F'^{p+r}. \end{cases}$$

This induces

$$\Phi : I^{p,q} \to I'^{p+r,q+r},$$

and conversely any $\Phi \in \mathrm{Hom}(V, V')$ satisfying this condition is a morphism of weight r. In general morphisms of mixed Hodge structures are *strict* in the sense that

$$\begin{cases} \Phi(W) \cap W'_{k+r} = \Phi(W_k), \\ \Phi(V) \cap F'^{p+r} = \Phi(F^p). \end{cases}$$

It is this strictness property together with the functoriality of mixed Hodge structures on the cohomology of complex algebraic varieties that underlies much of their use in algebraic geometry. In particular it is used to show that a sub-quotient of a mixed Hodge structure is itself a mixed Hodge structure.

- Associated to a mixed Hodge structure (V, W, F) is a set of pure Hodge structures $\{H^0, \ldots, H^m\}$ on the *associated graded*

$$\mathrm{Gr}_k^W(V) = W_k/W_{k-1} := H^k.$$

It is most frequently these, together with the above mentioned functoriality and strictness, that are most commonly used in algebro-geometric applications.[†]

[†]A notable exception is the variational theory of mixed Hodge structures whose associated graded Hodge structures are of Hodge-Tate type, meaning that H^k is isomorphic to a direct sum of $\mathbb{Q}(-k)$'s.

In these lectures it is the remaining extension data that is encoded in a mixed Hodge structure that will be our primary object of interest. For the special class of *limiting* mixed Hodge structures we shall see that this extension data has an intrinsic geometric structure not present for general Hodge structures or mixed Hodge structures.

Informal Definition:[‡] Given Hodge structures $\{H^0, \ldots, H^m\}$ the set $E = E(H^0, \ldots, H^m)$ of mixed Hodge structures (V, W, F) with

$$\mathrm{Gr}^W_\bullet(V) \cong \{H^0, \ldots, H^m\}$$

will be called the *extension data* associated to $\{H^0, \ldots, H^m\}$.

[‡]See [GGR] for the formal definition.

In a standard way one may formalize this by using the obvious notions of equivalence of mixed Hodge structures and of their associated gradeds. More important for these lectures will be the notion of *extension data of level* $\leqq k$. Intuitively this is just the set of k-fold iterated extensions of pure Hodge structures taken from $\{H^0, \dots, H^m\}$ in the category of mixed Hodge structures. For our computational purposes we shall deal with this in a very concrete fashion.

- *Extensions of level 1:* Let A, C be pure Hodge structures of weights $k-1, k$ respectively, and

  $$0 \to A \to B \to C \to 0$$

  an extension in the category of mixed Hodge structures.

This exact sequence splits over $\mathbb{Z}$ and also splits as an exact sequence of filtered complex vector spaces. But it does not split over both simultaneously, and the obstruction to doing is given by intrinsically interpreting the difference of these two and lies in the group

$$\text{(I.1)} \quad \mathrm{Ext}^1_{\mathrm{MHS}}(C, A) \cong \frac{\mathrm{Hom}_{\mathbb{C}}(C, A)}{F^0\,\mathrm{Hom}_{\mathbb{C}}(C, A) + \mathrm{Hom}_{\mathbb{Z}}(C, A)}.$$

We note that

- $\mathrm{Ext}^1_{\mathrm{MHS}}(C, A)$ is a compact, complex torus X whose tangent space at the identity is the quotient of a pure Hodge structure of weight -1 and with Hodge decomposition
(I.2)
$$\overbrace{(k-1, -k) + \cdots + (0, -1)} + \underbrace{(-1, 0)} + \cdots + (-k, k-1)$$

The part under the top bracket is $F^0\,\mathrm{End}(A\otimes C^\vee)$, and it is quotiented out to get $TX \cong \frac{\mathrm{End}(A\otimes C^\vee)}{F^0\,\mathrm{End}(A\otimes C^\vee)}$. We will denote by $J \subset X$ the sub-torus whose tangent space is the quotient of the maximal sub-Hodge structure of $\mathrm{End}(A\otimes C^\vee)$ lying over the lower bracket term above.[§]

- the level 1 extension data of (V, W, F) is naturally isomorphic to
(I.3)
$$\overset{k}{\oplus}\,\mathrm{Ext}^1_{\mathrm{MHS}}(W_k/W_{k-1}, W_{k-1}/W_{k-2}) \cong \overset{k}{\oplus}\,\mathrm{Ext}^1_{\mathrm{MHS}}(H^k, H^{k-1}).$$

[§]This maximal sub-Hodge structure is the largest sub-space of the $(-1,0)+(0,-1)$ part that is defined over $\mathbb{Q}$.

- *Extension data of level* $\leqq 2$. First we remark that in the category of mixed Hodge structures the higher Ext's are zero:

$$\operatorname{Ext}^q_{\mathrm{MHS}}(A, C) = 0, \quad q \geqq 2.$$

Therefore we shall use a more geometric description.*

Intuitively, level $\leqq 2$ extension data may be described as first using the set of all extension data for a mixed Hodge structure (V', W', F') whose weight filtration is $W'_0 \subset W'_1 \subset W'_2 \subset W'_3 = V$. Then the observation that a sub-quotient of a mixed Hodge structure is a mixed Hodge structure may be used to describe the extension data of level $\leqq 2$ for a general mixed Hodge structure.

*In some ways the Lie-theoretic description is the most satisfactory; cf. [GGR].

For the use in geometry what is important is the fibration

$$\left\{\begin{matrix}\text{extension data}\\ \text{of level} \leqq 2\end{matrix}\right\} \to \left\{\begin{matrix}\text{extension data}\\ \text{of level } 1\end{matrix}\right\}$$

where we shall refer to a typical fibre as *extension data of level* 2. With this understood we have

(I.4) *the extension data of level* 2 *is isomorphic to* $\overset{k}{\oplus} \mathrm{Ext}^1_{\mathrm{MHS}}(H^k, H^{k-2})$.

In more detail, the exact sequence

$$0 \to W_{k-2}/W_{k-3} \to W_k/W_{k-3} \to W_k/W_{k-1} \to 0$$

gives a class

$$e \in \mathrm{Ext}^1_{\mathrm{MHS}}(W_k/W_{k-1}, W_{k-2}/W_{k-3}).$$

Using that in the category of MHS's the higher Ext's are out and $\mathrm{Ext}^0_{\mathrm{MHS}}(H^k, H^{k-1}) = 0$ gives the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Ext}^1_{\mathrm{MHS}}(W_k/W_{k-1}, W_{k-2}/W_{k-3}) \\ &\to \mathrm{Ext}^1_{\mathrm{MHS}}(W_k/W_{k-2}, W_{k-2}/W_{k-3}) \\ &\to \mathrm{Ext}^1_{\mathrm{MHS}}(W_{k-1}/W_{k-2}, W_{k-2}/W_{k-3}) \to 0 \end{aligned}$$

that represents the differential of the fibration $E_2 \to E_1$ where the first term is the tangent space to a fibre.

- We note that

$$\mathrm{Ext}^1_{\mathrm{MHS}}(H^k, H^{k-2}) \cong \frac{\mathrm{Hom}_{\mathbb{C}}(H^k, H^{k-2})}{F^0 \,\mathrm{Hom}_{\mathbb{C}}(H^k, H^{k-2}) + \mathrm{Hom}_{\mathbb{Z}}(H^k, H^{k-1})}$$

is a complex Lie group that is a quotient of a $\mathbb{C}^m$ by a discrete subgroup that is a partial lattice, and whose tangent space at the identity is again the quotient of a pure Hodge structure of weight -2 with Hodge decomposition

$$
\text{(I.5)} \qquad \begin{aligned} &\overbrace{(k-2,-k)+\cdots+(0,-2)} \\ &\qquad + \underbrace{(-1,-1)} + (-2,0)+\cdots+(-k,k-2). \end{aligned}
$$

Specifically the tangent space $\frac{\mathrm{End}(H^{k-2}\otimes \check{H}^k)}{F^0\,\mathrm{End}(H^{k-2}\otimes \check{H}^k)}$ is obtained by quotienting out the part under the top bracket.

- In general the fibres of the filtration

$$
\left\{ \begin{matrix} \text{extension data} \\ \text{of level} \leqq \ell \end{matrix} \right\} \to \left\{ \begin{matrix} \text{extension data} \\ \text{of level} \leqq \ell - 1 \end{matrix} \right\}
$$

are isomorphic to $\overset{\ell}{\oplus}\, \mathrm{Ext}^1_{\mathrm{MHS}}(H^k, H^{k-\ell})$ and are complex Lie groups that are quotients of $\mathbb{C}^m$'s by partial lattices and whose complex tangent spaces are quotients of Hodge structures of weight $-\ell$.

- The above gives a description of the successive tangent spaces to the fibres of the tower of fibrations

$$E = E_m \to E_{m-1} \to \cdots \to E_1 \tag{I.6}$$

  where E_k are the set of extensions of level $\leqq k$ of mixed Hodge structures constructed from $\{H^0, \dots, H^m\}$. From (I.6), for any complex manifold B and holomorphic mapping from B to E we have a tower of holomorphic mappings

$$\begin{tikzcd}
 & E \arrow[d] \\
 & E_{m-1} \arrow[d] \\
B \arrow[uur, "\Phi_m"] \arrow[ur, "\Phi_{m-1}"] \arrow[r, dashed, no head] \arrow[dr, "\Phi_2"] \arrow[ddr, "\Phi_1"'] & \vdots \arrow[d] \\
 & E_2 \arrow[d] \\
 & E_1
\end{tikzcd}$$

We will show that in the situation arising from algebraic geometry there are very strong constraints on these mappings. Namely, arising naturally in algebraic geometry is the situation of a holomorphic mapping

$$\Phi : B \to E$$

where B is a smooth, possibly non-complete algebraic variety and where the differential of Φ satisfies the differential constraint

$$\boxed{\dot{F}^p \subseteq F^{p-1} \qquad \text{(IPR)}} \tag{I.7}$$

We may write this as $d\Phi \in F^{-1}\,\mathrm{End}(V)$.

Theorem 1.8: (i) *The differential of* Φ_1 *maps* TB *to a translate of the subspace given by the* $(-1,0)$ *part of the tangent space to* E_1 *as given in* (I.2). *Thus* Φ_1 *maps* B *to a translate of* J.

(ii) *On a fibre* S_1 *of* Φ_1 *the differential of* Φ_2 *maps* TS_1 *to a translate of the subspace given by the* $(-1,-1)$ *part in* (I.5). *More specifically, it maps* TS_1 *to a translate of the maximal sub-Hodge structure in the term over the bracket in* (I.5).

The proof results from the following points:

- the tangent spaces to complex manifolds are real vector spaces that have a complex structure;
- the differential of a holomorphic map is a complex linear mapping between real vector spaces that have complex structures; and
- the differential constraint (I.7).

In the situation at hand we are modding out the complexified vector space $\mathrm{End}(C \otimes A^\vee)$ by the part under the brackets $\overbrace{}$ in (I.2) and (I.5) for extension data of levels 1,2,

$$\overbrace{(k-3,-k)+\cdots+(1,-4)+(0,-3)}+(-1,-2)+ \cdots+(-k,k-3)$$

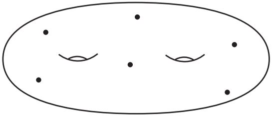
for level 3 and so forth. Moreover, the differential commutes with complex conjugation and satisfies the differential constraint (I.7) and thus lands in the term $(-1,-2)$. Combining these properties leads to a proof of the theorem.

Remark: The mapping $\Phi_1 : B \to J$ may be extended holomorphically to any smooth completion $\overline{B}$ of B.

Application: *A holomorphic mapping $\Phi : B \to E$ satisfying* (I.7) *is determined up to integration constants by the mapping to extension data of level $\leqq 2$.*

By determined up to integration constants we mean that the restriction of Φ to a fibre of Φ_k, $k \geqq 2$, is constant on connected components of that fibre.

Example I.9: X is a compact Riemann surface of genus $g \geqq 1$ and $D = \sum_i n_i p_i$ is a divisor of degree 0 and with support $|D| = \cup_{n_i \neq 0} p_i$.



We may (non-uniquely) write $\sum_\alpha p_\alpha - q_\alpha$ where the p_α, q_α are distinct points on X.

We then have ($\mathbb{Z}$-coefficients)

$$0 \to \frac{H^0(|D|)}{H^0(X)} \to H^1(X, |D|) \to H^1(X) \to 0.$$

The middle term has a mixed Hodge structure with weight filtration $W_0 \subset W_1$, Hodge filtration $F^1 \subset F^0$ and with

$$\mathrm{Gr}_0^W \cong \frac{H^0(|D|)}{H^0(X)} = H^0$$
$$\mathrm{Gr}_1^W \cong H^1(X) = H^1.$$

Using the Hodge decomposition

$$H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$$

where $H^{1,0} \cong H^0(\Omega^1_X)$ is the space of holomorphic 1-forms on X

$$\begin{aligned}
\mathrm{Ext}^1_{\mathrm{MHS}}(H^1, H^0) &= \frac{\mathrm{Hom}_{\mathbb{C}}(H^1, H^0)}{F^0 \,\mathrm{Hom}_{\mathbb{C}}(H^1, H^0) + \mathrm{Hom}_{\mathbb{Z}}(H^1, H^0)} \\
&\cong \big(H^{1,0}(X)^*/H_1(X, \mathbb{Z})\big) \otimes H^0_{\mathbb{Z}} \\
&= J(X) \otimes H^0_{\mathbb{Z}}
\end{aligned}$$

where $J(X)$ is the Jacobian variety of X. The extension class e is the direct sum of the images

$$\langle \mathrm{AJ}_X(p_\alpha - q_\alpha), \omega \rangle = \int_{q_\alpha}^{p_\alpha} \omega, \text{ mod periods}$$

where $\omega \in H^0(\Omega^1_X)$.

Example I.10: X is a smooth algebraic surface, where for the purposes of illustration we assume that $H^1(X) = 0$. Then using $\mathbb{Z}$-coefficients

$$H^2(X) = H^2(X)_{\mathrm{tr}} \oplus \mathrm{Pic}(X)$$

where $\mathrm{Pic}(X) = \mathrm{Hg}^1(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ are the cohomology classes of algebraic 1-cycles (linear combinations of algebraic curves) and is called the *algebraic part* of $H^2(X)$. The other piece $H^2(X)_{\mathrm{tr}} = H^2(X, \mathbb{Z}) \cap \mathrm{Pic}(X)^{\perp}$ is the orthogonal under the cup-product (or intersection pairing) of $\mathrm{Pic}(X)$ and is referred to as the *transcendental part* of $H^2(X)$.

Let $C \subset X$ be a smooth and irreducible algebraic curve. Then we have

$$0 \to H^1(C) \to H^2(X, C) \to \ker\{H^2(X) \to H^2(C)\} \to 0.$$

The last term is $H^2(X)_{\mathrm{tr}} \oplus C^\perp$ where $C^\perp \subset \mathrm{Hg}^1(X)$. The middle term is a mixed Hodge structure and the above is the sequence $0 \to W_1 \to W_2 \to W_2/W_1 \to 0$. The Ext^1 is a sum of two terms corresponding to the direct sum decomposition of Gr_2^W.

Tracing through the definition as in the previous example, the term in the level 1 extension data corresponding to $C^\perp$ is in

$$\mathrm{Hom}\big(C^\perp, J(C)\big).$$

The extension class is described as follows: An element in $C^\perp$ is represented by an algebraic 1-cycle D that intersects C properly in a 0-cycle $D \cdot C$ of degree zero, and the extension is given by

$$D \to \mathrm{AJ}_X(D \cdot C). \tag{I.11}$$

The other term in the Ext^1 is what is called a *membrane integral*. To describe it, again tracing through the definition it is determined by a mapping

$$F^1H^2(X)_{\mathrm{tr}} \otimes H^1(C)^* \to \mathbb{C}$$

given by

$$\omega \otimes \delta \to \int_\Delta \omega \tag{I.12}$$

where $\omega \in F^1H^2(X)_{\mathrm{tr}}$ and $\delta \in H_1(C)$ is a boundary $\delta = \partial\Delta$ of a 2-chain Δ in X. The integral is considered modulo periods $\int_\Gamma \omega$ where $\Gamma \in H_2(X)$. We will omit the argument that this is well-defined.

- **Remark.** For later use we consider the situation where the surface $X = X_0$ varies in a family X_t with the same curve C embedded as a curve $C_t \subset X_t$. This implies that the associated gradeds of the mixed Hodge structure on the $H^2(X_t, C_t)$ don't vary. Then the level 1 extension data varies with t in the fixed vector space $\mathrm{Ext}^1_{\mathrm{MHS}}(H^2, H^1)$. Our assumptions imply that the $\mathrm{Pic}(X_t)$ and subspace $C_t^\perp \subset \mathrm{Pic}(X_t)$ are constant rank lattices. The extension class of the family of mixed Hodge structures has the two parts, the part (I.4) given by

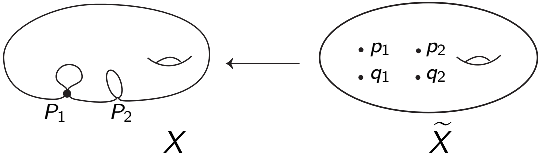
$$\mathrm{AJ}_{C_t}(D_t \cdot C_t) \in J(C_t) = J(C),$$

and the membrane integral part (I.12). As a consequence of Theorem I.8 we may infer that

(I.13) *The Abel-Jacobi part of the extension class may vary with t but the membrane integral part is constant.*

- **Variants of one of the above examples.** These are both examples of the mixed Hodge structure and level 1 extension for the relative cohomology of a smooth variety and a smooth subvariety. However our real interest is in singular varieties, and in preparation for this we will give a variant of the first of the above examples.

Example: $\widetilde{X} =$ smooth algebraic curve and we take d distinct ordered pairs $\{p_i, q_i\}$ of points and join them to points P_i to obtain a singular curve X with d nodes



Then $H^1(X)$ has a mixed Hodge structure with associated graded $\{H^0, H^1\}$ where $H^0 \cong \oplus \mathbb{Z}P_i$ and $H^1 \cong H^1(\widetilde{X})$. Using differential forms the Hodge filtration is given by $H^0(\Omega^1_{\widetilde{X}})$.† The level 1 extension data is isomorphic to $\overset{d}{\oplus} J(\widetilde{X})$ and the extension class is given by $\underset{\alpha}{\oplus} \mathrm{AJ}_{\widetilde{X}}(p_\alpha - q_\alpha)$.

†This means that it is the unique subspace $F^1 \subset H^1(X, \mathbb{C})$ with $F^1 \cap H^0_{\mathbb{C}} = (0)$ and $F^1 \xrightarrow{\sim} H^{1,0}(\widetilde{X}) \subset H^1_{\mathbb{C}}$.

II. Limiting Mixed Hodge structures

- Given V and a nilpotent operator $N \in \mathrm{End}(V)$ with $N^{m+1} = 0$, there exists a unique weight filtration

$$W_0(N) \subset \cdots \subset W_{2m}(N)$$

satisfying

$$\text{(II.1)} \qquad \begin{cases} N : W_k(N) \to W_{k-2}(N) \\ N^k : \mathrm{Gr}^{W(N)}_{m+k}(V) \xrightarrow{\sim} \mathrm{Gr}^{W(N)}_{m-k}(V). \end{cases}$$

A *limiting mixed Hodge structure* is a mixed Hodge structure $(V, W(N), F)$ such that

$$N : F^p \to F^{p-1},$$

or equivalently

$$N \in F^{-1}\,\mathrm{End}(V).$$

In terms of the Deligne decomposition

$$N : I^{p,q} \to I^{p-1,q-1}.$$

Using the second condition in (II.1)

$$H^{m+k} \cong H^{m-k}(-k); \tag{II.2}$$

we will frequently write the limiting mixed Hodge structure as

$$\{H^0, \ldots, H^{m-1}, H^m, H^{m-1}(-1), \ldots, H^0(-m)\}.$$

Example: Let $\mathcal{X}^* \to \Delta^*$ be a smooth family of algebraic varieties X_t over the punctured disc $\Delta^* = \{0 < |t| < 1\}$. Then $\lim_{t\to 0} H^n(X_t)$ may be defined and is a limiting mixed Hodge structure.[‡]

[‡]More precisely, it is an equivalence class of such where $(V, W(N), F) \sim (V, W(N), F')$ if $F' = \exp(\lambda N)F$ for some $\lambda \in \mathbb{C}$.

The N arises from the *monodromy* $T : H^n(X_{t_0}) \to H^n(X_{t_0})$. It is known (monodromy theorem) that T has a Jordan decomposition

$$T = T_s T_u$$

where the unipotent part $T_u = e^N$ with $N^{m+1} = 0$ for some $m \leqq n$. The semi-simple part T_s has all eigenvalues roots of unity; i.e., $T_s^{\mu} = \mathrm{Id}$ for some μ. By a base change $\tilde{t} = t^{\mu}$ we may assume that $T = e^N$.

- Given a limiting mixed Hodge structure as above, the action of N may be completed to the action of an sl_2, denoted $\{N, H, N^+\}$, where

$$\begin{cases} [H, N] = 2N, \\ [H, N^+] = -2N^+, \\ [N^+, N] = H. \end{cases}$$

The induced action on $\mathrm{Gr}^{W(N)}_{\bullet}(V)$ is semi-simple. The irreducible sl_2-module of dimension $\ell+1$ is isomorphic to the homogeneous polynomials in x, y of degree ℓ and where

$$N = x\partial_x, \quad N^+ = y\partial y, \quad H(x^i y^{\ell-i}) = (2i-\ell)x^i y^{\ell-i}.$$

The sequence $x^\ell, x^{\ell-1}y, \ldots, y^\ell$ is called an *N-string*, and the tops of the N-strings are called the *primitive pieces* of V.

- Assuming that $N \in \mathrm{End}(V, Q)$ we may choose the sl_2 to lie in $\mathrm{End}(V, Q)$. Decomposing V into a direct sum of irreducible sl_2-modules, on the primitive pieces of dimension $\ell+1$ the forms

$$Q_\ell(v, w) = Q(N^\ell v, w)$$

induce polarizations. In this way, $\mathrm{Gr}^{W(N)}_{\bullet}(V)$ becomes a direct sum of polarized Hodge structures; this construction is independent of the particular sl_2 containing N.

- Let $N_1, \ldots, N_k \in \mathrm{End}(V, Q)$ be commuting nilpotent endomorphisms and denote by

$$\sigma := \left\{ N_\lambda = \sum_{i=1}^{k} \lambda_i N_i, \;\; \lambda_i \in \mathbb{Q}^+ \right\}$$

  the positive cone they generate. We shall frequently refer to σ as a *monodromy cone*.

Theorem (Cattani-Kaplan): *If each $(V, Q, W(N_\lambda), F)$ is a limiting mixed Hodge structure, then $W(N_\lambda)$ is independent of the $N_\lambda \in \sigma$.*

We shall write the limiting mixed Hodge structure as $(V, W(\sigma), F)$. This is a purely linear algebra result, one whose only known proof is a Hodge theoretic argument that crucially involves the first and second Hodge-Riemann bilinear relations. We note that the primitive decomposition and polarizations on those factors depend on the particular N_λ.

Example: A smooth family $\mathfrak{X}^* \to \Delta^{*k}$ gives a monodromy cone and limiting mixed Hodge structure as above.

Theorem 2.3: *Let $(V, W(\sigma), F)$ be a limiting mixed Hodge structure with E_1 the compact complex torus giving the level 1 extension data and $J \subset E_1$ the sub-torus corresponding to the maximal sub-Hodge structures in the part $(0,-1)+(-1,0)$ of the Hodge structure $\mathrm{End}(H^1, H^0)$. Denote by $\mathrm{Pic}^+(J)$ the ample cone in $\mathrm{Pic}(J)/\mathrm{Pic}^0(J)$. Then there is an inclusion*

$$\check{\sigma} \hookrightarrow \mathrm{Pic}^+(J) \otimes \mathbb{Q}.$$

In other words, limiting mixed Hodge structures with fixed associated graded have the special geometric property that there is a distinguished cone of ample line bundles over the sub-torus of the level 1 extension data.

This situation typically arises when we have a complete family $\mathcal{X} \to \overline{B}$ of projective varieties whose fibres over a Zariski open $B \subset \overline{B}$ are smooth and where the varieties over a normal crossing divisor $Z := \overline{B} \backslash B$ are singular. Then the level 1 extension data of the limiting mixed Hodge structures over Z have distinguished semi-ample line bundles. In [GGR] it is proved that these line bundles are geometrically tied to the normal bundles in $\overline{B}$ of the stratification of Z. As discussed in loc. cit. they appear to be the main feature of the geometry of the period mapping at infinity.

- Rather than discuss the details of the linear algebra argument for the proof of Theorem II.3, we will discuss how the relevant natural maps arise, in particular why the *dual* of the monodromy cone appears. Restricting to the case where we have just one N with $W = W(N)$ the steps are

- $H_1(E_1, \mathbb{Z}) \cong \mathrm{End}_{\mathbb{Z}}(H^1, H^0) \subset \mathrm{Gr}^W_{-1} \mathrm{End}_{\mathbb{Z}}(V)$;
- using the duality induced by the form Q there is a natural map

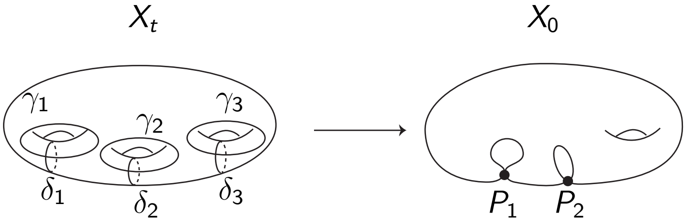
$$\mathrm{Gr}^W_{+2} \mathrm{End}_{\mathbb{Z}}(V) \to H^2(E_1, \mathbb{Z});$$

- tracing through the linear algebra one sees that N^+ maps to a class in $H^2(E_1, \mathbb{Q})$;
- using the symmetries induced by Q and the first Hodge-Riemann bilinear relation, a multiple of N^+ gives a class in $H^2(E_1, \mathbb{Z}) \cap H^{1,1}(E_1)$ and therefore defines (up to translation) a holomorphic line bundle $L_{N^+} \to E_1$;
- these last two steps use that for vector spaces A and B, $\Lambda^2 A \otimes S^2 B$ is a direct summand of $\Lambda^2(A \otimes B)$;
- the second Hodge-Riemann bilinear relation defines a metric in this line bundle whose curvature form is positive on $TJ \subset TE_1$, and using Kodaira's theorem this gives the proof of Theorem 2.3.

The details of this construction are in [GGR]. Below we shall give a coordinate calculation that illustrates how these bundles, which for reasons that will appear in the calculation we shall call *theta line bundles*, arise.

Our first example will be about algebraic curves. It will be in two parts, the first describing using pictures the limiting mixed Hodge structure associated to a stable, irreducible curve, and second being a period matrix calculation for this example that illustrates how the line bundle in Theorem II.3 initially arose.

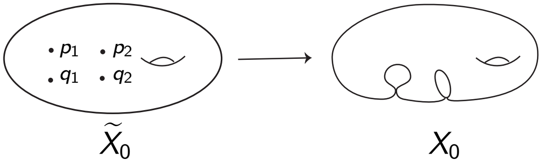
Example: We consider a family of smooth curves $X_t, t \in \Delta^*$, degenerating to a stable irreducible curve



In this picture the vanishing cycles δ_1, δ_2 on X_t contract to the nodes P_1, P_2 on X_0. Using the Picard-Lefschetz formula, the monodromy is given by

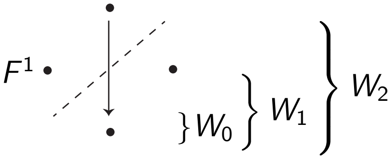
$$N\gamma_i = \delta_i, \quad i = 1, 2.$$

The desingularization of X_0 is



The limiting mixed Hodge structure has associated graded $\{H^0, H^1, H^0(-1)\}$ where $H^0_{\mathbb{Z}} = \mathbb{Z}P_1 \oplus \mathbb{Z}P_2$ and $H^1 = H^1(\widetilde{X}_0)$.

We have previously described the associated graded to the mixed Hodge structure on $H^1(X_0)$. For the limiting mixed Hodge structure H^1_{lim} on V the Hodge-Deligne diagram and weight and Hodge filtrations are



where the solid vertical arrow is N Here using previous notations $W_0 \cong H^0(X_{0,\text{sing}}) = H^0(|D|)$, $W_1 \cong H^1(\widetilde{X}_0, |D|)$ and $W_2/W_0 \cong H^1(\widetilde{X}_0 \backslash |D|)$. In terms of differential forms

- $F^1 \cong \underbrace{H^0(\Omega^1_{\widetilde{X}_0})}_{F^1 \cap W_1} + \underbrace{\mathbb{C}\eta_{p_1,q_1} + \mathbb{C}\eta_{p_2,q_2}}_{F^1 \cap W_2 / F^1 \cap W_1}$;
- $N(\eta_{p_i q_i}) = \operatorname{Res}_{p_i} \eta_{p_i,q_i}$;
- The notation here is that the η_{p_i,q_i} is a differential of the third kind with poles at p_i, q_i where the residues are ± 1; such an η_{p_i,q_i} is unique modulo $H^0(\Omega^1_{\widetilde{X}_0})$ and becomes unique if we normalize it to have the B-periods (the integrals over the γ's) to be zero.

We recall that $\operatorname{Ext}^1_{\mathrm{MHS}}(H^1, H^0) \cong J(\widetilde{X}_0)$ and the extension class is given by $\mathrm{AJ}_{\widetilde{X}_0}(\sum p_i - q_i) = \int_{q_1}^{p_1} \omega + \int_{q_2}^{p_2}$ mod periods, where ω is a basis for $H^0(\Omega^1_{\widetilde{X}_0})$.

As a special case of the general dualities, $\mathrm{Ext}^1_{\mathrm{MHS}}(H^0(-1), H^1)$ is dual to $\mathrm{Ext}^1_{\mathrm{MHS}}(H^1, H^0)$ and contains equivalent information. In this case tracing through the definition

$$\mathrm{Ext}^1_{\mathrm{MHS}}(H^1, H^0(-1)) \cong H^{0,1}(\widetilde{X}_0)/H^1(\widetilde{X}_0, \mathbb{Z})$$

and the extension class is given by

$$\left(\int_{\lambda_1} \eta_{p_1,q_1} + \cdots + \int_{\lambda_{2g}} \eta_{p_1,q_1}\right) + \left(\int_{\lambda_1} \eta_{p_2,q_2} + \cdots + \int_{\lambda_{2g}} \eta_{p_2,q_2}\right)$$

where $\lambda_1, \ldots, \lambda_{2g}$ is a standard basis for $H_1(\widetilde{X}_0, \mathbb{Z}) \cong H^1(\widetilde{X}_0, \mathbb{Z})$ ($g = 3$ in the case under discussion). The equivalence of these two expressions for the extension class is a consequence of the bilinear relation between differentials of the first and third kinds on a compact Reimann surface.

- When we come to the level 2 extension data, without going into the details we comment that in this example we really should consider a 2-parameter family obtained by independently smoothing the nodes. In this case we have monodromies N_1, N_2 and a 2-dimensional monodromy cone σ.

  A fibre of the map from extension data of level $\leqq 2$ to level 1 extension data is $\mathrm{Ext}^1_{\mathrm{MHS}}(H^0(-1), H^0)$. Here, again without going into the details, the symmetries arising from the bilinear form Q and from modding out by passing to equivalence classes of limiting mixed Hodge structures by rescaling the smoothing parameters t_1, t_2 should be taken into account. This will become clearer when we next redo this example using period matrix calculations. Thus although $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}^2(-1), \mathbb{Z}^2)$ $\cong \overset{4}{\otimes}\, \mathbb{C}/\mathbb{Z}$, what is left for the intrinsic part of the definition of the level 2 extension data is a single $\mathbb{C}/\mathbb{Z}$, and the extension class e_2 has the geometric interpretation

$$e_2 = \int_{q_2}^{p_2} \eta_{p_1,q_1} \equiv \int_{q_1}^{p_1} \eta_{p_2,q_2} \bmod \mathbb{Z}.$$

The equality results from the bilinear relation between differentials of the third kind. The integrals in e_2 are well defined modulo $\mathbb{Z}$, so that the quantity $\exp(2\pi i e_2) \in \mathbb{C}^*$ is then well defined and gives a "cross-ratio" of the two pairs of points (p_i, q_i).

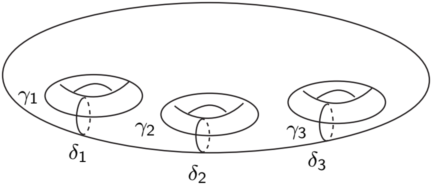
- The normalized period matrix for the above family of curves has the form

$$\Omega = \begin{pmatrix} I \\ M \end{pmatrix}$$

where

$$M = \begin{pmatrix} a_1 & a_2 & \lambda \\ b_{11} & b_{12} & a_1 \\ b_{21} & b_{22} & a_2 \end{pmatrix}, \qquad b_{12} = b_{21} \text{ and } \operatorname{Im}\lambda > 0.$$

Here the basis for $H_1(X_t, \mathbb{Z})$, $t = (t_1, t_2)$, is



where $N_i\gamma_i = \delta_i$,$i = 1, 2$. The basis $\omega_1, \omega_2, \omega_3$ for $H^0(\Omega^1_{X_t})$ has $\lim_{t\to 0} \omega_i = \eta_{p_i,q_i}$ for $i = 1, 2$.

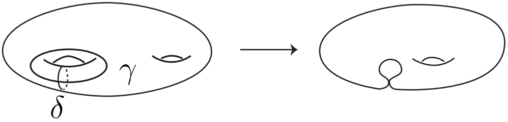
At $t_1 = t_2 = 0$ the H^1 in the limiting mixed Hodge structure is $H^1(\widetilde{X}_0)$, which determines the elliptic curve $\mathbb{C}/\mathbb{Z} + \mathbb{Z} \cdot \lambda$. At $t_1 = t_2 = 0$ the level 1 extension data is given by (a_1, a_2); the first above interpretation of this is in the third column on M, and the second dual interpretation is in the top row.

Turning to the level 2 extension data, it is encoded in the $b_{ij} = b_{ji}$. Setting $\ell(t) = \left(\frac{1}{2\pi i}\right) \log t$ one may show that

$$\begin{cases} b_{ii} = \ell(t_i) + (\text{holomorphic function}) & i = 1, 2 \\ b_{12} = b_{21} \text{ is holomorphic.} \end{cases}$$

The well-defined part of the level 2 extension data is then b_{12}, and one may check that $\exp(b_{12})$ is the e_2 given above.

- These considerations are local, and an interesting question is: What global information is there?[§] To discuss this we shall take the notationally simpler case of a family of $g = 2$ curves acquiring a node.



[§]A stratum in $\overline{\mathcal{M}}_2$ consists of the irreducible curves with one node as in the picture above, together with the curves  that have two nodes, and the reducible curves . The latter do not significantly effect the period matrix.

The period matrix is then

$$\Omega = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & \lambda \\ b & a \end{pmatrix}$$

where $b(t) = b(t) +$ (holomorphic). The global monodromy acting on Ω is given by $\Omega \to T\Omega$ where

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ m_1 & 1 & 0 & 0 \\ m_2 & 0 & 1 & 0 \\ n & m_2 & m_1 & 1 \end{pmatrix} \qquad m_i, n \in \mathbb{Z}.$$

Then

$$T\Omega = \begin{pmatrix} 1 & 0 \\ m_1 & 1 \\ a + m_2 & \lambda \\ n - m_1 a + b & a + m_2 - m_1\lambda \end{pmatrix}$$

which when the period matrix is renormalized is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a + m_2 - m_1\lambda & \lambda \\ n - 2m_1 a - m_1 m_2 + m_1^2\lambda & a + m_2 - m_1\lambda \end{pmatrix}.$$

The $a + m_2 - m_1\lambda$ term expresses that the level 1 extension data is globally well-defined in $\mathrm{Ext}^1_{\mathrm{MHS}}(H^1, H^0) \cong J(\widetilde{X}_0)$. The interesting term is the b, which under the action of monodromy gives

$$b \to b - 2m_1 a + m_1^2\lambda \mod \mathbb{Z}.$$

The classical theta function $\theta(a)$ satisfies

$$\left(\frac{1}{2\pi i}\right)\log\theta(a+\lambda) \equiv \left(\frac{1}{2\pi i}\right)\log\theta(a) - a + \frac{\lambda}{2} \mod \mathbb{Z},$$

while from the above

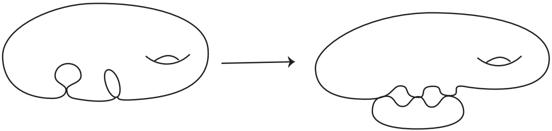
$$b(a+\lambda) \equiv b(a) - 2a + \lambda \mod \mathbb{Z}.$$

Thus

$$e^{2\pi i b} \textit{ transforms like } \theta(a)^2.$$

This is the origin of the theta line bundle constructed above. It also says that the level 2 extension data is intrinsically given as a section of the theta line bundle.

- This brings up the question of what happens to the extension data when we have a further degeneration? For example, in the $g = 3$ example above suppose we have a further degeneration



where the singular curves acquire an additional node whose vanishing cycle is $\delta_1 + \delta_2$. Thus we have N_1, N_2, N_3 where as a map from $\mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$ to $\mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2$

$$N_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Near $t_1 = t_2 = 0$ but keeping $t_3 \neq 0$ the B-part of the period matrix is (here omitting the holomorphic terms)

$$\begin{pmatrix} \ell(t_1) + \ell(t_3) & \ell(t_3) \\ \ell(t_3) & \ell(t_2) + \ell(t_3) \end{pmatrix}.$$

Then for the B-part of the period matrix we have

- $\exp(e_2) = \exp(\ell(t_3) + \cdots)$ is a non-vanishing section of the theta line bundle corresponding to $N_1^+ + N_2^+$;
- at $t_3 = 0$ the section vanishes; in order to define it we have to complete $\mathbb{C}^*$ to $\mathbb{C}^* \cup \{0\}$.

This is a harbinger of the general story of using the extension data to canonically define toroidal completions of period mappings. Basically the mechanism where the level 2 extension data over open strata maps to a product of $\mathbb{C}^*$'s, and when we cross into lower dimensional strata some of the $\mathbb{C}^*$'s become $\mathbb{C}^* \cup \{0\}$'s, will hold in general.

Example: We let X_1, X_2 be two smooth surfaces each containing a smooth curve C and

$$X = X_1 \cup_C X_2$$

the surface with a double curve obtained by gluing X_1 and X_2 along C. Then $H^2(X)$ has a mixed Hodge structure whose weight filtration is obtained from the Mayer-Vietoris sequence of $\{X, X_2 \amalg X_2, X_1 \cap X_2 = C\}$. The Hodge filtration will be described below.

However $H^2(X)$ is generally not part of a limiting mixed Hodge structure H^2_{lim}. For this to happen there has to be a *smoothing* of X, and in order for there to be such a smoothing $\mathfrak{X} \xrightarrow{\pi} \Delta$ of X, where all spaces and maps are smooth and $X_0 = X$, a necessary condition due to Bob Friedman is that

$$N_{C/X_1} \cong \check{N}_{C/X_2}. \tag{BF}$$

If there is a smoothing, then $dt\big|_{X_0}$ gives a nowhere vanishing section of $\check{N}_{C/X_1} \otimes \check{N}_{C/X_2}$. Conversely, if (BF) is satisfied, then there is a first order smoothing of X, and it is known that having only this is sufficient to be able to construct the limiting mixed Hodge structure that would arise from *any* complete smoothing of X.

To construct the mixed Hodge structure $H^2(X)$ we use the Mayer-Vietoris sequence and set

$$\begin{cases} H^1(C)_0 := H^1(C)/\left\{(\text{image of } H^1(X_1) \oplus H^1(X_2)\right\} \\ H^2(X)_0 := \ker\{H^2(X_1) \oplus H^2(X_2) \to H^2(C)\}. \end{cases}$$

Then we have

$$0 \to \underbrace{\underbrace{H^1(C)_0}_{W_1} \to H^2(X)}_{W_2} \to \underbrace{H^2(X)_0}_{\mathrm{Gr}_2^W} \to 0.$$

The interpretation of the extension data in $\mathrm{Ext}^1_{\mathrm{MHS}}(H^2(X)_0, H^1(C)_0)$ is very much like that given above for a smooth curve on a single smooth surface. Again we separate $H^2(X)_0$ into an algebraic part arising from the kernel of the map $\mathrm{Pic}(X_1) \oplus \mathrm{Pic}(X_2) \to H^2(C)$ given by $D_1 \oplus D_2 \to D_1 \cdot C - D_2 \cdot C$ (note the ordering of 1,2), and a transcendental part. If we deform X_1, X_2, C keeping the $\mathrm{Gr}^W_k H^2(X)$ fixed, then as above only the algebraic part will vary.

The Hodge filtration is defined using the isomorphism

$$H^2(X, \mathbb{C}) \cong \mathbb{H}^2 \left(\Omega^\bullet_{X_1 \amalg X_2} \to \Omega^\bullet_C \right),$$

where the map is the signed restriction, together with the filtration induced by the bétè filtration on the $\Omega^\bullet$'s. The notation here is that $\mathbb{H}$ denotes hypercohomology and $\Omega^\bullet_{X_1 \amalg X_2} \to \Omega^\bullet_C$ is a 2-term complex.

Turning to the level 2 extension data the fibre of the map from level $\leqq 2$ extension data to level 1 extension data is $\mathrm{Ext}^1_{\mathrm{MHS}}(H^1(-1), H^1)$. The tangent space here is a quotient of $\mathrm{End}_{\mathbb{C}}(H^1)/F^0\,\mathrm{End}_{\mathbb{C}}(H^1)$. The variable part of this again arises from the Hodge part $\mathrm{End}_{\mathbb{Q}}(H^1)$. For C general, $\mathrm{End}_{\mathbb{Q}}(H^1) \cong \mathbb{Q}$.

When we come to the case when X is to first order smoothable, the (BF) condition comes into play. Writing $H^2_{\mathsf{lim}} = \{H^1, H^2, H^1(-1)\}$ we have

$$H^1 = \mathrm{coker}\left\{\bigoplus_{\mathrm{i}=1}^{2} \mathrm{H}^1(\mathrm{X_i}) \to \mathrm{H}^1(\mathrm{C})\right\}$$

$$H^1(-1) = \mathrm{ker}\left\{H^1(C)(-1) \xrightarrow{G} \bigoplus_{i=1}^{2} H^3(X_i)\right\}$$

where G is the Gysin map. The H^2 is the cohomology at the middle of

$$H^0(C)(-1) \xrightarrow{G} \bigoplus_{i=1}^{2} H^2(X_i) \xrightarrow{R} H^2(C)$$

with R being the signed restriction mapping. That this is a complex is due to the topological consequence

$$\deg(N_{C/X_1}) + \deg(N_{C/X_2}) = 0$$

of BF.

Of interest is also how one may compute the Hodge filtration on $H^2_{\lim}$. Using notations that anticipate Lecture III, we set

$$\begin{cases} X^{[1]} = X_1 \amalg X_2 \\ X^{[2]} = X_1 \cap X_2 = C. \end{cases}$$

From the (BF) condition we may think of the first order smoothing as locally given by

$$t = x_1 x_2$$

where t and dt have global meaning along $X^{[2]}$. Then

$$\frac{dt}{t} = \frac{dx_1}{x_1} + \frac{dx_2}{x_2}$$

from which we may infer that there is a well-defined map of complexes

$$\text{(II.4)} \qquad \Omega^\bullet_{X^{[1]}}(\log X^{[2]}) \xrightarrow{\mathrm{Res}\wedge\frac{dt}{t}} \Omega^\bullet_{X^{[2]}}$$

where Res is the signed residue map.¶ A basic result is that

$$\text{(II.5)} \qquad H^2_{\mathrm{lim}} \cong \mathbb{H}^2\left(\Omega^\bullet_{X^{[2]}}(\log X^{[1]}) \xrightarrow{\mathrm{Res}\wedge\frac{dt}{t}} \Omega^\bullet_{X^{[2]}}\right),$$

and from this we may define the Hodge filtration on H^2_{lim} from the bétè filtration on the $\Omega^\bullet$'s.

To actually describe the Hodge filtration on H^2_{lim} we use (II.5) and also set

$$\Omega^\bullet_{X^{[1]}} \cup_{X^{[2]}} X^{[2]} = \ker\left(\bigoplus_{i=1}^{2} \Omega^\bullet_{X_i}\left(\log X^{[2]}\right) \to \Omega^{\bullet-1}_{X^{[2]}}\right).$$

¶To be precise, we should let $\mathcal{X}^{(1)} \xrightarrow{f} \Delta(\epsilon)$, $\epsilon^2 = 0$, be the first order smoothing of X and use $f_*\left(\Omega^\bullet_{\mathcal{X}^{(1)}/\Delta(\epsilon)}(\log X)\right)$ along $\epsilon = 0$.

Then using this notation the complexification of $0 \to W_1 \to H^2_{\mathrm{lim}} \to \mathrm{Gr}^W_3 \to 0$ is given by

$$0 \to \underbrace{\frac{\mathbb{H}^1(\Omega^\bullet_{X^{[2]}})}{H^1\left(\Omega^\bullet_{X^{[1]}} \cup_{X^{[2]}} X^{[2]}\right)}}_{W_1}$$

$$\to H^2_{\mathrm{lim}} \to \ker \underbrace{\left\{\mathbb{H}^2\left(\Omega^\bullet_{X^{[1]}} \cup_{X^{[2]}} X^{[2]}\right) \to \mathbb{H}^2(\Omega^\bullet_{X^{[2]}})\right\}}_{\mathrm{Gr}^W_2 \cong H^2(X^{[1]} \backslash X^{[2]})} \to 0.$$

It is the third term that is the analogue of the differentials of the third kind that appeared in the previous example of algebraic curves.

Finally a preview of some of what will be in Lecture III. Suppose we vary $X_{i,t}$, $i = 1, 2$, and C_t. In order to vary the limiting mixed Hodge structure we must maintain the (BF) condition above. Then this assumption must enter into any cohomological calculation of the variation of the extension data. This will be our next topic.

Example: We will give an illustration of Theorem I.8.

- In order to have a LMHS H^2_{lim} where there is extension data of level $\geqq 3$ we must have $N^2 \neq 0$;
- In the geometric situation where we have a family of surfaces $\mathfrak{X} \to \Delta$ that is in standard form; i.e., $\mathfrak{X}$ is smooth, the X_t are smooth for $t \neq 0$ and X_0 is a reduced normal crossing divisor X_i, setting

$$X^{[k]} = \coprod_I \left(\bigcap_{i_j \in I} X_{i_1} \cap \cdots \cap X_{i_k} \right), \quad I = (i_1, \dots, i_k),$$

  we have

$$N^k \neq 0 \implies X^{[k+1]} \neq \emptyset;$$

- In order to get non-trivial extension data for H^2_{lim} in the geometric case, we must have $N^2 \neq 0$, which gives $X^{[3]} \neq \emptyset$, i.e. we must have triple points;

- The simplest $H^2_{\lim}$ with $N^2 \neq 0$ are those of *Hodge-Tate type*; i.e.,

$$H^2_{\lim} = \{H^0, H^0(-1), H^0(-2)\};$$

- An example of smoothable X_0 with this $H^2_{\lim}$ is the union $\cup_{\alpha=1}^m H_\alpha$ of $m \geqq 4$ hyperplane $\mathbb{P}^2$'s in general position in $\mathbb{P}^3$;
- If $\omega_t \in H^0(\Omega^2_{X_t})$ has a limit

$$\omega_0 \in H^0\left(\Omega^2_{X^{[1]}}(\log D)\right)$$

where D is the normal crossing divisor induced from $X^{[2]}$, then ω_0 has log poles with opposite residue along the double curve part of D. At the triple points $P_{\alpha\beta\gamma} = X_\alpha \cap X_\beta \cap X_\gamma$ the form ω_0 has double residues $\mathrm{Res}^{(2)}_{P_{\alpha\beta\gamma}}(\omega_0)$;

- Over $\mathbb{C}$, $H^0(-2)$ is generated by the ω_0's, H^0 is generated by the $P_{\alpha\beta\gamma}$'s, and

$$N(\omega_0) = \sum \mathrm{Res}^{(2)}_{P_{\alpha\beta\gamma}}(\omega_0).$$

This is the geometric side of the picture. When one extracts from this the period matrix side the essential part, meaning the part

$$\{H^0, H^0(-1), H^0(-2)\}$$

of the limiting mixed Hodge structure where $H^2 = H^0(-1) \oplus H'^2$ but the H'^2 is not part of the N-string, the resulting period matrix will have the form

$$\Omega = \begin{pmatrix} I \\ A \\ B \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & I \\ 0 & -I & 0 \\ I & 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix}$$

where HRI and HRII are satisfied. Thus

$$\Omega(t) = \begin{pmatrix} I \\ A = \ell(t)I + A_0 \\ B = \frac{\ell(t)^2}{2} I + \ell(t)B_1 + B_2 \end{pmatrix}$$

where A_0, B_1, B_2 are holomorphic. The level 2 extension data is given by A, and B gives the level 4 extension data.[‖]

[‖] As a side remark, in general the power of $\ell(t)$ in level k extension data is $[k/2]$.

Now we come to the main point. HRI gives

$$B + {}^tB = {}^tAA$$

so that the symmetric part of the level 4 extension data is determined algebraically from the level 2 part. However the IPR gives

$$dB = {}^tAdA$$

from which we see that the full level 4 extension data is only determined up to interpretation constants by the level 2 stuff.

Example: For configurations of d hyperplanes in general position in $\mathbb{P}^2$,

- for $d \leqq 5$ the configuration is rigid; there are no parameters in the limiting mixed Hodge structures;

- for $d \geqq 6$ there are parameters. For example, for $d = 6$ if we pick two of the H_α's and intersect them to get a $\mathbb{P}^1$, then the remaining H_β's meet the $\mathbb{P}^1$ in four distinct points and the level 2 extension data is the logarithm of their cross ratio. For any $d \geqq 6$ the level 2 extension data is a collection of logarithms of cross ratios;
- the level 4 extension data may be expressed as Aamoto integrals, which we will not go into here.

In general there is a vast literature on variations of mixed Hodge structure whose associated graded are of Hodge-Tate type. The arithmetic aspect of this subject is of particular interest. The extension data is expressed explicitly in terms of higher logarithms and the periods are special combinations of values of multi-zeta functions. Some references are given in [GRR].

We are not aware of literature about the special features of variations of *limiting* mixed Hodge structures.

III. First order variation of mixed Hodge structures and the associated extension data

- We begin with some general remarks about the use of Hodge theory in algebraic geometry. This was the topic of the lectures given last year (see the references at the end); there the applications of Hodge theory centered around its role in the topology of algebraic varieties. Among other things it was illustrated how just the existence of a functorial Hodge structure on the cohomology of a smooth variety leads to vanishing theorems on cohomology and vice versa. Originally due to Kodaira and Spencer in the 1950s, this technique has been significantly refined over the years.

Hodge theory may also be used directly in the study of the geometry of algebraic varieties. The classic example here is due to Riemann. Associated to the polarized Hodge structure on $H^1(C)$ for a smooth algebraic curve, which is a linear algebraic object, Riemann constructed a geometric object given by the theta divisor $\Theta \subset J(C)$ in the Jacobian variety of the curve. He then showed that much of the geometry of C could be constructed from the geometry of Θ. Among the penultimate results here are the Torelli theorem $(J(C), Q)$ determines C), and Riemann's singularity theorem relating the singularities of Θ to the special divisors on C.

However, because of the IPR the method of associating a geometric object to the cohomology of a smooth algebraic variety is only possible when the period domain is Hermitian symmetric (the classical case), which only occurs for very special varieties. Perhaps the central point of these lectures is that to *limiting mixed Hodge structures*, which arise in families of varieties whose general member is smooth, one may associate geometric objects arising from the extension data in the limiting mixed Hodge structures. We feel that the geometric applications of this construction are probably in their early days.

Another use of Hodge theory in the study of the geometry of algebraic varieties is to extract geometric information from the linear algebraic data given by the first variation of a Hodge structure, a so-called *infinitesimal variation of Hodge structure.* Among other things this method has been used to derive Torelli-type results and to study the *Noether-Lefschetz loci*, defined as the subvarieties of parameter spaces of families of varieties that have special geometric properties, such as the existence of algebraic cycles not present in general members of the family. What has yet to be explored is the use of infinitesimal variations centered at singular varieties X_0 in a family $\{X_t\}$ whose general member is smooth. Interestingly here there are two rather different cases:

(i) the limiting mixed Hodge structure $H^n(X_t)_{\lim}$ is a pure Hodge structure;

(ii) the limiting mixed Hodge structure is not a pure Hodge structure.

Case (i) arises when the action of monodromy on $H^n(X_t)$ around $t = 0$ is finite; using the above notation $T = T_s e^N$ we have $N = 0$. This does *not* necessarily mean that $H^n(X_0)$ has a pure Hodge structure, at least over $\mathbb{Z}$.**

Roughly speaking case (ii) breaks into three sub-cases:

(iia) the study of the associated graded to the limiting mixed Hodge structures when we have a smoothable equisingular deformation of X_0;

By a *smoothable equisingular deformation* we mean an equisingular deformation $X_{0,s}$ of X_0 such that the $X_{0,s}$ are smoothable for all s. This case is similar to the study of ordinary period mappings.

**However for many varieties with quotient singularities, $H^n(X_0)$ has a pure structure over $\mathbb{Q}$.

(iib) The study of the variation of the extension data when we have an equisingular deformation of X_0 whose associated limiting mixed Hodge structures have constant associated graded pure Hodge structures;

(iic) a smoothing deformation of X_0.

Although there are well-developed cohomological techniques for using variational methods centered around a smooth variety, this is not the case when the variety is singular. In order to be able to carry out calculations in the singular case it is necessary to be able to compute the limiting mixed Hodge structure. For this the traditional method is to use semi-stable reduction (SSR).

Given a family $\mathcal{X}' \xrightarrow{\pi'} \Delta'$ that is smooth over ${\Delta'}^*$, SSR is the process of using blowing up, base change and normalization to arrive at a picture

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\pi} & \Delta \quad \ni t \\ \downarrow & & \downarrow \\ \mathcal{X}' & \xrightarrow{\pi'} & \Delta' \quad \ni t' = t^{\mu} \end{array}$$

which is a cyclic unbranched curving over $\Delta^* \to {\Delta'}^*$; $\mathcal{X}$ is smooth and $X_0 = \cup X_i$ is a reduced normal crossing divisor.

Computation of limiting mixed Hodge structures:

We will give an algorithm that is useful in practice for computation of limiting mixed Hodge structures. We use the notations

$$\begin{aligned}
X^{[1]} &= \coprod_i X_i = \widetilde{X} && \text{(normalization of } X_0\text{)} \\
X^{[2]} &= \coprod_{i<j} X_i \cap X_j = \widetilde{X}_{0,\mathrm{sing}} && \text{(normalization of } X_{0,\mathrm{sing}}\text{)} \\
X^{[3]} &= \coprod_{i<j<k} X_i \cap X_j \cap X_k \\
&\ \ \vdots
\end{aligned}$$

and

$$D = \sum_i X_i = \text{ divisor on } X_0.$$

Given a normal crossing variety X_0 as above, there is the (BF) condition that X_0 be smoothable to first order. This means that there is a smooth scheme $\mathcal{X}_\epsilon \to \Delta(\epsilon)$, $\epsilon^2 = 0$, and with $\mathcal{X}_{\epsilon,\mathrm{red}} = X_0$. Concretely it means that local smoothings

$$x_{i_1} \cdots x_{i_k} = \epsilon$$

of X_0 can be patched together modulo ϵ^2 to give a global smoothing. The (BF) condition is:

- If a smoothing $\mathcal{X} \to \Delta$ exists, then dt gives a global non-vanishing section of the co-normal bundle $\mathcal{I}_{X_0}/\mathcal{I}^2_{X_0}$; thus

$$\text{(III.1)} \qquad \mathcal{O}_D(X_0) = \left(\mathcal{I}_{X_0}/\mathcal{I}^2_{X_0}\right)^* \otimes \mathcal{O}_D \cong \mathcal{O}_D.$$

- For any normal crossing variety X_0 the *infinitesimal normal bundle* $\mathcal{O}_D(X_0)$ may be defined, and (III.1) is the necessary and sufficient condition that there exist a first order smoothing of X_0.

- To define $\mathcal{O}_D(X_0)$, from the exact sequence

$$0 \to \mathcal{I}_{X_0}/\mathcal{I}^2_{X_0} \to \Omega^1_{\mathcal{X}} \otimes \mathcal{O}_{X_0} \to \Omega^1_{X_0} \to 0$$

where $\Omega^1_{\mathcal{X}}$ and $\Omega^1_{X_0}$ are the Kähler differentials, one may show that as $\mathcal{O}_{X_0}$-modules

$$\mathrm{Ext}^1_{\mathcal{O}_{X_0}}\left(\Omega^1_{X_0}, \mathcal{O}_{X_0}\right) \cong \left(\mathcal{I}_{X_0}/\mathcal{I}^2_{X_0}\right)^* \otimes \mathcal{O}_D.$$

We then define the *infinitesimal normal bundle* by

$$\mathcal{O}_D(X_0) = \mathrm{Ext}^1_{\mathcal{O}_{X_0}}\left(\Omega^1_{X_0}, \mathcal{O}_{X_0}\right),$$

and (III.1) is the condition for X_0 to have a first order smoothing.

One part of the main computational techniques that will be used is that if the condition (III.1) is satisfied, then in terms of the groups $H^a(X^{[b]})(-c)$ one may compute what would be the associated graded and extension data to the limiting mixed Hodge structure if a smoothing exists. More precisely,

- if (III.1) is satisfied, then there exists a limiting mixed Hodge structure whose associated graded may be computed from the $H^a(X^{[b]})(-c)$'s;
- this limiting mixed Hodge structure is well defined upon the choice of a trivialization $\mathcal{O}_D(X_0) \cong \mathcal{O}_D$; and it is the limiting mixed Hodge structure associated to *any* smoothing $\mathcal{X} \to \Delta$ if such exists;
- the fibres of the mappings $E_k \to E_{k-1}$ constructed from the extension data may also be computed from the $H^a(X^{[b]})(-c)$'s.

If we have a semi-stable reduction $\mathfrak{X} \to \Delta$, then there is an equivalence class of limiting mixed Hodge structures at the origin. The above asks for a description of this limiting mixed Hodge structure in terms of cohomology groups computed from the components X_i of the normal crossing divisor $X_0 = \cup X_i$. In brief outline this goes as follows.

(i) For the associated graded to the limiting mixed Hodge structure we shall describe a pre-complex constructed from the groups $H^q(X^{[p]})(-r)$ and where the differential is given by a sum of mappings $R + G$ where R is a signed restriction mapping and G is a Gysin mapping. The condition

$$(R + G)^2 = 0$$

that the pre-complex be a complex is a consequence of the condition (III.1) that X_0 be to first order smoothable. In (III.2) below for the case $n = 2$ we shall give a schematic algorithm that describes how in practice one may carry out this computation.

(ii) The level 1 extension data can be described in terms of the relative cohomology groups $H^q(X^{[p]}, X^{[p+1]})(-r)$. We will not give the general formula but will describe how it may be derived and then we shall use that description in the special cases that arise later.

(iii) The level 2 extension data may also be described in terms of relative cohomology groups arising from the stratification given by the $X^{[p]}$'s. Again we shall not give the general formula but shall use special cases as the need arises in the examples.

(iv) Finally, as has been seen in Lecture II, it will not be necessary to consider extension data of levels $k \geqq 3$.

Assuming the condition (III.1) we will give a diagram that describes pictorially how in general one may compute the associated graded to the limiting mixed Hodge structure $H^2_{\lim}$ for a family $\mathcal{X} \to \Delta$ of surfaces in which we have semi-stable-reduction. Following that we will discuss a couple of special cases that illustrate the general situation.

(III.2)

$$\mathrm{Gr}_0^W \qquad \underbrace{H^0(X^{[1]})} \to \underbrace{H^0(X^{[2]})} \to \underbrace{H^0(X^{[3]})}$$

$$\mathrm{Gr}_1^W \qquad \underbrace{H^1(X^{[1]})} \to \underbrace{H^1(X^{[2]})}$$

$$\mathrm{Gr}_2^W \qquad \underbrace{H^0(X^{[2]})(-1)} \to \underbrace{H^2(X^{[1]}) \oplus H^0(X^{[3]})(-1)} \to \underbrace{H^2(X^{[2]})}$$

$$\mathrm{Gr}_3^W \qquad \underbrace{H^1(X^{[2]})(-1)} \to \underbrace{H^3(X^{[1]})}$$

$$\mathrm{Gr}_4^W \qquad \underbrace{H^0(X^{[3]})(-2)} \to \underbrace{H^2(X^{[2]})(-1)} \to \underbrace{H^4(X^{[1]})}$$

$$\begin{array}{ll} \underbrace{\qquad\qquad} = H^0_{\lim} & \underbrace{\qquad\qquad} = H^3_{lim} \\ \underbrace{\qquad\qquad} = H^1_{\lim} & \underbrace{\qquad\qquad} = H^4_{\lim} \\ \underbrace{\qquad\qquad} = H^2_{\lim} & \end{array}$$

The N maps are given by all arrows $H^a(X^{[b]})(-c-1) \to H^a(X^{[b]})(-c)$ that can be drawn and that are between non-zero groups. The rules are

- the horizontal rows form complexes where the maps are either "$R =$ signed restriction" or "$G =$ Gysin," whichever makes sense at a particular spot.

Example 1: The top row is all R's and the bottom row is all G's.

Example 2:

$$
\text{(III.3)} \qquad
\begin{tikzcd}[column sep=small]
 & H^2(X^{[1]}) \arrow[dr, "R"] & \\
H^0(X^{[2]})(-1) \arrow[ur, "G"] \arrow[dr, "R"'] & \oplus & H^2(X^{[2]}) \\
 & H^0(X^{[3]})(-1) \arrow[ur, "G"'] &
\end{tikzcd}
$$

where

$$H^0(X^{[2]})(-1) \ni \alpha \to \begin{pmatrix} G\alpha \\ R\alpha \end{pmatrix}$$

and

$$\begin{matrix} H^2(X^{[1]}) \\ \oplus \\ H^0(X^{[3]})(-1) \end{matrix} \ni \begin{pmatrix} \beta' \\ \beta'' \end{pmatrix} \to R\beta' + G\beta''.$$

The cohomology of these complexes at the appropriate color gives the associated gradeds of the corresponding $H^q_{\lim}$'s.

Example 1:

$$\mathrm{Gr}_0^W H^0_{\lim} = \ker\{H^0(X^{[1]}) \to H^0(X^{[2]})\},$$
$$\mathrm{Gr}_0^W H^1_{\lim} = \ker\{H^0(X^{[2]}) \to H^0(X^{[3]})\}/\mathrm{im}\{H^0(X^{[1]}) \to H^0(X^{[2]})\},$$
$$\mathrm{Gr}_0^W H^2_{\lim} = H^0(X^{[3]})/\mathrm{im}\{H^0(X^{[2]}) \to H^0(X^{[3]})\}.$$

Example 2: We will only illustrate the main case

$$\mathrm{Gr}_2^W H^2_{\lim} = \{\text{cohomology at the middle spot of (III.3)}\}.$$

Special case: $X = X_1 \cup_D X_2$ where X_1, X_2 are smooth surfaces and $D = X_1 \cap X_2$ is a smooth double curve. Then

$$\mathrm{Gr}_1^W H^2_{\lim} \cong H^1(D)/\operatorname{Im}\{H^1(X_1) \oplus H^1(X_2)\},$$
$$\mathrm{Gr}_2^W H^2_{\lim} \cong \text{cohomology of (III.3)},$$
$$\mathrm{Gr}_3^W H^2_{\lim} \cong \ker\{H^1(D)(-1) \to H^3(X^{[1]})\}.$$

- The final rule is that map N is given by the process described above where all non-zero maps $H^a(X^{[b]})(-c-1) \to H^a(X^{[b]})(-c)$ are used.

As a final note we remark that when $X = X_1 \cup X_2 \cup X_3$ the condition that (III.3) be a complex reduces to the triple point formula.

Extensions of period mappings

Given a pair $(\overline{B}, Z)$ and a variation of Hodge structure $(\mathcal{V}, \mathcal{F}, \nabla; B)$[††]

$$\text{(III.4)} \qquad \Phi : B \to P \subset \Gamma \backslash D$$

with unipotent monodromy generators around the branches Z_i of Z there is the canonical Deligne extension $(\mathcal{V}_e, \mathcal{F}_e, \nabla_e; \overline{B})$ of the variation of Hodge structure to $\overline{B}$. We will be primarily interested in the case where $\overline{B}$ is projective, so we assume this to be the case.

[††]Notations are from [GGR].

There are two canonical extensions

$$
\text{(III.5)} \qquad \begin{cases} \Phi^T : \overline{B} \to \overline{P}^T & \text{(maximal extension)} \\ \overline{\Phi}^S : \overline{B} \to \overline{P}^S & \text{(minimal extension)} \end{cases}
$$

of (III.4). We will define these at the set-theoretic level. It is conjectured, and proved in some special cases, that each of $\overline{P}^T$ and $\overline{P}^S$ are projective algebraic varieties. This result is not needed here; basically we will use $\overline{P}^T$ and $\overline{P}^S$ as guides as what to expect for the boundary of moduli spaces and for partial desingularizations of that boundary.

Definitions III.6: (i) $\overline{P}^T$ is the set of equivalence classes of limiting mixed Hodge structures arising from the canonical extension $(\mathcal{V}_e, \mathcal{F}_e, \nabla_e; \overline{B})$ of the variation of Hodge structure over B. (ii) $\overline{P}^S$ is the set of associated graded polarized Hodge structures arising from the limiting mixed Hodge structures arising from $(\mathcal{V}_e, \mathcal{F}_e, \nabla_e; \overline{B})$.

Remarks:

- Both $\overline{P}^T$ and $\overline{P}^S$ may be given the structure of compact Hausdorff topological spaces in which P is a dense open set. In [GGR] it is proved that the normalization $\hat{\overline{P}}^T$ may be defined and is a compact complex analytic variety (conjecturally $\hat{\overline{P}}^T = \overline{P}^T$).
- By passing to the associated gradeds of the limiting mixed Hodge structures there are period mappings

$$\Phi_I : Z_I^* \to P_I \subset \Gamma_I \backslash D_I$$

such that $\overline{P}^S = P \cup \left(\bigcup_I P_I\right)$ is a compact Hausdorff space stratified by complex analytic varieties. In the classical case and when Γ is an arithmetic group, by using Borel's extension theorem $\overline{P}^S$ may be identified with the Satake-Baily-Borel completion of P.

We shall refer to $\overline{P}^S$ as the *Satake-Baily-Borel completion* of the image $P \subset \Gamma\backslash D$ of the period mapping, and to $\overline{P}^T$ as the *toroidal-like completion* of P.

- There is a diagram of set-theoretic maps

$$
\begin{tikzcd}
 & P_n = \overline{P}^T \arrow[d] \\
 & P_{n-1} \arrow[d] \\
\overline{B} \arrow[uur, "\Phi^T=\Phi_{2n}"] \arrow[ur, "\Phi_{2n=1}"'] \arrow[dr, "\Phi_1"] \arrow[ddr, "\Phi_0=\Phi^S"'] & \vdots \arrow[d] \\
 & P_1 \arrow[d] \\
 & P_0 = \overline{P}^S
\end{tikzcd} \tag{III.7}
$$

where

$$\Phi_m(b) = \left\{ \begin{matrix} \text{set of extension data of level } \leqq m \text{ in the} \\ \text{limiting mixed Hodge structure at } b \in Z \end{matrix} \right\}.$$

Then from Lecture II we have

(III.8) The restriction of Φ_1 to a fibre of Φ_0 maps the closure of that fibre to an abelian variety;

(III.9) The restriction of Φ_2 to a fibre of Φ_1 maps to a product of $\mathbb{C}^*$'s;

(III.10) In the classical case $n = 1$. In general the mappings Φ_m, $m \geqq 3$, are determined up to "integration constants" by Φ_0, Φ_1, Φ_2.

Below we will give a schematic for computing the differential of Φ^T. More precisely, the differential maps to a filtered object and the induced maps on the associated graded will be described and illustrated in a geometric example.

Differential of the period mapping and deformation theory

We recall the basic setup from (III.7) above:

(III.11)
$$\begin{tikzcd} & \overline{\mathcal{P}}^T \arrow[dd] \\ \overline{B} \arrow[ur, "\Phi^T"] \arrow[dr, "\Phi^S"'] & \\ & \overline{\mathcal{P}}^S \end{tikzcd}$$

where $(\overline{B}, Z)$ is a pair where Φ^T and Φ^S are the canonical extensions of a period mapping

$$\Phi : B \to P \subset \Gamma\backslash D \tag{III.12}$$

with $B = \overline{B}\backslash Z$. Typically B will be a Zariski open in a desingularized KSBA moduli space $\mathcal{M}$ and $\overline{B}$ will be a desingularization of the canonical completion $\overline{\mathcal{M}}$ of $\mathcal{M}$. A significant method in the general study of period mappings (III.12) has been the use of the differential of Φ. In the geometric case when there is a family $\mathcal{X} \to B$ the differential of the period mapping has a cohomological expression whose geometric interpretation has been instrumental in many of the applications of Hodge theory to algebraic geometry.

It is natural to first seek to extend the definition of the differential to the completed map in (III.11), and then in the geometric case to express the extended differential map cohomologically and use its geometric interpretations to derive properties about the boundary structure of $\overline{B}$ and subsequently about that of $\overline{\mathcal{M}}$. Here we shall summarize the main points used in the study of the boundary structure of moduli spaces of surfaces of general type.

Specifically, over B there are two equivalent approaches to the study of a family of Hodge structures. One is via variations of Hodge structure (and the associated Higgs bundle construction), and the other is via period mappings.

The first order information (differential) in each is equivalent. Assuming unipotent monodromies around the branches of Z when we extend the variation of Hodge structure to $\overline{B}$ there is the canonical extension $(\mathcal{V}_e, \mathcal{F}_e, \nabla_e, \overline{B})$ and corresponding Higgs bundle construction, and there are the extended period mappings (III.11). In the first order variation of each there is an additional ingredient, namely the induced weight filtrations on each.
For the purposes of these lectures we shall define the differential at $b \in \overline{B}$ to be the mapping

$$T_{\overline{B}}(-\log Z)_b \xrightarrow{\delta} F^{-1}\,\mathrm{End}(\mathcal{E}_{e,b}) \tag{III.13}$$

where $\mathcal{E}_e = \mathrm{Gr}_{\mathcal{F}_e}(\mathcal{V}_e)$. This formulation will be particularly useful for computation of geometric examples.

We shall summarize the resulting structure in the cases of weights 1 and 2. At a point of the boundary $Z = \overline{B} \backslash B$ the differential of the mapping Φ^T has a weight filtration. For weight n the weights w of Φ^T_* satisfy $-(n+1) \leqq w \leqq +(n-1)$. The associated graded to the weight w part of the differential is given by maps

$$I^{p,q} \to I^{p-1,q+w+1}$$

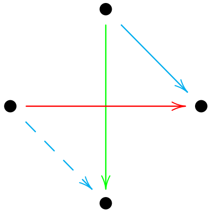
in the Deligne decomposition of the limiting mixed Hodge structure at the point of Z. For the cases $n = 1, 2$ we shall give a schematic depicting this structure. In the geometric case the $I^{p,q}$ are interpreted cohomologically and this leads to a Kodaira-Spencer type interpretation of the graded pieces of the differential of the extended period mapping. An application of this will be a cohomological expression for the first order variation of the level k extension data when that data of levels less than k are held constant.

We note that for an ordinary period map to polarized Hodge structures of weight n, the differential is determined by the maps $V^{(n-p,p)} \to V^{(n-p-1,p+1)}$ for $0 \leqq p \leqq \left(\frac{n}{2}\right)$. Here for the cases of curves and surfaces the differential will be determined by the maps

$$I^{n,q} \to I^{n-1,q+w+1}, \qquad 0 \leqq q \leqq n$$

and so we shall only depict those:

(III.14) $\qquad n = 1$



The interpretations are

$\longrightarrow$ is weight 0 and reflects the first order variation of the associated graded to the limiting mixed Hodge structure; thus the map Φ_*^S is given by $\longrightarrow$ in the above schematic.

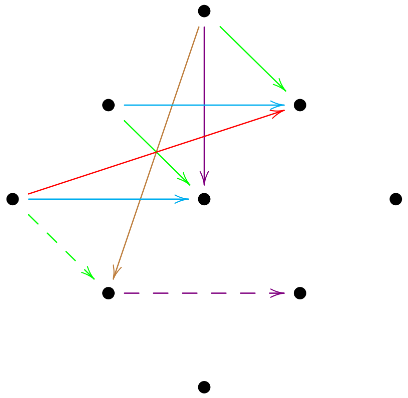
$\longrightarrow$ is weight -1 and reflects the first variation in the level 1 extension data whose associated graded to the limiting mixed Hodge structures are held constant;

$\dashrightarrow$ contains the same information, albeit in dual form, as $\searrow$;

$\longrightarrow$ is weight -2 and reflects the first variation in the level 2 extension data when the associated graded to the limiting mixed Hodge structures and the level 1 extension data are held constant.

We note that there are no elements of positive weight in (III.14) and that the lowest weight is -2. These properties are characteristic of the classical case. The differential of the map $\overline{P}^T \to \overline{P}^S$ is represented pictorially by mapping (III.14) to the red arrow $\longrightarrow$.

We next turn to the $n = 2$ case

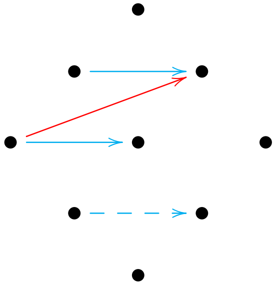
(III.15) 

The interpretations are

$\longrightarrow$ has weight $+1$. It is not present in the classical case, and has the Lie-theoretic interpretation that to first order the point $\Phi^T(b) \in \check{D}$ moves out of the Schubert cycle in which it lies.

The arrows $\longrightarrow$, $\longrightarrow$ and $\longrightarrow$ have the same interpretation as in the $n = 1$ case. The $\longrightarrow$ has weight -3 and reflects the first order variation of the level 3 extension data when to first order $\Phi^T(b)$ remains in its Schubert cycle, and the associated graded to the limiting mixed Hodge structure together with the first two levels of extension data remain fixed. We have seen that as a consequence of the IPR it is determined by the arrows of lower weight.

The differential of the map $\overline{P}^T \to \overline{P}^S$ at $\Phi_T(b)$ is given pictorially by

(III.15) $\longrightarrow$ 

In the geometric case the $I^{p,q}$'s are given by the algorithm represented pictorially in (III.2). The maps giving these arrows will then be expressed by multiplication by Kodaira-Spencer type cohomology classes, and we now illustrate one of these.

Example: We will cohomologically interpret the arrows $\longrightarrow$, $\longrightarrow$ and $\longrightarrow$ in (III.15) in the case when

$$X = X_1 \cup_C X_2$$

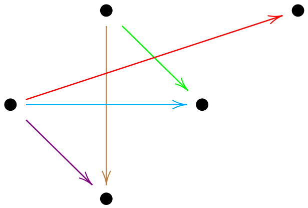
consists of two smooth surfaces X_1, X_2 joined along a double curve C. We use $C_i \subset X_i$ for the curve C in the surface X_i $(i = 1, 2)$. The condition (III.1) is

$$N_{C_1/X_1} \cong \check{N}_{C_2/X_2},$$

or equivalently

(III.16) $$\mathcal{O}_C(C_1) \cong \mathcal{O}_C(-C_2).$$

The relevant parts of (III.15) are



The dots are given by the prescription below (III.2). Using this a part of the red arrow will be a mapping

(III.17) $$H^0(\Omega^2_{X_1}) \to H^1(\mathcal{O}_C)(-1)/H^1(\Omega^1_{X_2}).$$

Note the exchange between X_1 and X_2; this will be a reflection of (III.1).

To describe (III.17) we will use the cohomology mappings arising from the commutative diagram

$$
\begin{array}{ccccccccc}
 & & \Omega^2_{X_1} & \rightarrow & \Omega^2_{X_1}\big|_{C_1} & \cong & \Omega^1_{C_1}(-C_1) & & \\
 & & 0 & & & & & & \\
 & & \downarrow & & & & \wr\| & & \\
 & & \Omega^1_{X_2} & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \rightarrow & \Omega^1_{X_2}(\log C_2) & \rightarrow & \Omega^1_{X_2}(C_2) & \rightarrow & \Omega^1_{C_2}(C_2) & \rightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & \mathcal{O}_{C_2}(-1) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0. & & & & & &
\end{array}
$$

The horizontal isomorphism on the top is adjunction, and the vertical isomorphism on the right uses (III.1). The composition of the maps on cohomology give the map

$$\begin{array}{ccc} H^0(\Omega^2_{X_1}) & \longrightarrow & \ker\left\{H^1(\mathcal{O}_{C_2})(-1) \to H^2(\Omega^1_{X_2})\right\} \\ \cap & & \cap \\ I^{2,0} & & I^{1,2} \end{array}$$

An explicit example where this map is non-zero is given by an I-surface having a simple elliptic singularity.

We next turn to the mapping $\longrightarrow$. In general this mapping is defined only if the mapping $\longrightarrow$ is zero, which will be the case if $N = 0$. Thus one may think of X as giving an equi-singular deformation $X_t = X_{1,t} \bigcup_{C_t} X_{2,t}$. Then

$$\longrightarrow : H^0(\Omega^2_{X_1}) \to H^1(\Omega^1_{X_1})$$

is the usual derivative of a period mapping. We note that the image of this mapping lies in

$$(C_1)^{\perp} \subset H^1(\Omega^1_{X_1})$$

reflecting the assumption that C_1 deforms along with X_1. For the mapping

$$\searrow : I^{2,1} \to I^{1,2} \tag{III.18}$$

we first note that it is defined when both $\longrightarrow$ and $\longrightarrow$ are zero. Geometrically we imagine a family

$$X_t = X_1 \bigcup_{t,C} X_2$$

where $X_1, X_2,\ C$ are constant but the gluing of X_1 and X_2 along C varies with t.

Now

$$\mathrm{Ext}^1_{\mathrm{MHS}}(H^1(-1), H^2) \cong \frac{H^1 \,\widecheck{(-1)} \otimes H^2}{F^0(H^1 \,\widecheck{(-1)} \otimes H^2) + (H^1 \,\widecheck{(-1)} \otimes H^2)_{\mathbb{Z}}} .$$

This is a compact complex torus having a summand that is an abelian variety J with tangent space

$$TJ \cong \mathrm{Hom}(I^{0,1}, I^{2,2}_{\mathbb{Z}} \otimes \mathbb{C}).$$

Using the duality $\check{I}^{0,1} \cong I^{2,1}$ we shall give the geometric interpretation of (III.18) under the simplifying assumption that the $H^1(X_i) = 0$ for $i = 1, 2$. Then $J = J(C)$ is the Jacobian variety of C. We set

$$(\mathrm{Pic} X_1 \oplus \mathrm{Pic} X_2)^0 = (C_1 \oplus C_2)^{\perp}.$$

For the family of embeddings $j_t : C \hookrightarrow X_1 \times X_2$ there is a mapping

$$\alpha_t : (\mathrm{Pic} X_1 \oplus \mathrm{Pic} X_2)^0 \to J(C)$$

and unwinding the definitions the mapping $\longrightarrow$ may be identified with the derivative of α_t. In words

> *Fixing X_1, X_2, C and mapping the gluing of X_1, X_2 along a family of different embeddings of C in these surfaces, a part of the variation in the first order extension data is measured by the variation when the $\mathrm{Pic}(X_i)$ map to $J(C)$.*

From a different perspective this result was discussed in Lecture II.

To interpret the arrow $\searrow$ we note that because we have a limiting mixed Hodge structure there is a duality between $\mathrm{Ext}^1_{\mathrm{MHS}}(H^1(-1), H^2)$ and $\mathrm{Ext}^1_{\mathrm{MHS}}(H^2, H^1)$. Thus this arrow contains no new information beyond $\searrow$.

That leaves the interpretation of $\downarrow$. Here we recall from Lecture II that it is shown that for a variation of Hodge structure on $\overline{B}\backslash B$ the level 1 extension data gives a cone σ of line bundles over a compact complex torus T and that the fibres of Φ^S map to a sub-torus J of T over which the line bundles $L \in \sigma$ are ample. The level 2 extension data then maps the fibres of Φ_1 to nowhere vanishing sections of these line bundles.

Derivative of the period mapping when the monodromy is finite

The above has been concerned with the differential of the period mapping when the monodromy is unipotent so that there is a limiting mixed Hodge structure. In the uses of Hodge theory to study moduli there are many interesting cases where a family of smooth varieties acquires singularities with finite, even perhaps trivial, monodromy. What can one say about the period mapping and its differential in these cases? Here we shall give a brief discussion and geometric illustration of this question.

The Hodge-theoretic situation is typically given by a period mapping

$$\Phi : \Delta^* \to \Gamma\backslash D \tag{III.19}$$

where Γ is generated by a semi-simple $T = T_s$ where $T^m = \mathrm{Id}$; i.e., the eigenvalues of T are m^{th} roots of unity. In the geometric case we might have a family of smooth surfaces $\mathcal{X}^* \to \Delta^*$ that fills in to $\mathcal{X} \to \Delta$ where the central fibre X_0 has a canonical or a quotient singularity. In the first case X_0 may be considered as an interior point in a moduli space whereas in the second case it may correspond to a boundary point.

Given (III.19) there is a diagram

(III.20)
$$\begin{tikzcd} \widetilde{\Delta}^* \arrow[r, "\widetilde{\Phi}"] \arrow[d] & D \arrow[d] \\ \Delta^* \arrow[r, "\Phi"] & \Gamma\backslash D \end{tikzcd}$$

where $\widetilde{\Delta}^* \to \Delta^*$ is the finite cyclic covering given by $t = \tilde{t}^m$. As will be illustrated later, in the geometric case even if $m = 1$ it may be necessary to do a base change to have a semi-stable reduction that is necessary in order to carry out the calculations (e.g., a Wahl singularity).

Given (III.20) it is a classical result from Hodge theory that *the mapping* $\widetilde{\Phi} : \widetilde{\Delta}^* \to D$ *extends to*

$$\widetilde{\Phi} : \widetilde{\Delta} \to D.$$

In other terms, $\Gamma\backslash D$ is what is sometimes referred to as a V-manifold, and (III.19) extends to a holomorphic mapping from the complete disc Δ to this V-manifold.

(III.21) *Provisional definition:* The differential of (III.19) at the origin is given by $\widetilde{\Phi}_* : T_{\{0\}}\widetilde{\Delta} \to T_{\widetilde{\Phi}(0)}D$.

We say provisional because the correct definition has not yet been fully worked out. Perhaps it should use the framework provided by the theory of stacks.

Example: Suppose that $\mathcal{X} \to \Delta$ is a family of surfaces where X_t is smooth for $t \neq 0$ while X_0 has an ordinary double point p (an A_1-singularity). Then the monodromy is a Picard-Lefschetz transformation and $m = 2$. In this case the standard semi-stable reduction gives a family $\widetilde{\mathcal{X}} \xrightarrow{\tilde{\pi}} \widetilde{\Delta}$ where $\widetilde{\mathcal{X}}$ is smooth and the central fibre

$$\tilde{\pi}^{-1}(0) = X_1 \bigcup_C X_2$$

where X_1 is the standard desingularization of $X_0, X_2 = \mathbb{P}^2$ and $C \cong \mathbb{P}^1$. The curve $C_1 \subset X_1$ is a curve with $C_1^2 = -2$ and $C_2 \subset \mathbb{P}^2$ is a conic with $C_2^2 = 2$.

Using methods similar to those in (III.17) above, the derivative of the period mapping is computed from the cohomology of the diagram

$$
\text{(III.22)}\qquad
\begin{tikzcd}
0 \arrow[r] & K_{X_1}(-C) \arrow[r] & K_{X_1} \arrow[r] & K_{X_1}\big|_{C_1} \arrow[r] \arrow[d, phantom, "\wr\|"] & 0 \\
 & & & \mathcal{O}_C \arrow[d, phantom, "\wr\|"] & \\
0 \arrow[r] & \Omega^1_{X_2} \arrow[r] & \Omega^1_{X_2}(\log C_2) \arrow[r] & \mathcal{O}_{C_2}(-1) \arrow[r] & 0.
\end{tikzcd}
$$

For the bottom row, if (x, y) are local coordinates in $\mathbb{P}^2$ such that $C_2 = \{y = 0\}$, then the first mapping is the inclusion and the second map is

$$a(x, y)\frac{dx}{y} + b(x, y)dy \to a(x, 0).$$

The cohomology of (III.22) gives

$$H^0(K_{X_1})(-C_1) \to H^0(K_{X_2}) \to H^0(\mathcal{O}_{C_2}(-1)) \xrightarrow{\delta} H^1(\Omega^1_{X_2})$$

where $\delta(1) = [C_2]$ is the fundamental class of C_2. The composition

$$H^0(K_{X_1}) \to [C_2]$$

is the image of d/dt under Φ_*.

Application: We let X be a smooth surface of general type with $p_g(X) \neq 0$. Let $\mathcal{M}$ be the corresponding KSBA moduli space and suppose that a point $x \in \mathcal{M}$ corresponds to a surface X_0 with an A_1-singular point p. Then

> *If p is not a base point of the canonical series $|K_{X_0}|$, in a neighborhood of x in $\mathcal{M}$ the locus of surfaces having an A_1-singularity is a reduced divisor.*

To complete the argument for this statement one has to show

(i) in the moduli space $\mathcal{M}$ the condition that the Chern class of the line bundle $L_1 \to X_1$ remain a Hodge class is a reduced divisor in $\mathcal{M}$; and

(ii) under a 1-parameter deformation $L_{1,t} \to X_{1,t}$ of $L_1 \to X_1$, the section $s \in H^0(\mathcal{O}_{X_1}(L_1))$ that defines C_1 deforms to a section $s_t \in H^0(\mathcal{O}_{X_{1,t}}(L_{1,t}))$.

It is the condition (i) that results from the above cohomological calculation. The condition (ii) may be proved by a standard argument.

The above is what one expects. The interest is not so much the result itself but rather the method of proof that is much more generally applicable.

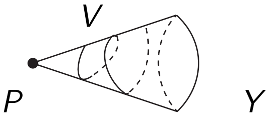
One intuition behind that above definition of the derivative of the period mapping is this: In the above example if $\delta_t \in H_2(X_t, \mathbb{Z})$ is the vanishing cycle and $\omega_t \in H^0(\Omega^2_{X_t})$, then the period

$$\begin{aligned}\int_{\delta_t} \omega_t &= \omega_0(p) \cdot t^{1/2} + (\text{higher order terms}) \\ &= \omega_0(p)\tilde{t} + \mathcal{O}(\tilde{t}^2).\end{aligned}$$

Thus the derivative at $t = 0$ is just $\omega_0(p)$.

Example: Let X be an I-surface with a $\frac{1}{4}(1,1)$ singularity p. Recall that there are several ways to construct (X, p). To describe one of these we use the following notation:

- $\mathbb{P}(1,1,2)$ has coordinates (x_1, x_2, y) and is embedded in $\mathbb{P}^3$ as a quadric cone with vertex P



- $\mathbb{P}(1,1,2,5)$ has coordinates (x_1, x_2, y, z); any non-singular Gorenstein I-surface Y has the equation

$$z^2 = a_0 y^5 + a_2(x_1, x_2) y^4 + \cdots + a_{10}(x_1, x_2)$$

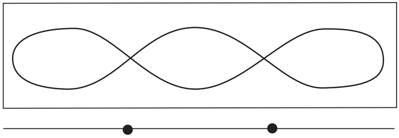
  where $a_{2k}(x_1, x_2)$ is homogeneous of degree $2k$ and $a_0 \neq 0$. This is because for Gorenstein I-surfaces X the pluricanonical ring $R(X) = \oplus H^0(mK_X)$ has the postulated form; i.e., generators and relations appear only in places where they are forced to do so.
- $Y \to \mathbb{P}(1,1,2) \subset \mathbb{P}^3$ is branched over a quintic $V \in |\mathcal{O}_{\mathbb{P}^3}(5)|$ where $P \notin V$.

- As $a_0 \to 0$ the quintic V passes through the vertex $P = (0,0,1,0)$ and the limit surface X ceases to be Gorenstein but acquires a $\frac{1}{4}(1,1)$ singularity over P.
- For $a_0 = 0$ and $(x_1, x_2) \neq (0,0)$ over the fibre of projection $\mathbb{P}(1,1,2) \to \mathbb{P}^1$ we have a double cover of a quartic intersection with the line over (x_1, x_2). The discriminant of the quartic has terms like $a_2^3 a_{10}^3, a_4^2 a_6^2 a_0^2, \ldots,$; they all have the same degree $3 \cdot 2 + 3 \cdot 10$, $2 \cdot 4 + 2 \cdot 6 + 2 \cdot 8, \cdots = 36$. Hence, when $a_0 = 0$ outside of what happens at P we have a 2:1 covering of $\mathbb{P}(1,1,2) \backslash P$ with four branch points over the intersection of V with the rulings of the quadric. In general there are 36 such rulings where two of the branch points come together; i.e., the corresponding curves acquire nodes.

- Over $x_1 = x_2 = 0$ we have $z = 0$ which gives the singular point $(0,0,1,0)$ on X. Here $\mathbb{P}(1,1,2,5)$ has a $\frac{1}{2}(1,1,1)$ singularity of index 2. A weighted blowup puts in a Veronese surface X_2 under the map

  $$(x_1, x_2, 0, z) \to \{\text{all quadratic monomials in } x_1, x_2, z\};$$

  the blown up surface X_1 intersects the Veronese X_2 in the conic hyperplane section C given by $z^2 - a_2(x_1, x_2) = 0$. Here we assume a_2 is non-degenerate. The conic C maps 2:1 to the $\mathbb{P}^1$ coming from x_1, x_2; i.e., we have $z = \pm\sqrt{a_2(x_1, x_2)}$.
- Thus $X \xrightarrow{f} \mathbb{P}^1$ is an elliptic surface having a bi-section



with branch points at the zeroes of $a_2(x_1, x_2)$.

- For the numerology we have for $C_{X_i} = C \subset X_i$
  - $\begin{cases} C_1^2 = -4 \\ C_2^2 = 4 \end{cases}$
  - $K_{X_1} \cdot C_{X_2} = 2$ (thus $K_{X_1} \cdot C_{X_1} + C_{X_1}^2 = -2$)
  - $f_* \omega_{X_1/\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}E}(2)$.

We know that $p_g(X_1) = 2$ since the monodromy $T = \mathrm{Id}$.
We now denote by C the $\mathbb{P}^1$ in either X_1 or X_2 and consider the surface $X_0 = X_1 \cup_C X_2$. Then H^2_{lim} is the cohomology of

$$H^0(C)(-1) \xrightarrow{G} H^2(X_1) \oplus H^2(X_2) \xrightarrow{R} H^2(C)$$

where G is the direct sum of Gysin maps and R is the sum of signed restriction maps. Thus in the diagram (III.15) the image of the dotted arrow is spanned by

$$[C]^{\perp} \in H^1(\Omega^1_{X_1}) \cong H^2(\text{Veronese}) \cong \mathbb{C}^2.$$

This has dimension 1, from which we may conclude

> For the moduli of I-surfaces there is 1-condition to have a $\frac{1}{4}(1,1)$ singularity. This condition is detected Hodge theoretically by the presence of an additional Hodge class in $\Phi(0) = H^2_{\lim}$.

Note: There is an interesting dimension count going on here. Namely,

- elliptic surfaces Y with $q(Y) = 0$, $p_g(Y) = 2$ have $30 = 10 \cdot \chi(\mathcal{O}_Y)$ moduli. Specifically, in the case at hand (here $T_X =$ tangent bundle)

$$h^1(T_Y) = 30 \text{ and } h^0(T_Y) = 0;$$

- the "expected codimension"[‡‡] in moduli for Y to have a line bundle $L \to Y$ with

$$L^2 = -4, \quad L \cdot K_Y = 2$$

is $p_g(Y) = 2$;
- if $h^0(L) \neq 0$, then $h^0(L) = 1$ and there is a unique curve $C \in |L|$; from

$$C^2 = -4, \quad C \cdot K_Y = 2$$

we infer that the arithmetic genus $p_a(C) = 0$, and one expects that generically $C = \mathbb{P}^1$;

[‡‡]By "expected codimension" we mean that quantity given by a naive dimension count (p_g conditions in this case). In many interesting cases a correction to the expected dimension count must be added. This is an example of an improper intersection of the image of a period mapping with a Mumford-Tate subdomain. (Actually we should refer to "codimension counts," where the naive one gives an upper bound on the codimension of the subvariety in quotations.

- for (Y, C) as above we may contract C to give an I-surface with a $\frac{1}{4}(1,1)$ singularity.

With the details to be given elsewhere, one has

— $\dim\{\text{pairs } (Y, L) \text{ as above}\} = 30 - 2 = 28$;

— among such pairs (Y, L) it is one condition to have $h^0(L) \neq 0$;

— the space of pairs (Y, C) has dimension 27;

— but $\dim \mathcal{M}_I = 28$, so the above explains why imposing a $\frac{1}{4}(1,1)$ singularity is one condition in moduli of I-surfaces.*

*This is an example where there is one condition that the Hodge class remain effective under a deformation of the desingularization of an I-surface where the Hodge class also deforms.

Note: In general there is one condition on $\omega \in H^0(K_{X_1})$ that

$$(d/dt)(\omega) = 0 \text{ in } [C]^\perp \subset \mathrm{Hg}^1(X_2) \cong \mathbb{Z}^2.$$

For I-surfaces with a $\frac{1}{4}(1,1)$ singularity, $|K_X|$ is in 1-1 correspondence with the rulings of the quadric, and the above condition is equivalent to the corresponding ruling being tangent to the quintic at the vertex.

References

[GGR] Mark Green, Phillip Griffiths, and Colleen Robles, *Towards a maximal completion of a period map*. arXiv:2010.06720

Additional references that provide background for some of the material in these lectures:

Isolated Hypersurface Singularities. Lectures given at the University of Miami during April, 2019. https://albert.ias.edu/handle/20.500.12111/7867

Using Hodge theory to detect the structure of a compactified moduli space. Talk given on November 27, 2019 at the IMSA conference held at IMATE at UNAM, Mexico City. https://hdl.handle.net/20.500.12111/7873

Positivity and Vanishing Theorems. Lectures given at the University of Miami, Spring 2020.
https://hdl.handle.net/20.500.12111/7881

Period Mapping at Infinity. IMSA talk on 5/6/20.
https://hdl.handle.net/20.500.12111/7910

Completions of Period Mappings. IMSA talk on 10/5/20.
https://hdl.handle.net/20.500.12111/7922