A SMOOTHNESS RESULT IN DEFORMATION THEORY

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Outline

I. Introduction

- For compact, complex manifold X the cohomology group
$$H^1(T_X)$$
parametrizes the space of first order deformations of the complex structure on X.
- The obstruction to lifting $\theta_1 \in H^1(T_X)$ to a second order deformation of X is the bracket
$$[\theta_1, \theta_1] \in H^2(T_X).$$
- If this vanishes, there is an obstruction in $H^2(T_X)$ to extending to a third order deformation of X, etc.[1]
- If $H^2(T_X) = 0$, then lifting to all orders may be done; in many examples $H^2(T_X) \neq 0$ but nonetheless all the obstructions vanish.
- A traditional problem is to give criteria that this happens so that the Kuranishi space $\mathrm{Def}(X)$ is smooth and has tangent space

$$T\,\mathrm{Def}(X) = H^1(T_X). \tag{1}$$

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[1]The map $H^1(T_X) \otimes H^1(T_X) \to H^2(T_X)$ is symmetric.

- One problem is that in general the group $H^2(T_X)$ does not have a geometric interpretation.
- For surfaces it is dual to $H^0(K_X \otimes \Omega^1_X)$, but this does not seem to be of much help.
- The main classical result due to Bogomolov-Tian-Todorov (BTT) is that (1) holds if X is Kähler and we are in the Calabi-Yau case where the canonical bundle

$$K_X \cong \mathcal{O}_X. \tag{2}$$

- In this case $H^2(T_X) \cong H^2(\Omega^{n-1}_X)$ is a summand of $H^{n+1}(X, \mathbb{C})$ and therefore it does have geometric meaning.
- Recently the mechanism behind the proof of the BTT theorem has been formalized leading to a proof of the

(3) Theorem: *The obstruction space* $\mathrm{Ob}(X)$ *to deforming a compact Kähler manifold lies in*

$$\ker\{H^2(T_X) \to \oplus \operatorname{Hom}(H^q(\Omega^p_X), H^{q+2}(\Omega^{p-1}_X))\}$$

where the map is induced by the contraction

$$\rfloor : T_X \otimes \Omega^p_X \to \Omega^{p-1}_X.$$ [2]

- The BTT result is an immediate consequence of this using $\Omega^n_X \cong \mathcal{O}_X$ and taking $q = 0$, $p = n$ so that the map in (3) becomes the identity.
- There is also a relative version of the above where we take the deformation of a pair (X, D) with $D \subset X$ a global normal crossing divisor (NCD); the log-Calabi-Yau case is when $D \in |-K_X|$.
- In this case there are two types of deformations:
  - (i) D remains a NCD with the same number of components; the Zariski tangent space is

$$H^1(T_X(-\log D));$$

  - (ii) D may be partially smoothed; then

$$\mathbb{E}\mathrm{xt}^1(\Omega^1_X(\log D), \mathcal{O}_X)$$

[2] $\mathrm{Ob}(X) = 0 \iff \mathrm{Def}(X)$ contains an open neighborhood of the origin in $H^1(T_X)$.

is the Zariski tangent space.

In [KKP] there is a variant, motivated by mirror symmetry, of the relative case.

- Finally there is the case where X may be singular with dualizing sheaf $\omega_X \cong \mathcal{O}_X$. Here one looks for deformations that smooth X. The classical case is when X is a d-semi-stable normal crossing variety (NCV).[3] Then there is an extension of the BTT result ([F] when $\dim X = 2$, [KN] for any $\dim X$). The recent paper [CLM] contains a result that extends many earlier results. The paper [FL] is complementary to these works; it deals with the case where X has isolated singularities.
- The purpose of this talk us to discuss the proof of (3) above and the extension of this result to (i) in the relative case. We will not take up (ii) or the case of an X with general singularities.
- The approach we shall take will be concrete and analytic; we will try to isolate the essential points.[4]

II. Deformation theory

- X is a fixed, compact C^∞ manifold.
- An *almost complex structure* ω is given by a sub-bundle $T_\omega \subset T_{\mathbb{R},X} \otimes \mathbb{C}$ such that

$$T^*_\omega \oplus \overline{T}^*_\omega \xrightarrow{\sim} T^*_{\mathbb{C},X}.$$

[3]This means that $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \cong \mathcal{O}_D$ where D is the singular locus of X. In the local to global spectral sequence for Ext we have

$$\begin{array}{ccccc} H^0(\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)) & \longrightarrow & \mathbb{E}\mathrm{xt}^1(\Omega^1_X, \mathcal{O}_X) & \longrightarrow & H^1(\mathrm{Ext}^0(\Omega_X, \mathcal{O}_X)) \\ \wr\| & & \wr\| & & \wr\| \\ H^0(\mathcal{O}_D) & \longrightarrow & T\,\mathrm{Def}(X) & \longrightarrow & T\,\mathrm{Def}_{\mathrm{es}}(X) \end{array}$$

where the first term corresponds to first order smoothings of X.

[4]There is an extensive formal homological framework for the obstructions. The intent here is to use direct "blue collar" analytic methods as in the original BTT proofs.

Relative to a suitable covering of X by open sets U in each one there will be a set ω^i of smooth complex-valued 1-forms that at each point gives a basis for T^*_ω.

- The almost complex structure is *integrable* if the Frobenius condition

$$d\omega^i \equiv 0 \left\langle \omega^i \right\rangle \tag{4}$$

is satisfied; here $\langle\omega^i\rangle$ is the ideal in the C^∞ forms $A^*(U)$ generated by the ω^i.

- By the Newlander-Nirenberg theorem an integrable almost structure defines a complex manifold; locally in U there are functions z^i such that

$$\operatorname{span}\{dz^i\} = \operatorname{span}\{\omega^i\}.$$

- Holomorphic functions f are defined by

$$df \equiv 0 \operatorname{span}\{\omega^i\};$$

coordinate changes from z^i in U to $\tilde{z}^i$ in $\widetilde{U}$ are holomorphic.

- A *deformation* of the almost complex structure is given in each U by

$$\begin{aligned} \omega^i(t) &= dz^i + t\varphi^i_{1\bar{j}} d\bar{z}^j + t^2 \varphi^i_{2\bar{j}} d\bar{z}^j + \cdots \\ &= dz^i + \varphi(t). \end{aligned}$$

- The compatibility condition in intersections of open sets gives that

$$\varphi(t) \in A^{0,1}(T_X)[t];$$

i.e., $\varphi(t) = \varphi^i_{\bar{j}}(t)\partial/\partial z^i \otimes d\bar{z}^j$ is a global $(0,1)$ form with values in the complex tangent bundle T_X.[5]

- The integrability condition (4) is the $\overline{\partial}$-*Maurer-Cartan equation*

$$\boxed{\overline{\partial}\varphi(t) + \frac{1}{2}[\varphi(t), \varphi(t)] = 0;} \tag{5}$$

here the bracket is induced from the usual one on sections of T_X.

[5]Intrinsically the bundle of tangent spaces to the Grassmannian bundle $T_\omega \subset T_{\mathbb{C},X}$ is $\operatorname{Hom}(T_\omega, T_{\mathbb{C},X}/T_\omega) \cong \operatorname{Hom}(T_\omega, \overline{T}_\omega)$. This is the intrinsic interpretation of φ. We will not deal with convergence issues. So far as I know in all cases where a formal power series construction is possible a convergent one may be done.

- The first few equations in (5) are

$$\begin{gathered}\overline{\partial}\varphi_1 = 0\\ \overline{\partial}\varphi_2 + \frac{1}{2}[\varphi_1, \varphi_1] = 0\\ \overline{\partial}\varphi_3 + [\varphi_1, \varphi_2] = 0\\ \vdots\end{gathered}$$

- The first of these gives that φ_1 defines the *Kodaira-Spencer class* in $H^1(T_X)$.
- The second shows that the bracket $[\varphi_1, \varphi_1] \in H^2(T_X)$ where it gives the first obstruction to extending the almost complex structure given by φ_1 to an integrable complex structure.
- The next one gives the second obstruction in $H^2(T_X)$, etc.
- $(A^{0,\bullet}(T_X), \overline{\partial}, [\ ,\ ])$ is a *differential graded Lie algebra* (dgla). We will say a little bit more about these below. What is needed is a formalism that systematically deals with the obstructions that arise when we try to solve the Maurer-Cartan equation for $\varphi(t)$'s in a dgla. Such exists as a general theory; I am not aware of its applications to deformation theory of smooth algebraic varieties that in examples go significantly beyond what we will discuss here.

III. Absolute case of the BTT theorem

- As with obstruction theory in topology there isn't much to be able to say in general about a direct step-by-step approach to constructing $\varphi(t)$. However, if we can map the obstructions to something topological, then maybe Hodge theory can be used to get a handle on them. Here a basic idea is that *the differential of the period mapping suggests how to construct such mappings.*
- This differential is especially simple in the Calabi-Yau case. Let $\Omega \in H^0(\Omega^n_X)$ be a generator; locally there are coordinates such that

$$\Omega = dz^1 \wedge \cdots \wedge dz^n.$$

- The contraction mapping
$$\rfloor : T_X \otimes \Omega_X^n \xrightarrow{\sim} \Omega_X^{n-1}$$
induces an isomorphism
$$A^{0,q}(T_X) \xrightarrow{\sim} A^{n-1,q}(X) \tag{6}$$
which commutes with $\overline{\partial}$; then we have
$$H^q(T_X) \xrightarrow{\sim} H^q(\Omega_X^{n-1}). \tag{7}$$
- A natural question is what the bracket $[\ ,\ ]$ becomes under the identifications (6) and (7), a question to which we now turn.
- To give the basic identity (8) below that makes everything work, for $\psi \in A^{n-1,q}(X)$ we define
$$\operatorname{div}\psi \in A^{0,q}(X)$$
by
$$\operatorname{div}\psi \cdot \Omega = \partial\psi.$$
- In coordinates for
$$\begin{aligned}\psi &= \psi_{i,\bar{J}}(\partial/\partial z^i \rfloor dz^1 \wedge \cdots \wedge dz^n) \otimes d\bar{z}^J \\ &= \psi_{i,\bar{J}}\left((-1)^{i-1} dz^1 \wedge \cdots \wedge \widehat{dz}^i \wedge \cdots \wedge dz^n\right) \otimes d\bar{z}^J; \\ \operatorname{div}\psi &= \partial_{z^i}\psi_{i,\bar{J}} d\bar{z}^J.\end{aligned}$$
- For $\psi, \eta \in A^{0,1}(T_X)$ we have
$$\boxed{[\psi,\eta]\rfloor\Omega = \partial(\psi\wedge\eta\rfloor\Omega) + (\operatorname{div}\psi)\eta\rfloor\Omega + (\operatorname{div}\eta)\psi\rfloor\Omega.} \tag{8}$$
Here $\psi \wedge \eta$ is defined by independently wedging the T_X and $(0,1)$ parts; the proof is by calculation in local coordinates.
- As a corollary we have
$$\operatorname{div}\psi = \operatorname{div}\eta = 0 \implies [\psi,\eta]\rfloor\Omega = \partial(\psi\wedge\eta\rfloor\Omega). \tag{9}$$
This is a first hint that for CY varieties bracketing induces a trivial action on cohomology; to implement this suggestion we will use the

PROPOSITION 10: *If X is Kähler, then any class in $H^1(T_X)$ is represented by a form $\varphi \in A^{0,1}(T_X)$ with*

$$\operatorname{div} \varphi = 0.$$

Proof. For a choice of Kähler metric there is a unique harmonic form φ representing the class. This form satisfies both $\overline{\partial}\varphi = 0$ and the adjoint equation

$$\overline{\partial}^* \varphi = 0.$$

We will show that

$$\overline{\partial}^* \varphi = 0 \implies \operatorname{div} \varphi = 0.$$

In local coordinates $\overline{\partial}^*$ is a first order differential operator. Since the metric is Kähler around any point we may choose holomorphic coordinates so that the metric is

$$\sum dz^i \otimes d\bar{z}^i + (\text{second order terms}).$$

Evaluated at the origin the formula for $\overline{\partial}^*$ on X is the same as for the Euclidean metric, and for this one $\overline{\partial}^* \varphi$ is visibly equal to $\operatorname{div} \varphi$. □

THEOREM 11 (BTT): *The deformations of a Kähler Calabi-Yau variety are unobstructed.*

Proof. Using (6) above we have

$$A^{0,\bullet}(T_X) \cong A^{n-1,\bullet}(X). \tag{12}$$

With this identification the right-hand side of (12) becomes a differential graded Lie algebra. Using (9) the subspace

$$\ker \partial \subset A^{n-1,\bullet}(X)$$

is a differential graded Lie sub-algebra in which the bracket maps by

$$[\ ,\ ] : \operatorname{Ker}(\partial) \to \operatorname{Im}(\partial).$$

The $\partial\overline{\partial}$-lemma implies that for $\psi \in A^{n-1,\bullet}(X)$

$$\overline{\partial}\psi = 0 \text{ and } \psi \in \operatorname{Im}(\partial) \implies \psi = \overline{\partial}\eta$$

for some $\eta \in A^{n-1,\bullet}(X)$. As a consequence the induced graded Lie algebra $H^\bullet(\Omega_X^{n-1})$ is *abelian.*

By (10) we may take for $\varphi_1 \in A^{0,1}(T_X)$ a set of forms such that $\partial(\varphi_1 \rfloor \Omega) = 0$. Then the iterative construction of $\varphi(t)$ satisfying the $\overline{\partial}$-Maurer-Cartan equation (5) takes place in an abelian dgla; hence it is solvable. $\square$

(13) • For later reference we note that proof analysis shows we do not need the full strength of the $\partial\overline{\partial}$-lemma. Rather it suffices that on $A^{\bullet,\bullet}(X)$ the Hodge→de Rham spectral sequence degenerates at E_1.

IV. Main theorem and its proof

- A *differential graded Lie algebra* $(\mathcal{A}^\bullet, d, [\ ])$ is a graded vector space $\mathcal{A} = \underset{q \geqq 0}{\oplus} \mathcal{A}^q$ with

$$d : \mathcal{A}^q \to \mathcal{A}^{q+1}, \quad d^2 = 0$$
$$[\ ,\ ] : \mathcal{A}^p \otimes \mathcal{A}^q \to \mathcal{A}^{p+q}$$

where properties analogous to those of the dgla $(A^{0,\bullet}(T_X), \overline{\partial}, [\ ,\ ])$ are assumed to hold. Important among these is the graded Jacobi identity.[6]

To each power series $\varphi(t)$ in $t_1, \dots, t_m$ with coefficients in $\mathcal{A}$ and zero constant term and that converges in a neighborhood of the origin we associate the germ of subvariety defined by the Maurer-Cartan equation

$$d\varphi(t) + \frac{1}{2}[\varphi(t), \varphi(t)] = 0. \tag{14}$$

- There is a notion of equivalence modeled on that for $A^{0,\bullet}(T_X)$ induced by $\mathrm{Diff}(X)$.[7]
- Under conditions that are again modeled on those satisfied by $A^{0,\bullet}(T_X)$ there is a versal object (the Kuranishi space) given by an analytic

[6] There is a vast literature about these objects. The notes [M] and the references cited therein are a good source.

[7] Actually one uses $\exp(C^\infty$ sections of $T_X)$. The infinitesimal action of $\theta \in A^{0,0}(T_X)$ on $\varphi \in A^{0,1}(T_X)$ is

$$\dot{\varphi} = [\theta, \varphi] - \overline{\partial}\theta.$$

This carries over to a general dgla.

subvariety

$$\mathrm{Def}(\mathcal{A}) \subset H^1(\mathcal{A}).$$

- In addition to the deformation functor $\mathcal{A} \to \mathrm{Def}(\mathcal{A})$ there is an obstruction functor given by

$$\varphi(t) \to \text{LHS of (14)} \subset H^2(\mathcal{A});$$

here there is a versal object given by a subvariety

$$\mathrm{Ob}(\mathcal{A}) \subset H^2(\mathcal{A}).$$

- There is an obvious notion of morphism

$$f : \mathcal{A} \to B \tag{15}$$

of dgla's; here the important property is that (15) induces

$$\begin{cases} f_{\mathrm{Def}} : \mathrm{Def}(\mathcal{A}) \to \mathrm{Def}(B), \\ f_{\mathrm{Ob}} : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(B). \end{cases} \tag{16}$$

- A morphism (15) of dgla's is a *quasi-isomorphism* if the induced maps

$$H^q(\mathcal{A}) \xrightarrow{\sim} H^q(B)$$

are isomorphisms; in this case the maps (16) are bi-regular.
- An example of a quasi-isomorphism is the inclusion

$$\ker \partial \hookrightarrow A^{\bullet,\bullet}(X)$$

for the dgla $(A^{\bullet,\bullet}(X), \overline{\partial}, \wedge)$ where X is compact Kähler; as previously noted proof analysis shows that we only need the degeneration at E_1 of the Hodge$\to$ de Rham spectral sequence.[8]
- We now give a sketch of the proof of Theorem 3; the details are in [M]. The contraction map

$$A^{0,i}(T_X) \times A^{p,q}(X) \to A^{p-1,q+i}(X)$$

induces

$$\begin{cases} \iota : A^{0,\bullet}(T_X) \to \mathrm{Hom}(A^{\bullet,\bullet}(X), A^{\bullet,\bullet}(X)), \\ \iota_\varphi(\omega) = \varphi \rfloor \omega \end{cases}$$

[8]For the gradation $A^{\bullet,q}(X)$ has degree q.

with the properties

$$\begin{cases} \iota_{\overline{\partial}\varphi} = [\overline{\partial}, \iota_\varphi], \\ \iota_{[\varphi,\psi]} = [\iota_\varphi, [\partial, \iota_\psi]] = [[\iota_\varphi, \partial], \iota_\psi]. \end{cases} \tag{17}$$

- Define
$$K^\bullet\left(\ker\partial, \frac{A^{\bullet,\bullet}(X)}{\partial A^{\bullet,\bullet}(X)}\right) = \overset{i}{\oplus}\operatorname{Hom}^{i-1}\left(\ker\partial, \frac{A^{\bullet,\bullet}(X)}{\partial A^{\bullet,\bullet}(X)}\right),$$
and make it into a dgla $(K^\bullet, \delta, \{\ ,\ \})$ by setting
$$\begin{cases} \delta f = \overline{\partial} f + (-1)^i f\overline{\partial}, \quad f \in \operatorname{Hom}^{i-1}(\bullet,\bullet), \\ \{f,g\} = f\partial g - (-1)^{\deg(f)\deg(g)} g\partial f. \end{cases}$$

Basic calculation: $\iota : A^{0,\bullet}(T_X) \to K\left(\ker\partial, \frac{A^{\bullet,\bullet}(X)}{\partial A^{\bullet,\bullet}(X)}\right)$ *is a morphism of dgla's.*

This is a consequence of (17).

- By the $\partial\overline{\partial}$-lemma
$$H^{p,q}_{\overline{\partial}}(\ker(\partial)) \cong H^{p,q}\frac{(A^{\bullet,\bullet}(X)}{\partial A^{\bullet,\bullet}(X)} \cong H^q(\Omega^p_X)$$
$$\implies H^2(K) \cong \bigoplus_{p+q=r+s-1} \operatorname{Hom}\left(H^p(\Omega^q_X), H^r(\Omega^s_X)\right).$$
- On the other hand the dgla
$$L = \left\{ f \in K : f(\ker(\partial)) \subset \frac{\ker(\partial)}{\partial A^{\bullet,\bullet}(X)}, f(\partial A^{\bullet,\bullet}(X)) = 0 \right\}$$
$$\cong K^\bullet\left(\frac{\ker(\partial)}{\partial A^{\bullet,\bullet}(X)}, \frac{\ker(\partial)}{\partial A^{\bullet,\bullet}(X)}\right) \qquad (\{\ ,\ \} = 0 \text{ here})$$
is *abelian* and thus $\operatorname{Ob}(L)$ is trivial.
- The inclusion
$$L \hookrightarrow K$$
is a quasi-isomorphism
$$\implies \operatorname{Def}(L) \cong \operatorname{Def}(K)$$
$$\implies \operatorname{Ob}(\operatorname{Def}(K)) \text{ is trivial}$$
$$\implies \operatorname{Ob}(\operatorname{Def}(X)) \to \operatorname{Ob}(\operatorname{Def}(K)) \text{ is the trivial mapping}$$
$$\implies \text{result.}$$

□

- What is going on here? For $\varphi \in H^1(T_X)$ and $\omega \in H^p(\Omega^q_X))$ we want

$$[\varphi, \varphi] \rfloor \omega = 0. \tag{18}$$

From the second equation in (17) the left-hand side in (18) is

$$[\iota_\varphi, [\partial, \iota_\varphi]] \cdot \omega.$$

Using $\partial\omega = 0$ and working modulo $\mathrm{Im}(\partial)$ the term

$$\varphi \rfloor \partial(\varphi \rfloor \omega)$$

appears twice but with opposite signs, hence is zero. Intuitively
  - the operation $\omega \to \varphi \rfloor \omega$ takes cohomology classes to cohomology classes;
  - $\partial(\varphi \rfloor \omega)$ is zero in cohomology;
  - thus $\varphi \rfloor \partial(\varphi \rfloor \omega)$ is zero in cohomology.
- Proof analysis shows we need
  - $\alpha_1 \in A^{0,q}(\Omega^p_X)$ and $\overline{\partial}\alpha_1 = 0$ gives that $\partial\alpha_1$ is $\overline{\partial}$-closed in $A^{0,q}(\Omega^{p+1}_X)$;
  - degeneration at E_1 of the Hodge-de Rham spectral sequence implies that

$$\partial\alpha_1 = \overline{\partial}\alpha_2 \ \text{ for } \ \alpha_2 \in A^{0,q-1}(\Omega^{p+1}_X);$$

  - i.e., $\overline{\partial}$-closed and ∂-exact $\implies$ $\overline{\partial}$-exact;
  - above calculation takes place in ∂-exact forms.

V. The relative case

- We will use C^∞ forms, $\overline{\partial}$, the C^∞ log-complex etc. In [I] there is a more complete discussion using only holomorphic objects and a more extensive homological formalism.
- We assume given a fixed pair (X, Y) of C^∞ manifolds; will consider almost complex structures on X that induce almost complex structures on Y.
- Locally we have $\{\omega^i, \omega^\alpha\}$ such that

$$\omega^i\big|_Y = 0. \tag{19}$$

- Assume that we have an integrable almost complex structure pair; may choose local holomorphic coordinates z^i, w^α such that $Y = \{z^i = 0\}$.

- Using (19) a nearby almost complex structure for the pair is

$$\begin{aligned}\omega^i &= dz^i + \varphi^i_{\bar{j}} d\bar{z}^j + z^j \varphi^i_{j\bar{\alpha}} d\bar{w}^\alpha,\\ \omega^\alpha &= dw^\alpha + \varphi^\alpha_{\bar{i}} d\bar{z}^i + \varphi^\alpha_{\bar{\beta}} d\bar{w}^\beta.\end{aligned}$$

  This leads to a deformation theory for the pair.
- Note that φ induces a form in $\mathcal{I}_Y \cdot A^{0,1}(N_{Y/X})$; in case codim $Y = 1$, $\mathcal{I}_Y$ is a line bundle and

$$\varphi \in A^{0,1}(T_X(-\log Y)).$$

  May now take Y to be a normal crossing divisor D and use as ansatz

$$\varphi \in A^{0,1}(T_X(-\log D)). \tag{20}$$

- Leads to a deformation theory with

$$\begin{cases}\mathrm{Def}(X,D) \subset H^1(T_X(-\log D)),\\ \mathrm{Ob}(X,D) \subset H^2(T_X(-\log D)).\end{cases} \tag{21}$$

- If $D \in |-K_X|$ then have

$$A^{0,\bullet}(T_X(-\log D)) \xrightarrow{\sim} A_X^{n-1,\bullet}(\log D) \tag{22}$$

  where the right-hand side is in the C^∞ log complex.
- If X is projective the Hodge$\to$de Rham spectral sequence degenerates at E_1.
- May repeat the above argument to have the BTT result for log Calabi-Yau varieties

$$\mathrm{Def}(X,D) \textit{ is smooth with tangent space } H^1(T_X(-\log D))$$

  (cf. [I] for a detailed algebraic proof).
- Remark that deformations of a pair (X,D) where we do not require that it remain a local product, i.e., we allow partial smoothing of D, may be obstructed (cf. [FPR]).
- There is an important variant (KKP) of the above result: A base-point-free anti-canonical pencil in $|-K_X|$ gives

$$f : X \to \mathbb{P}^1, \quad f^*\mathcal{O}_{\mathbb{P}^1}(1) = -K_X.$$

Assume $f^{-1}(\infty) := D$ is a NCD and that for $Y := X - D$ the restriction of f to Y gives

$$w : Y \to \mathbb{C}.$$

Assume w is smooth outside a compact subset of Y; the smooth fibres are Calabi-Yau varieties.

Example: $Y = \mathbb{C}^{*n}$ and $w(z) = \sum z_i + \frac{1}{z_1 \cdots z_n}$.

- Define $\mathrm{Def}(X, D)_D \subset \mathrm{Def}(X, D)$ to be the deformations $(\mathcal{X}, \mathcal{D}) \xrightarrow{\pi} \mathcal{S}$ such that

$$\pi^{-1}(S) \cap \mathcal{D} = D \times S$$

(the deformation is anchored at ∞).

These are Landau-Ginzberg models that are ubiquitous in mirror symmetry.

THEOREM: $\mathrm{Def}(X, D)_D$ *is unobstructed.*[9]

———<○>———

- Am not sure if in the literature there is an extension of the full statement in Theorem 3 to the relative case.
- A main issue: Very many naturally occurring deformation problems are unobstructed but have the relevant $H^2 \neq 0$. Basically one has to work a bit to find natural obstructed deformation problems.[10]

  Is there a refinement of Theorem 3 that would give more applicable criteria for smoothness?
- Finally, even if there are obstructions it may be that there are only primary ones; i.e., the Kuranishi space is given by the quadratic equations arising from the kernel of

$$H^1 \otimes H^1 \to H^2.$$

[9]The results in [KKP] are more extensive that this. Of particular interest is the subcomplex of the log-complex using log forms ψ such that $df \wedge \psi$ is also a log form.

[10]It seems that one has to work hard to find examples where $\mathrm{Def}(X)$ is non-reduced. Thus generally $\mathrm{Def}(X)$ tends to be a variety whose singular points correspond to X's with special geometric properties.

One says that the deformation problem is formal. Here the basic example is the result of Goldman-Millson (*Publ. Math. IHES*, tome 67 (1988)):

Let $(\mathbb{V};\nabla) \to X$ be the flat vector bundle underlying a variation of Hodge structure over a compact Kähler manifold. If

$$\rho : \pi_1(X) \to \mathrm{Aut}(V)$$

is the monodromy representation, then $\mathrm{Def}(\rho)$ is defined by the quadratic equations arising from the kernel of

$$\mathrm{Sym}^2\{H^1(X,\mathbb{V})\} \to H^2(X,\mathbb{V}).$$

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