Atypical Hodge Loci

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References

*Talk based on the paper [BKU] and related works given in the references in that work, and on extensive discussions with Mark Green and Colleen Robles.
I. Introduction

**Q:** What can one say about Hodge loci?

- \( B \) is a smooth, connected quasi-projective variety;
- \( \nabla \to B \) is the local system underlying a variation of polarized Hodge structure of weight \( n \);
- geometric case; smooth projective family \( X \to B \) and

\[
\nabla_b = H^n(X_b, \mathbb{Q})_{\text{prim}};
\]

- \( \text{HL}(B) = \) set of \( b \in B \) with more Hodge classes in \( \nabla_b^\otimes := \bigoplus (\otimes^k \nabla_b) \) than at a very general point of \( b \);
- Cattani-Deligne-Kaplan: \( \text{HL}(B) \) is a countable union of proper algebraic subvarieties;
- in geometric case assuming the Hodge conjecture there are extra classes of algebraic cycles in \( X \times \cdots \times X \)'s.
Q: What can we say about $\text{HL}(B)$?

- very informally stated the main result in [BKU] is

\[(I.1) \quad \text{For } n \geq 3 \text{ and aside from exceptional degenerate cases, every irreducible component of } \text{HL}(B)_{\text{pos}} \text{ has strictly larger than the expected codimension;}
\]

- know of no conceptual reason why in the non-classical case there should be more than the expected amount of algebraic cycles;

- proof uses integrability conditions for the differential constraint imposed by transversality in the non-classical case;

- sufficient condition for result is

$$g^{-k,k} \neq 0, \quad \text{some } k \geq 3;$$

- notation and criterion for this given below.

- implied by coupling length $\geq 3$
II. Two examples

* \( X = X_\beta, \ T = T_\beta B \) and \( T \to H^1(T_X) = T\text{ Def}(X) \);
* \( V^p,q = H^q(\Omega^p_X) \) and \( T \to \bigoplus \text{Hom}(V^p,q, V^{p-1,q+1}) \) is Kodaira-Spencer mapping giving first variation of Hodge decomposition of a class in \( H^n(X) \);
* for \( X \) a surface, \( \theta \in T, \lambda \in H^1_g(X) \) and \( \theta \cdot \lambda \in H^{0,2}(X) \) gives the first order deviation from \( \lambda \) remaining a Hodge class in the direction \( \theta \);
* \( \text{NL}_\lambda \subset B \) is the Noether-Lefschetz locus where \( \lambda \) remains a Hodge class; assume reduced and define \( T_\lambda \subset T = \ker\{\theta \to \theta \cdot \lambda\} \);
* for \( X \subset \mathbb{P}^3 \) of degree \( d \geq 4 \) in the estimate

\[
d - 3 \leq \text{codim}_B \text{NL}_\lambda \leq \binom{d - 1}{3} = h^{2,0}(X)
\]

both bounds are achieved (Green; lower bound \( \iff \) \( X \) contains a line).
• now let \( \dim X = 4, \lambda \in H^g_2(X)_{\text{prim}} \); in first approximation

\[
\text{codim } \text{NL}_\lambda \leq h^{1,3}(X) + h^{0,4}(X);
\]

• but \( \theta \cdot \lambda \in H^{1,3}(X) \) so this estimate must be refined to

\[
(*) \quad \text{codim}_B \text{NL}_\lambda \leq h^{1,3}(X).
\]

**Definition:** The right-hand side of (*) is the expected codimension of \( \text{NL}_\lambda \) in \( B \).

• **Integrability:** With \( T_\lambda \subset T \) as above set

\[
\sigma_\lambda = \text{Image}\{ T_\lambda \otimes H^{4,0}(X) \to H^{3,1}(X) \}.
\]

**Observation:**

\[
\text{codim}_B \text{NL}_\lambda \leq h^{1,3}(X) - \dim \sigma(\lambda).
\]
Proof: For $\theta \in T_\lambda$, $\theta' \in T$, $\omega \in H^{4,0}(X)$

$$\langle \theta \omega, \theta' \lambda \rangle = \langle \omega, \theta \theta' \lambda \rangle$$

$$= \langle \omega, \theta' \theta \lambda \rangle \quad \text{(integrability)}$$

$$= 0.$$ 

Thus the number of conditions on $\theta' \in T$ to be in $T_\lambda$ is

$$\leq h^{1,3} - \dim \sigma(\lambda).$$

• Note: For the first example of $X \subset \mathbb{P}^3$ the expected codimension drops for geometric reasons: if $L \subset X$ is a line with Hodge class $\lambda$ and if $\omega \in H^0(\Omega^2_X) \cong H^0(\mathcal{O}_X(d - 4))$, then if $L \subset (\omega)$,

$$\langle \theta \lambda, \omega \rangle = \langle \lambda, \theta \cdot \omega \rangle = \int_L \theta |\omega = 0$$

for all $\theta \in H^1(T_X)$; thus such $\omega$'s do not contribute to the equations defining $\mathbb{N}_\lambda$. In the second example the drop by $\sigma(\lambda)$ in the expected codimension is for Hodge theoretic reasons.
III. Statement of main result

- Polarized Hodge structure \((V, Q, F^\bullet)\) of weight \(n\)
  - non-degenerate \(Q : V \otimes V \to \mathbb{Q}\),
    \[ Q(u, v) = (-1)^n Q(v, u); \]
  - \(F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_\mathbb{C}, F^p \oplus \overline{F}^{n-p+1} \xrightarrow{\sim} V_\mathbb{C} \) for \(0 \leq p \leq n; \)
  - \(V_{p,q} = F^p \cap \overline{F}^q, V_\mathbb{C} = \bigoplus V_{p,q}\) with \(\overline{V}_{p,q} = V_{q,p}\);
  - Hodge-Riemann bilinear relations;
  - \(n = 2m, Hg^m(V) = V^{m,m} \cap V. \)
  - Lie algebra \(g \subset \text{End}(V, Q)\) and \(g_\mathbb{C} = \bigoplus g^{-k,k}\) where
    \[
    g^{-k,k} := g^{-k} = \{ A \in g_\mathbb{C} : A(V_{p,q}) \subset V^{p-k,q+k} \}
    \]
    \[
    \ell(g) = \min\{ k : g^{-k} \neq (0) \};
    \]
    \[
    \ell(g) \geq 3 \implies n \geq 3.
    \]
• **Mumford-Tate group**
  
  - $V^\otimes := \oplus (\otimes^k V)$;
  - Hodge tensors $H^g\bullet(V) = \oplus H^{g, kn/2}(V)$;
  - $MT(V) \subset \text{Aut}(V, Q)$ is $\text{Fix}(H^g\bullet(V))$;
  - is a reductive $\mathbb{Q}$-algebraic group $H$;
  - finite cover of $H$ is $\mathbb{C}^* \times H_0$ where $H_0$ is semi-simple; for simplicity of exposition we will assume $H$ is semi-simple; essential ideas appear in this case;
Example: Assume $\lambda, Q$ generate the algebra of Hodge tensors, $\text{MT}(V) = H_\lambda = \text{Fix } \lambda \subset \text{Aut}(V, Q)$.

- **Variation of Hodge structure** $(V, F^\bullet; B)$
  - $B$ and $V \to B$ as above;
  - $F^\bullet$ is a filtration of $V := V_{\mathbb{C} \otimes \mathbb{C}O_B}$ inducing a polarized Hodge structure on each $V_b$ (understood there is $Q : V \otimes V \to \mathbb{Q}$);
  - $\nabla F^p \subset F^{p-1} \otimes \Omega^1_B$ (transversality);
  - for $b_0 \in B$ and $V = V_{b_0}$ we have the monodromy group $\Gamma \subset \text{Aut}(V, Q)$;
  - the $\mathbb{Q}$-Zariski closure $\overline{\Gamma} = \text{semi-simple } \mathbb{Q}$-algebraic group that is a factor of the MT-group of $(V, Q, F^b_\bullet)$ at a very general point of $B$. 
Period mappings

- $G = \text{semi-simple } \mathbb{Q}\text{-algebraic group and } D = \text{period domain of polarized Hodge structures of a given type and with generic Mumford-Tate group } G$;
- $D = G(\mathbb{R})/G_0$, $G_0$ compact;
- period mapping $\Phi : B \to \Gamma \backslash D$;
- for simplicity of exposition we will assume that $\Gamma^Q := G = \text{Mumford-Tate group of the polarized Hodge structure at a very general point of } B$;

This is justified because the image $\Phi(B)$ of the period mapping is contained in a translate of a $\Gamma^Q(\mathbb{R})$-orbit.

- $T_{F_0}D \cong g_C/F^0g_C \cong \bigoplus_{k>0} g^{-k,k}$ where $g_C \subset \text{End}(V_C)$ and $\theta \in g^{-k,k}$ satisfies $\theta(F^p) \subset F^{p-k}$;
- $\Phi_* : T_bB \to g^{-1,1}$ (transversality);
- image is an abelian subalgebra $a \subset g^{-1,1}$ (integrability);
- notation: $\Phi(B) = P \subset \Gamma \backslash D$, $\tilde{P} \subset D$ inverse image of $P$. 
• **Hodge loci**
  
  - if \( \lambda, Q \) generate \( \text{Hg}^\bullet(V_b^\otimes) \), then

  \[
  \text{NL}_\lambda^0 = \Phi^{-1}(\Phi(B) \cap G_\lambda(\mathbb{R}) \cdot o)^0 \quad (^0 = \text{identity component});
  \]

  - thus Noether-Lefschetz loci are (translates of) orbits of particular Mumford-Tate subgroups of \( G \);
  
  - this is the crucial conceptual point; **Hodge loci are intersections with translates of Mumford-Tate sub-domains.**

**Definition:** If \( H \subset G \) is a Mumford-Tate group and

\[
D_H = H(\mathbb{R}) \cdot o, \quad \Gamma_H = \Gamma \cap H
\]

\( \Phi^{-1}(\Phi(B) \cap (\Gamma_H \setminus D_H))^0 \) is a **special subvariety** of \( B \).

• thus special subvarieties of \( B \) are those \( b \in B \) where the algebra of Hodge tensors is strictly larger than at a general point of \( B \).
• set $P_H = P \cap (\Gamma_H \setminus D_H)$, then the standard codimension of an intersection inequality is

\[
\text{(†)} \quad \text{codim}_{\Gamma \setminus D} P_H \leq \text{codim}_{\Gamma \setminus D} P + \text{codim}_{\Gamma \setminus D}(\Gamma_H \setminus D_H);
\]

**Definition**: Special subvariety is **atypical** if we have strict inequality in (†).

• Thus atypical means we have more Hodge tensors than predicted by the usual expected dimension count formula.

**Example**: Notations as in the second example above

\[
\text{codim}_{\Gamma \setminus D}(\Phi(B) \cap (\Gamma_\lambda \setminus D_\lambda)) = \text{codim}_{\Gamma \setminus D} \Phi(B) + \text{codim}_{\Gamma \setminus D}(\Gamma_\lambda \setminus D_\lambda) - \dim \sigma(\lambda).
\]
Theorem ([BKU]): If $\ell(g) \geq 3$, then every special subvariety of $B$ is atypical.

Example: For smooth $X \subset \mathbb{P}^{n+1}$, $n \geq 3$ and $\text{deg}(X) \geq 6$ every special subvariety is atypical.

Reason for result: $M$ is a manifold and $I'$, $I''$ distributions in $TM$ given by $\{\omega'_i\}$, $\{\omega''_\alpha\}$; $N'$, $N''$ variable integral manifolds of $I'$, $I''|_{N'}$'s; want to estimate codimension of the $N' \cap N''$'s; integrability conditions given by $d\omega'_i$ may impose linear relations on the $\omega''_\alpha|_{N'}$; leads to more than expected number of $N' \cap N''$'s.
IV. Proof of the main result

- Assume equality holds in \((\mathfrak{h})\) and will arrive at a contraction to the assumption \(g^{-k} \neq 0\);
- pass to tangent spaces and use

\[
\begin{align*}
\dim T_0D &= g^{-}, \\
\dim T_0D_H &= \mathfrak{h}^{-}, \\
T_0\tilde{P} &\subset g^{-1}, \\
T_0(\tilde{P} \cap D_H) &= T_0\tilde{P} \cap \mathfrak{h}^{-1}
\end{align*}
\]

\[
\sum_{k \geq 2} \dim \mathfrak{h}^{-k} + \text{codim}_{\mathfrak{h}^{-1}}(T_0\tilde{P} \cap \mathfrak{h}^{-1}) = \sum_{k \geq 2} \dim g^{-k} + \text{codim}_{g^{-1}} T_0\tilde{P}
\]

\[
\dim \mathfrak{h}^{-k} \leq \dim g^{-k}
\]

\[
\text{codim}_{\mathfrak{h}^{-1}}(T_0\tilde{P} \cap \mathfrak{h}^{-1}) \leq \text{codim}_{g^{-1}} T_0\tilde{P}
\]

\[
\implies \begin{cases} 
\mathfrak{h}^{-k} = g^{-k}, & k \geq 2 \\
\text{codim}_{\mathfrak{h}^{-1}} T_0\tilde{P} \cap \mathfrak{h}^{-1} = \text{codim}_{g^{-1}} T_0\tilde{P}.
\end{cases}
\]
We want to conclude

\[ h^{-1} = g^{-1} \]

\[ \Downarrow \]

\[ T_0D_H = T_0D \implies D_H = D. \]

- **Basic idea**
  - $h, g$ are reductive Lie algebras in which $h^-, g^-$ are parabolic sub-algebras;
  - there is maximal torus $t_R \subset g_R$ relative to which $h^-, g^-$ are direct sums of negative root spaces;
  - the root space and Hodge decomposition of $h^-, g^-$ align in the sense that each of $h^{-k}, g^{-k}$ are direct sums of negative root spaces;
  - this is the first key point where the Lie theory and Hodge theory interact; the complex structure on $T_0D \cong g_R/g^0_R$ is given by an $E \in t_R$ whose centralizer is $g^0_R$ and whose eigenspace decomposition on $g_R/g^0_R$ is a direct sum of non-trivial conjugate root subspaces (cf. [R] for details);
– let $\beta_i$, $i \in I$, be the simple positive roots with corresponding root space $g_{\beta_i} \subset g^+$; denote by $J \subset I$ the subset, possibly empty, where $g_{\beta_i} \subset h^+$; since every positive root is a sum of the $\beta_i$, it will suffice to show that

$$J = I;$$

in this case we will have $(\sharp\sharp)$;

– note that $g^-$ is generated by $g^{-1} \iff$ all $g_{-\beta_i} \subset g^{-1}$.

**Second key point**

– this is where integrability comes in; $\tilde{P} \subset D$ is an integral manifold of the $G(\mathbb{R})$-invariant distribution of $T(D)$ given by $g^{-1} \subset g^- = T_0D$; the real Lie algebra generated by the brackets in $g^{\pm 1}$ corresponds to a reductive subgroup $G'_\mathbb{R} \subset G(\mathbb{R})$ and $\tilde{P} \subset G'_\mathbb{R} \cdot e$; thus we may assume

$$g^{-1} \text{ bracket generates } g^-$$

(cf. [R] for details);
suitably interpreted the previous considerations apply also to \( \tilde{P} \cap D_H \); the upshot is that in effect we may assume this bracket generating property also for \( \mathfrak{h}^{-1} \) and \( \mathfrak{h}^- \);

we now assume (b) also does not hold, and from this note that

\((bb)\quad [\mathfrak{g}_{\beta_i} \cdot \mathfrak{g}_{\beta_j}] = 0, \quad j \in J \text{ and } i \in I \setminus J.\)

Indeed, if this bracket is non-zero, then since \( \mathfrak{h}^{-2} = \mathfrak{g}^{-2} \) it belongs to \( \mathfrak{h}^2 \) and the non-zero root space

\[ \mathfrak{g}_{\beta_i} = [[\mathfrak{g}_{\beta_i}, \mathfrak{g}_{\beta_j}], \mathfrak{g}_{-\beta_j}] \in \mathfrak{h}^1 \]

which is a contradiction;
– for the final step, if $g^3 \neq 0$, we have $\beta_1, \beta_2, \beta_3$ such that $\beta_1 + \beta_2 + \beta_3$ is a root. Then

$$[g_{\beta_1+\beta_2+\beta_3}, g_{-\beta_1-\beta_2}] = g_{-\beta_3} \in \mathfrak{h}^1;$$

thus

$$J \neq \emptyset.$$ 

If $J \neq I$, then (bb) gives a contradiction to the fact that the highest root is $\sum_{i \in I} n_i \beta_i$ with all $n_i > 0$.

– If $n = 2$, the argument works all the way up to the last step where, as in the first example, we do have $J = \emptyset$ (and $g^{-2} \neq 0 \iff h^{2,0} > 0$ for $G = \text{SO}(2a, b)$).
Examples: \( X \subset \mathbb{P}^{n+1} \) smooth degree \( d \) hypersurface

\[ F(x) = 0 \]

where \( F(x) \) homogeneous of degree \( d \). For

\[
\begin{cases}
S^\bullet = \mathbb{C}[x_0, \ldots, x_{n+1}], \\
J^\bullet = \text{Jacobian ideal } \{ F_{x_0}, \ldots, F_{x_{n+1}} \}, \\
R^\bullet = S^\bullet / J^\bullet_F
\end{cases}
\]

there is an isomorphism

\[ H^{p, n-p}(X)_{\text{prim}} \cong R^{(n-p)d+n-2}; \]

tangent space family of \( X \)'s is

\[ T \cong R^d \]
and

\[(\#\#) \quad T \otimes H^{p,n-p}(X)_{\text{prim}} \to H^{p-1,n-p+1}(X)_{\text{prim}}\]

given by multiplication of polynomials

\[R^d \otimes R^{(n-p)d+n-2} \to R^{(n-p+1)d+n-2}.\]

\(X\) non-singular gives (Macauly’s theorem) that

\[\text{the mappings (\#\#) are non-zero whenever both sides are non-zero.}\]

\(G\) is Mumford-Tate group for the period mapping of \(X\)'s, then

\[R^d \to \mathfrak{g}^{-1,1} \subset F^{-1} \text{End}(V, Q).\]
Image is an *abelian* sub-algebra $\mathfrak{a} \subset \mathfrak{g}^{-1,1} \subset \mathfrak{g}_\mathbb{C}$, induces

$$\text{Sym}^k \mathfrak{a} \to \mathfrak{g}^{-k,k}$$

giving

$$\text{Sym}^k \mathfrak{a} \to \mathfrak{g}^{-k,k} \subset \bigoplus \text{Hom} \left( H^{p,n-p}(X)_{\text{prim}}, H^{p-k,n-p+k}(X)_{\text{prim}} \right)$$

which is a subspace of

$$\text{Sym}^k R^d \otimes R^{(n-p)d+n-2} \to R^{(n-p+k)d+n-2}$$

given by multiplication of polynomials. Conclude that map is non-zero whenever both sides are non-zero, which gives $\mathfrak{g}^{-3,3} \neq 0$ for $n \geq 3$, $d \geq 6$. \(\square\)
- In the second example at the beginning

\[ \sigma(\lambda) = 0 \iff g^{-3} = 0. \]

- in general coupling length defined by

\[ \zeta(a) = \max\{ m : \text{Sym}^m a \to \text{Hom}(\nabla_b^{n,0}, \nabla_b^{n-m,m}) \neq 0 \} \]

at a general point of \( B \). Then

\[ \zeta(a) \geq \ell(g). \]

There are many examples where \( \zeta(a) \geq 3 \); e.g., hypersurfaces as above, CY’s of dimension \( \geq 3 \) whose Yukaya coupling is \( \neq 0 \).
V. Motivic Hodge structures

- Recent posting arXiv.org/abs/2308.16164 by Tobias Kreutz gives an interesting application of the method in [BKU].
- Idea is nice; statement of the result is not complete because it does not use integrability of transversality; following is an amended version.
- Polarized Hodge structure (PHS) \((V, F^\bullet, Q) := H \text{ comes from geometry}\) if
  - (first approximation) \(H = H^n(X)\) for a smooth projective variety \(X\),
  - actual definition is motivic; basically \(H\) is made up of sub-quotients of the above.
- These objects have Mumford-Tate groups \(G\) and corresponding Mumford-Tate domains \(D\) with compact dual \(\bar{D} = G(\mathbb{C})/P\); this is a homogenous algebraic variety defined over \(\mathbb{Q}\).
• If $D$ is non-classical then most points of $D$ do not come from geometry; intuitive reason is that because of the differential constraint the image of a period mapping does not contain an open set; the set of points of coming from geometry is the complement of a countable union of proper analytic subvarieties.

• Nobody has exhibited an explicit $H$ not coming from geometry; assuming the generalized Hodge conjecture (GHC) and the version due to André of Grothendieck’s generalized period conjecture (GPC), Kreutz gives a necessary condition that $H$ come from geometry.

• With terms to be explained the result is

\[
\text{tr deg}(H) < L(g) \implies H \text{ does not come from geometry}.
\]

• $H$ is defined over a field $k$ if equivalently
  
  \begin{itemize}
  
  \item $F^p \subset V \otimes_{\mathbb{Q}} k$,
  
  \item $F^\bullet \in \tilde{D}(k)$.
  \end{itemize}
Then the definition
\[ \text{tr} \deg(H) := \min \text{tr} \deg(k) \]
makes sense.

As above \( H \in \tilde{D} = G(\mathbb{C})/P \) where \( G = \) Mumford-Tate group of \( H \), and we define
\[ L(g) := \min \{ \text{codim}_{g/g_0} a : a \subset g^{-1,1} \text{ is abelian} \}. \]
Then
\[
\begin{cases}
L(g) = 0 & \iff D \text{ is a Hermitian symmetric domain} \\
L(g) > 0 & \iff D \text{ is non-classical}.
\end{cases}
\]

**Theorem:** Assuming (GHC) and (GPC), if
\[ \text{tr} \deg(H) < L(g) \]
the \( H \) does not come from geometry.

Equivalently,
\[ H \text{ comes from geometry} \implies \text{tr} \deg(H) \geq L(g). \]
• For $X$ defined over $\overline{\mathbb{Q}}$ the GPC roughly says that the relations over $\overline{\mathbb{Q}}$ satisfied by the period matrix are reflected in the Mumford-Tate group of the PHS. The extension of the GHC to a general $X$ is due to André is essential for the proof.

• The argument also gives for $H = H^n(X)$ with Mumford-Tate domain $D$ and assuming GPC

$$\text{tr} \deg H < \dim D \implies X \text{ is not defined over } \overline{\mathbb{Q}}.$$

**Example:** $n = 2$ and $H$ has Hodge numbers $(2, b, 2)$

$$\text{tr} \ deg(H) \leq b \implies \begin{cases} H \text{ does not come} \\ \text{from geometry} \end{cases}.$$

$n = 3$ and $H$ has Hodge numbers $(1, 1, 1, 1)$

$$\text{tr} \ deg(H) \leq 2 \implies \begin{cases} H \text{ does not come} \\ \text{from geometry} \end{cases}.$$
References
