Atypical Hodge Loci

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Outline

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*Talk based on the paper [BKU] and related works given in the references in that work, and on extensive discussions with Mark Green and Colleen Robles.

I. Introduction

Q: *What can one say about Hodge loci?*

- B is a smooth, connected quasi-projective variety;
- $\mathbb{V} \to B$ is the local system underlying a variation of polarized Hodge structure of weight n;
- geometric case; smooth projective family $\mathcal{X} \xrightarrow{\pi} B$ and

$$\mathbb{V}_b = H^n(X_b, \mathbb{Q})_{\mathrm{prim}};$$

- $\mathrm{HL}(B) =$ set of $b \in B$ with more Hodge classes in $\mathbb{V}_b^{\otimes} := \overset{k}{\oplus}(\overset{k}{\otimes}\,\mathbb{V}_b)$ than at a very general point of b;
- Cattani-Deligne-Kaplan: $\mathrm{HL}(B)$ is a countable union of proper algebraic subvarieties;
- in geometric case assuming the Hodge conjecture there are extra classes of algebraic cycles in $\underbrace{X \times \cdots \times X}_{k}$'s.

Q: *What can we say about* $\mathrm{HL}(B)$*?*

- very informally stated the main result in [BKU] is

(I.1) *For* $n \geqq 3$ *and aside from exceptional degenerate cases, every irreducible component of* $\mathrm{HL}(B)_{\mathrm{pos}}$ *has strictly larger than the expected codimension;*

- know of no conceptual reason why in the non-classical case there should be more than the expected amount of algebraic cycles;
- proof uses integrability conditions for the differential constraint imposed by transversality in the non-classical case;
- sufficient condition for result is

$$\mathfrak{g}^{-k,k} \neq 0, \qquad \text{some } \; k \geqq 3;$$

- notation and criterion for this given below.
- implied by coupling length $\geqq 3$

II. Two examples

- $X = X_b$, $T = T_b B$ and $T \to H^1(T_X) = T\,\mathrm{Def}(X)$;
- $V^{p,q} = H^q(\Omega_X^p)$ and $T \to \oplus \mathrm{Hom}(V^{p,q}, V^{p-1,q+1})$ is Kodaira-Spencer mapping giving first variation of Hodge decomposition of a class in $H^n(X)$;
- for X a surface, $\theta \in T$, $\lambda \in \mathrm{Hg}^1(X)$ and $\theta \cdot \lambda \in H^{0,2}(X)$ gives the first order deviation from λ remaining a Hodge class in the direction θ;
- $\mathrm{NL}_\lambda \subset B$ is the Noether-Lefschetz locus where λ remains a Hodge class; assume reduced and define $T_\lambda \subset T = \ker\{\theta \to \theta \cdot \lambda\}$;
- for $X \subset \mathbb{P}^3$ of degree $d \geqq 4$ in the estimate

$$d - 3 \leqq \mathrm{codim}_B \mathrm{NL}_\lambda \leqq \binom{d-1}{3} = h^{2,0}(X)$$

  both bounds are achieved (Green; lower bound $\iff X$ contains a line);

- now let $\dim X = 4$, $\lambda \in \mathrm{Hg}^2(X)_{\mathrm{prim}}$; in first approximation

$$\operatorname{codim} \mathrm{NL}_\lambda \leqq h^{1,3}(X) + h^{0,4}(X);$$

- but $\theta \cdot \lambda \in H^{1,3}(X)$ so this estimate must be refined to

$$(*) \qquad \operatorname{codim}_B \mathrm{NL}_\lambda \leqq h^{1,3}(X).$$

Definition: The right-hand side of $(*)$ is the *expected codimension* of NL_λ in B.

- *Integrability:* With $T_\lambda \subset T$ as above set

$$\sigma_\lambda = \mathrm{Image}\{ T_\lambda \otimes H^{4,0}(X) \to H^{3,1}(X)\}.$$

Observation:

$$\operatorname{codim}_B \mathrm{NL}_\lambda \leqq h^{1,3}(X) - \dim \sigma(\lambda).$$

Proof: For $\theta \in T_\lambda$, $\theta' \in T$, $\omega \in H^{4,0}(X)$

$$\begin{aligned} \langle \theta\omega, \theta'\lambda \rangle &= \langle \omega, \theta\theta'\lambda \rangle \\ &= \langle \omega, \theta'\theta\lambda \rangle \quad \text{(integrability)} \\ &= 0. \end{aligned}$$

Thus the number of conditions on $\theta' \in T$ to be in T_λ is $\leqq h^{1,3} - \dim \sigma(\lambda)$. □

- **Note:** For the first example of $X \subset \mathbb{P}^3$ the expected codimension drops for geometric reasons: if $L \subset X$ is a line with Hodge class λ and if $\omega \in H^0(\Omega_X^2) \cong H^0(\mathcal{O}_X(d-4))$, then if $L \subset (\omega)$,

$$\langle \theta\lambda, \omega \rangle = \langle \lambda, \theta \cdot \omega \rangle = \int_L \theta \rfloor \omega = 0$$

for all $\theta \in H^1(T_X)$; thus such ω's do not contribute to the equations defining NL_λ. In the second example the drop by σ_λ in the expected codimension is for Hodge theoretic reasons.

III. Statement of main result

- *Polarized Hodge structure* $(V, Q, F^\bullet)$ *of weight* n
  - non-degenerate $Q : V \otimes V \to \mathbb{Q}$,
    $Q(u, v) = (-1)^n Q(v, u)$;
  - $F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_\mathbb{C}$, $F^p \oplus \overline{F}^{n-p+1} \xrightarrow{\sim} V_\mathbb{C}$ for $0 \leqq p \leqq n$;
  - $V^{p,q} = F^p \cap \overline{F}^q$, $V_\mathbb{C} = \oplus V^{p,q}$ with $\overline{V^{p,q}} = V^{q,p}$;
  - Hodge-Riemann bilinear relations;
  - $n = 2m$, $\mathrm{Hg}^m(V) = V^{m,m} \cap V$.
  - Lie algebra $\mathfrak{g} \subset \mathrm{End}(V, Q)$ and $\mathfrak{g}_\mathbb{C} = \oplus \mathfrak{g}^{-k,k}$ where

$$\mathfrak{g}^{-k,k} := \mathfrak{g}^{-k} = \{A \in \mathfrak{g}_\mathbb{C} : A(V^{p,q}) \subset V^{p-k,q+k}\}$$

$$\ell(\mathfrak{g}) = \min\{k : \mathfrak{g}^{-k} \neq (0)\};$$

  - $\ell(\mathfrak{g}) \geqq 3 \implies n \geqq 3$.

- *Mumford-Tate group*
  - $V^{\otimes} := \overset{k}{\oplus}(\otimes^k V)$;
  - Hodge tensors $\mathrm{Hg}^{\bullet}(V) = \overset{k}{\oplus}\,\mathrm{Hg}^{kn/2}(V)$;
  - $\mathrm{MT}(V) \subset \mathrm{Aut}(V, Q)$ is $\mathsf{Fix}(\mathrm{Hg}^{\bullet}(V))$;
  - is a reductive $\mathbb{Q}$-algebraic group H;
  - finite cover of H is $\mathbb{C}^{*k} \times H_0$ where H_0 is semi-simple; for simplicity of exposition we will assume H is semi-simple; essential ideas appear in this case;

Example: Assume λ, Q generate the algebra of Hodge tensors, $\mathrm{MT}(V) = H_\lambda = \mathsf{Fix}\ \lambda \subset \mathrm{Aut}(V, Q)$.

- *Variation of Hodge structure* $(\mathbb{V}, \mathcal{F}^\bullet; B)$
  - B and $\mathbb{V} \to B$ as above;
  - $\mathcal{F}^\bullet$ is a filtration of $\mathcal{V} := \mathbb{V}_{\mathbb{C} \otimes \mathbb{C}} \mathcal{O}_B$ inducing a polarized Hodge structure on each $\mathbb{V}_b$ (understood there is $Q : \mathbb{V} \otimes \mathbb{V} \to \mathbb{Q}$);
  - $\nabla \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1_B$ (transversality);
  - for $b_0 \in B$ and $V = V_{b_0}$ we have the monodromy group $\Gamma \subset \mathrm{Aut}(V, Q)$;
  - the $\mathbb{Q}$-Zariski closure $\overline{\Gamma}^{\mathbb{Q}}$ = semi-simple $\mathbb{Q}$-algebraic group that is a factor of the MT-group of $(V, \mathbb{Q}, F^\bullet_b)$ at a very general point of B.

- *Period mappings*
  - G = semi-simple $\mathbb{Q}$-algebraic group and D = period domain of polarized Hodge structures of a given type and with generic Mumford-Tate group G;
  - $D = G(\mathbb{R})/G_0$, G_0 compact;
  - period mapping $\Phi : B \to \Gamma\backslash D$;
  - for simplicity of exposition we will assume that $\overline{\Gamma}^{\mathbb{Q}} := G$ =Mumford-Tate group of the polarized Hodge structure at a very general point of B;

  This is justified because the image $\Phi(B)$ of the period mapping is contained in a translate of a $\overline{\Gamma^{\mathbb{Q}}}(\mathbb{R})$-orbit.

  - $T_{F_0}D \cong \mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}} \cong \overset{k>0}{\oplus}\, \mathfrak{g}^{-k,k}$ where $\mathfrak{g}_{\mathbb{C}} \subset \mathrm{End}(V_{\mathbb{C}})$ and $\theta \in \mathfrak{g}^{-k,k}$ satisfies $\theta(F^p) \subset F^{p-k}$;
  - $\Phi_* : T_bB \to \mathfrak{g}^{-1,1}$ (transversality);
  - image is an abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}^{-1,1}$ (integrability);
  - notation: $\Phi(B) = P \subset \Gamma\backslash D$, $\widetilde{P} \subset D$ inverse image of P.

- *Hodge loci*
  - if λ, Q generate $\mathrm{Hg}^\bullet(V_b^\otimes)$, then

    $$\mathrm{NL}_\lambda^0 = \Phi^{-1}(\Phi(B) \cap G_\lambda(\mathbb{R}) \cdot o)^0 \qquad (^0 = \text{identity component});$$

  - thus Noether-Lefschetz loci are (translates of) orbits of particular Mumford-Tate subgroups of G;
  - this is the crucial conceptual point; **Hodge loci are intersections with translates of Mumford-Tate sub-domains.**

Definition: If $H \subset G$ is a Mumford-Tate group and

$$D_H = H(\mathbb{R}) \cdot o, \qquad \Gamma_H = \Gamma \cap H$$

$\Phi^{-1}(\Phi(B) \cap (\Gamma_H \backslash D_H))^0$ is a **special subvariety** of B.

- thus special subvarieties of B are those $b \in B$ where the algebra of Hodge tensors is strictly larger than at a general point of B.

- set $P_H = P \cap (\Gamma_H \backslash D_H)$, then the standard codimension of an intersection inequality is

$$(\natural) \quad \operatorname{codim}_{\Gamma\backslash D} P_H \leqq \operatorname{codim}_{\Gamma\backslash D} P + \operatorname{codim}_{\Gamma\backslash D}(\Gamma_H \backslash D_H);$$

Definition: Special subvariety is **atypical** if we have strict inequality in $(\natural)$.

- Thus atypical means we have more Hodge tensors than predicted by the usual expected dimension count formula.

Example: Notations as in the second example above

$$\operatorname{codim}_{\Gamma\backslash D}(\Phi(B) \cap (\Gamma_\lambda \backslash D_\lambda)) = \operatorname{codim}_{\Gamma\backslash D} \Phi(B) + \operatorname{codim}_{\Gamma\backslash D}(\Gamma_\lambda \backslash D_\lambda) - \dim \sigma(\lambda).$$

Theorem ([BKU]): *If $\ell(\mathfrak{g}) \geqq 3$, then* **every** *special subvariety of B is atypical.*

Example: For smooth $X \subset \mathbb{P}^{n+1}$, $n \geqq 3$ and $\deg(X) \geqq 6$ every special subvariety is atypical.

Reason for result: M is a manifold and I', I'' distributions in TM given by $\{\omega'_i\}$, $\{\omega''_\alpha\}$; N', N'' variable integral manifolds of $I', I''|_{N'}$'s; want to estimate codimension of the $N' \cap N''$'s; integrability conditions given by $d\omega'_i$ may impose linear relations on the $\omega''_\alpha|_{N'}$; leads to more than expected number of $N' \cap N''$'s.

IV. Proof of the main result

- Assume equality holds in (♮) and will arrive at a contraction to the assumption $\mathfrak{g}^{-k} \neq 0$;
- pass to tangent spaces and use

$$\dim T_0 D = \mathfrak{g}^-,$$
$$\dim T_0 D_H = \mathfrak{h}^-,$$
$$T_0 \widetilde{P} \subset \mathfrak{g}^{-1},$$
$$T_0(\widetilde{P} \cap D_H) = T_0 \widetilde{P} \cap \mathfrak{h}^{-1}$$

$$\sum_{k \geqq 2} \dim \mathfrak{h}^{-k} + \mathrm{codim}_{\mathfrak{h}^{-1}}(T_0 \widetilde{P} \cap \mathfrak{h}^{-1}) = \sum_{k \geqq 2} \dim \mathfrak{g}^{-k} + \mathrm{codim}_{\mathfrak{g}^{-1}} T_0 \widetilde{P}$$
$$\dim \mathfrak{h}^{-k} \leqq \dim \mathfrak{g}^{-k}$$
$$\mathrm{codim}_{\mathfrak{h}^{-1}}(T_0 \widetilde{P} \cap \mathfrak{h}^{-1}) \leqq \mathrm{codim}_{\mathfrak{g}^{-1}} T_0 \widetilde{P}$$
$$\Longrightarrow \begin{cases} \mathfrak{h}^{-k} = \mathfrak{g}^{-k}, \quad k \geqq 2 \\ \mathrm{codim}_{\mathfrak{h}^{-1}} T_0 \widetilde{P} \cap \mathfrak{h}^{-1} = \mathrm{codim}_{\mathfrak{g}^{-1}} T_0 \widetilde{P}. \end{cases}$$

We want to conclude

$$(\natural\natural)\qquad \begin{array}{ccccc} \mathfrak{h}^{-1} & = & \mathfrak{g}^{-1} & & \\ & \Downarrow & & & \\ T_0 D_H & = & T_0 D & \Longrightarrow & D_H = D. \end{array}$$

- *Basic idea*
  - $\mathfrak{h}, \mathfrak{g}$ are reductive Lie algebras in which $\mathfrak{h}^-, \mathfrak{g}^-$ are parabolic sub-algebras;
  - there is maximal torus $\mathfrak{t}_\mathbb{R} \subset \mathfrak{g}_\mathbb{R}$ relative to which $\mathfrak{h}^-, \mathfrak{g}^-$ are direct sums of negative root spaces;
  - the root space and Hodge decomposition of $\mathfrak{h}^-, \mathfrak{g}^-$ align in the sense that each of $\mathfrak{h}^{-k}, \mathfrak{g}^{-k}$ are direct sums of negative root spaces;
  - this is the first key point where the Lie theory and Hodge theory interact; the complex structure on $T_0 D \cong \mathfrak{g}_\mathbb{R}/\mathfrak{g}^0_\mathbb{R}$ is given by an $E \in \mathfrak{t}_\mathbb{R}$ whose centralizer is $\mathfrak{g}^0_\mathbb{R}$ and whose eigenspace decomposition on $\mathfrak{g}_\mathbb{R}/\mathfrak{g}^0_\mathbb{R}$ is a direct sum of non-trivial conjugate root subspaces (cf. [R] for details);

- let β_i, $i \in I$, be the simple positive roots whith corresponding root space $\mathfrak{g}_{\beta_i} \subset \mathfrak{g}^+$; denote by $J \subset I$ the subset, possibly empty, where $\mathfrak{g}_{\beta_i} \subset \mathfrak{h}^+$; since every positive root is a sum of the β_i, it will suffice to show that

$$(\flat) \qquad J = I;$$

  in this case we will have $(\natural\natural)$;
- note that $\mathfrak{g}^-$ is generated by $\mathfrak{g}^{-1} \iff$ all $\mathfrak{g}_{-\beta_i} \subset \mathfrak{g}^{-1}$.

- *Second key point*
  - this is where integrability comes in; $\widetilde{P} \subset D$ is an integral manifold of the $G(\mathbb{R})$-invariant distribution of $T(D)$ given by $\mathfrak{g}^{-1} \subset \mathfrak{g}^- = T_0 D$; the real Lie algebra generated by the brackets in $\mathfrak{g}^{\pm 1}$ corresponds to a reductive subgroup $G'_{\mathbb{R}} \subset G(\mathbb{R})$ and $\widetilde{P} \subset G'_{\mathbb{R}} \cdot o$; thus we may assume

$$\mathfrak{g}^{-1} \text{ bracket generates } \mathfrak{g}^-$$

  (cf. [R] for details);

- suitably interpreted the previous considerations apply also to $\widetilde{P} \cap D_H$; the upshot is that in effect we may assume this bracket generating property also for $\mathfrak{h}^{-1}$ and $\mathfrak{h}^{-}$;
- we now assume ($\flat$) also does not hold, and from this note that

$$(\flat\flat) \qquad [\mathfrak{g}_{\beta_i} \cdot \mathfrak{g}_{\beta_j}] = 0, \quad j \in J \text{ and } i \in I \backslash J.$$

  Indeed, if this bracket is non-zero, then since $\mathfrak{h}^{-2} = \mathfrak{g}^{-2}$ it belongs to $\mathfrak{h}^2$ and the non-zero root space

$$\mathfrak{g}_{\beta_i} = [[\mathfrak{g}_{\beta_i}, \mathfrak{g}_{\beta_j}], \mathfrak{g}_{-\beta_j}] \in \mathfrak{h}^1$$

  which is a contradiction;

- for the final step, if $\mathfrak{g}^3 \neq 0$, we have $\beta_1, \beta_2, \beta_3$ such that $\beta_1 + \beta_2 + \beta_3$ is a root. Then

$$[\mathfrak{g}_{\beta_1+\beta_2+\beta_3}, \mathfrak{g}_{-\beta_1-\beta_2}] = \mathfrak{g}_{-\beta_3} \in \mathfrak{h}^1;$$

thus

$$J \neq \emptyset.$$

If $J \neq I$, then ($\flat\flat$) gives a contradiction to the fact that the highest root is $\sum_{i \in I} n_i \beta_i$ with all $n_i > 0$.
- If $n = 2$, the argument works all the way up to the last step where, as in the first example, we do have $J = \emptyset$ (and $\mathfrak{g}^{-2} \neq 0 \iff h^{2,0} > 0$ for $G = \mathrm{SO}(2a, b)$).

- *Examples:* $X \subset \mathbb{P}^{n+1}$ smooth degree d hypersurface

$$F(x) = 0$$

where $F(x)$ homogeneous of degree d. For

$$\begin{cases} S^\bullet = \mathbb{C}[x_0, \cdots, x_{n+1}], \\ J^\bullet = \text{Jacobian ideal } \{F_{x_0}, \cdots, F_{x_{n+1}}\}, \\ R^\bullet = S^\bullet / J_F^\bullet \end{cases}$$

there is an isomorphism

$$H^{p,n-p}(X)_{\mathrm{prim}} \cong R^{(n-p)d+n-2};$$

tangent space family of X's is

$$T \cong R^d$$

and

$$(\sharp\sharp) \qquad T \otimes H^{p,n-p}(X)_{\mathrm{prim}} \to H^{p-1,n-p+1}(X)_{\mathrm{prim}}$$

given by multiplication of polynomials

$$R^d \otimes R^{(n-p)d+n-2} \to R^{(n-p+1)d+n-2}.$$

X non-singular gives (Macauly's theorem) that

> *the mappings* ($\sharp\sharp$) *are non-zero whenever both sides are non-zero.*

G is Mumford-Tate group for the period mapping of X's, then

$$R^d \to \mathfrak{g}^{-1,1} \subset F^{-1}\,\mathrm{End}(V, Q).$$

Image is an *abelian* sub-algebra $\mathfrak{a} \subset \mathfrak{g}^{-1.1} \subset \mathfrak{g}_{\mathbb{C}}$, induces

$$\mathrm{Sym}^k \mathfrak{a} \to \mathfrak{g}^{-k,k}$$

giving

$$\mathrm{Sym}^k \mathfrak{a} \to \mathfrak{g}^{-k,k} \subset \oplus \mathrm{Hom}\left(H^{p,n-p}(X)_{\mathrm{prim}}, H^{p-k,n-p+k}(X)_{\mathrm{prim}}\right)$$

which is a subspace of

$$\mathrm{Sym}^k R^d \otimes R^{(n-p)d+n-2} \to R^{(n-p+k)d+n-2}$$

given by multiplication of polynomials. Conclude that map is non-zero whenever both sides are non-zero, which gives $\mathfrak{g}^{-3,3} \neq 0$ for $n \geqq 3, d \geqq 6$. □

- In the second example at the beginning

$$\sigma(\lambda) = 0 \iff \mathfrak{g}^{-3} = (0).$$

- in general *coupling length* defined by

$$\zeta(\mathfrak{a}) = \max\{m : \operatorname{Sym}^m \mathfrak{a} \to \operatorname{Hom}(\mathbb{V}_b^{n,0}, \mathbb{V}_b^{n-m,m}) \neq 0\}$$

at a general point of B. Then

$$\zeta(\mathfrak{a}) \geqq \ell(\mathfrak{g}).$$

There are many examples where $\zeta(\mathfrak{a}) \geqq 3$; e.g., hypersurfaces as above, CY's of dimension $\geqq 3$ whose Yukaya coupling is $\neq 0$.

V. Motivic Hodge structures

- Recent posting arXiv.org/abs/2308.16164 by Tobias Kreutz gives an interesting application of the method in [BKU].
- Idea is nice; statement of the result is not complete because it does not use integrability of transversality; following is an amended version.
- Polarized Hodge structure (PHS) $(V, F^\bullet, Q) := H$ *comes from geometry* if
  - (first approximation) $H = H^n(X)$ for a smooth projective variety X,
  - actual definition is motivic; basically H is made up of sub-quotients of the above.
- These objects have Mumford-Tate groups G and corresponding Mumford-Tate domains D with compact dual $\check{D} = G(\mathbb{C})/P$; this is a homogenous algebraic variety defined over $\overline{\mathbb{Q}}$.

- If D is non-classical then most points of D do not come from geometry; intuitive reason is that because of the differential constraint the image of a period mapping does *not* contain an open set; the set of points of coming from geometry is the complement of a countable union of proper analytic subvarieties.
- Nobody has exhibited an explicit H not coming from geometry; assuming the generalized Hodge conjecture (GHC) and the version due to André of Grothendieck's generalized period conjecture (GPC), Kreutz gives a necessary condition that H come from geometry.
- With terms to be explained the result is

$$\boxed{\operatorname{tr}\deg(H) < L(\mathfrak{g}) \implies H \text{ does not come from geometry.}}$$

- H is defined over a field k if equivalently
  - $F^p \subset V \otimes_{\mathbb{Q}} k$,
  - $F^\bullet \in \check{D}(k)$.

- Then the definition

$$\operatorname{tr}\deg(H) := \min \operatorname{tr}\deg(k)$$

  makes sense.
- As above $H \in \check{D} = G(\mathbb{C})/P$ where $G =$ Mumford-Tate group of H, and we define

$$L(\mathfrak{g}) := \min \left\{ \operatorname{codim}_{\mathfrak{g}/\mathfrak{g}^0} \mathfrak{a} : \mathfrak{a} \subset \mathfrak{g}^{-1,1} \text{ is abelian} \right\}.$$

  Then

$$\begin{cases} L(\mathfrak{g}) = 0 \iff D \text{ is a Hermitian symmetric domain} \\ L(\mathfrak{g}) > 0 \iff D \text{ is non-classical.} \end{cases}$$

Theorem: *Assuming (GHC) and (GPC), if*

$$\operatorname{tr}\deg(H) < L(\mathfrak{g})$$

the H does **not** *come from geometry.*

- Equivalently,

$$H \textit{ comes from geometry} \implies \operatorname{tr}\deg(H) \geqq L(\mathfrak{g}).$$

- For X defined over $\overline{\mathbb{Q}}$ the GPC roughly says that the relations over $\overline{\mathbb{Q}}$ satisfied by the period matrix are reflected in the Mumford-Tate group of the PHS. The extension of the GHC to a general X is due to André is essential for the proof.
- The argument also gives for $H = H^n(X)$ with Mumford-Tate domain D and assuming GPC

$$\mathrm{tr}\deg H < \dim D \implies X \textit{ is } \textbf{not} \textit{ defined over } \overline{\mathbb{Q}}.$$

Example: $n = 2$ and H has Hodge numbers $(2, b, 2)$

$$\mathrm{tr}\deg(H) \leqq b \implies \left\{ \begin{matrix} H \text{ does not come} \\ \text{from geometry} \end{matrix} \right\}.$$

$n = 3$ and H has Hodge numbers $(1, 1, 1, 1)$

$$\mathrm{tr}\deg(H) \leqq 2 \implies \left\{ \begin{matrix} H \text{ does not come} \\ \text{from geometry} \end{matrix} \right\}.$$

References

[BKU] G. Baldi, B. Klingler, and E. Ullmo, On the distribution of the Hodge locus, arXiv:2107.08838, 2021.

[R] C. Robles, Schubert varieties as variations of Hodge structure, *Selecta Math. (N.S.)* **20**(3) (2014), 719–768.