AFFINE INSERTION AND PIERI RULES FOR THE AFFINE GRASSMANNIAN

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Abstract. We study combinatorial aspects of the Schubert calculus of the affine Grassmannian Gr associated with $SL(n,\mathbb{C})$. Our main results are:

- Pieri rules for the Schubert bases of $H^*(\mathrm{Gr})$ and $H_*(\mathrm{Gr})$, which expresses the product of a special Schubert class and an arbitrary Schubert class in terms of Schubert classes.
- A new combinatorial definition for k-Schur functions, which represent the Schubert basis of $H_*(\mathrm{Gr})$.
- A combinatorial interpretation of the pairing $H^*(\mathrm{Gr}) \times H_*(\mathrm{Gr}) \to \mathbb{Z}$.

These results are obtained by interpreting the Schubert bases of Gr combinatorially as generating functions of objects we call strong and weak tableaux, which are respectively defined using the strong and weak orders on the affine symmetric group. We define a bijection called affine insertion, generalizing the Robinson-Schensted Knuth correspondence, which sends certain biwords to pairs of tableaux of the same shape, one strong and one weak. Affine insertion offers a duality between the weak and strong orders which does not seem to have been noticed previously.

Our cohomology Pieri rule conjecturally extends to the affine flag manifold, and we give a series of related combinatorial conjectures.

Contents

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1. Introduction

Let Gr denote the affine Grassmannian of $G = SL(n,\mathbb{C})$. Let $\{\xi^w \in H^*(\mathrm{Gr},\mathbb{Z}) \mid w \in \tilde{S}_n^0\}$ and $\{\xi_w \in H_*(\mathrm{Gr},\mathbb{Z}) \mid w \in \tilde{S}_n^0\}$ denote the corresponding Schubert bases in cohomology and homology, where $\tilde{S}_n^0$ is the subset of the affine symmetric group $\tilde{S}_n$ containing affine Grassmannian elements (elements of minimal length in their cosets in $\tilde{S}_n/S_n$). Quillen (unpublished), and Garland and Raghunathan [6] showed that the affine Grassmannian is homotopy-equivalent to the group $\Omega SU(n,\mathbb{C})$ of based loops into $SU(n,\mathbb{C})$, and thus $H_*(\mathrm{Gr})$ and $H^*(\mathrm{Gr})$ acquire structures of dual Hopf-algebras. In [3], Bott calculated $H^*(\mathrm{Gr})$ and $H_*(\mathrm{Gr})$ explicitly – they can be identified with a quotient $\Lambda^{(n)}$ and a subring $\Lambda_{(n)}$ of the ring Λ of symmetric functions. Using an algebraic construction known as the *nil-Hecke ring*, Kostant and Kumar [10] studied the Schubert bases of $H^*(\mathrm{Gr})$ (in fact for flag varieties of Kac-Moody groups) and Peterson [26] studied the Schubert bases of $H_*(\mathrm{Gr})$. Lam [13], confirming a conjecture of Shimozono, identified the Schubert classes ξ^w and ξ_w explicitly as symmetric functions in $\Lambda_{(n)}$ and $\Lambda^{(n)}$.

In cohomology, the Schubert classes ξ^w are given by the *dual k-Schur functions* $\{\tilde{F}_w \mid w \in \tilde{S}_n^0\} \subseteq \Lambda^{(n)}$, introduced in [18] by Lapointe and Morse. In [12], these symmetric functions (also called *affine Schur functions*) were shown to be the Grassmannian case of the affine Stanley symmetric functions $\tilde{F}_w(x)$, where w varies over $\tilde{S}_n$. The symmetric functions $\tilde{F}_w(x)$ are generating functions of objects which we call *weak tableaux*.

In homology, the Schubert classes ξ_w are given by the *k-Schur functions* $\{s_\lambda^{(k)}(x) \mid \lambda_1 < n = k+1\} \subseteq \Lambda_{(n)}$. The k-Schur functions were first introduced by Lapointe, Lascoux, and Morse [14] for the study of Macdonald polynomials [24], though so far a direct connection between Macdonald polynomials and the affine Grassmannian has yet to be established. A number of conjecturally equivalent definitions of k-Schur functions have been presented (see [14, 15, 17, 18]). In this article, a k-Schur function will always refer to the definition of [17, 18] and we can thus view $\{s_\lambda^{(k)}(x) \mid \lambda_1 < n = k+1\}$ as the basis of $\Lambda_{(n)}$ dual to the basis $\{\tilde{F}_w \mid w \in \tilde{S}_n^0\}$ of $\Lambda^{(n)}$. One of our main results (Theorem 5.10) is that k-Schur functions are the generating functions of objects we call *strong tableaux*.

In this paper, we study the topology of Gr by studying the combinatorics of these symmetric functions, as generating functions of weak and strong tableaux.

A weak tableau is a chain $w = w_0 \preceq w_1 \preceq \cdots \preceq w_r = v$ in the left weak order of $\tilde{S}_n$ such that every pair $w_i \preceq w_{i+1}$ is a *weak strip*, denoted $w_i \rightsquigarrow w_{i+1}$. A pair $w \preceq v$ is a weak strip if $vw^{-1} \in \tilde{S}_n$ is a *cyclically decreasing permutation*, a notion introduced in [12]. The generating function $\mathrm{Weak}_{v/w}(x)$ of weak tableaux, with fixed starting and finishing permutations w and v, is called a *weak Schur function*. In general, $\mathrm{Weak}_{v/w}(x)$ coincides with the affine Stanley symmetric function $\tilde{F}_{vw^{-1}}(x)$. When the affine permutations $w, v \in S_n \subset \tilde{S}_n$ are usual Grassmannian permutations (minimal length coset representatives in $S_n/(S_k \times S_{n-k})$), weak Schur functions reduce to usual (skew) Schur functions. In the case of affine Grassmannian permutations, weak Schur functions are dual k-Schur functions and weak tableaux are *k-tableaux* (introduced in [16]).

The definition of a strong tableau depends on an additional parameter $l \in \mathbb{Z}$, corresponding to a choice of maximal parabolic in $\tilde{S}_n$. A *marked strong cover* $C = (w \xrightarrow{i,j} v)$ consists of $w, v \in \tilde{S}_n$ and $(i,j) \in \mathbb{Z}^2$ such that v covers w (denoted $w \lessdot v$) in the strong order (Bruhat order) with $w\, t_{ij} = v$, and the *straddling condition* $i \le l < j$ holds. C is said to be marked at $w(j)$. Note that not all covers $w \lessdot v$ can be marked so as to straddle l. A *strong strip* S from $x = \mathrm{inside}(S)$ to $y = \mathrm{outside}(S)$ is a sequence of marked strong covers $x = x_0 \xrightarrow{i_1,j_1} x_1 \xrightarrow{i_2,j_2} x_2 \longrightarrow \cdots \xrightarrow{i_r,j_r} x_r = y$ such that the markings increase. Finally, a *strong tableau* is a chain $w = w_0 \le w_1 \le \cdots \le w_r = v$ in the strong order of $\tilde{S}_n$ such that every pair $w_i \le w_{i+1}$ has been given the structure of a strong strip.

The generating function $\mathrm{Strong}_{v/w}(x)$ of strong tableaux with fixed starting and finishing permutations w and v, is called a *strong Schur function*. Again when $w, v \in S_n \subset \tilde{S}_n$ are usual Grassmannian permutations, strong Schur functions reduce to usual (skew) Schur functions. Strong tableaux, in the case of $S_n \subset \tilde{S}_n$, are closely related to chains in the k-Bruhat order (with $k = l$) of Bergeron and Sottile [1]. An important difference is that our strong covers can be marked in more than one way, reflecting the fact that affine Chevalley coefficients are not multiplicity-free (see 3.4). In other words, we are dealing with markings of the "l-straddling strong order" on $\tilde{S}_n$.

Our main theorem (Theorem 5.1) is an algorithmically defined bijection called *affine insertion*. In its simplest case, affine insertion establishes a bijection between nonnegative integer matrices with row sums less than n, and pairs (P, Q) where P is a strong tableau, Q is a weak tableau, both tableaux start at the identity $\mathrm{id} \in \tilde{S}_n$, and both end at the same permutation $v \in \tilde{S}_n^0$. This bijection reduces to the usual row-insertion Robinson-Schensted-Knuth (RSK) algorithm (see [5]) as $n \to \infty$. As a corollary of the affine insertion bijection we obtain Cauchy identities (Theorem 5.3) and Pieri rules (Theorem 5.4 and 5.6) for weak and strong Schur functions. Affine insertion also implies in particular that the number of pairs (P, Q) of a standard strong tableau and a standard weak tableau of the same "shape" (starting at the identity id and ending at some permutation $v \in \tilde{S}_n^0$) and size m is given by $m!$. We concentrate on geometric aspects here, leaving the investigation of such enumerative consequences for a later work.

Affine insertion exhibits a duality between the weak and strong orders which does not seem to have been studied before, even in the case of the finite symmetric group S_n. This duality is a combinatorial version of the pairing between homology $H_*(\mathrm{Gr})$ and cohomology $H^*(\mathrm{Gr})$ of the affine Grassmannian.

The construction and proof of the affine insertion algorithm is reduced to a "local rule" using the technology of Fomin's *growth diagrams* [4]. The local rule, which is constructed directly on the level of affine permutations, represents the most involved part of this paper. Our local rule has many elements which will be familiar to experts of Schensted insertion, including local bijections analogous to boxes being bumped to the next row, or boxes not interfering with each other. The strong covers $x \lessdot y$ of $\tilde{S}_n$ in this article roughly correspond to boxes in the traditional language of Young tableaux. A novel and rather mysterious kind of "replacement bump" appears when one considers consecutive strong covers $x \lessdot x' = x \cdot t_{ij} \lessdot x'' = x' \cdot t_{i'j'}$, where the reflections t_{ij} and $t_{i'j'}$ do not commute.

As corollaries of the affine insertion theorem, one deduces Pieri rules for the $H^*(\mathrm{Gr})$ and $H_*(\mathrm{Gr})$. Let $c_{0,m} = s_{m-1} \cdots s_1 s_0 \in \tilde{S}_n$ where $\{s_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ denote the simple generators of $\tilde{S}_n$. We obtain an affine homology Pieri rule in $H_*(\mathrm{Gr})$ (Theorem 5.11):

$$\xi_{c_{0,m}}\, \xi_w = \sum_{w \rightsquigarrow z} \xi_z,$$

where the sum runs over weak strips $w \rightsquigarrow z$ of size m and an affine cohomology Pieri rule for $H^*(\mathrm{Gr})$ (Theorem 5.12):

$$\xi^{c_{0,m}}\, \xi^w = \sum_S \xi^{\mathrm{outside}(S)},$$

where the sum runs over strong strips S of size m with $\mathrm{inside}(S) = w$. The affine cohomology Pieri rule is an affine analogue of the usual Pieri rule for the Schubert calculus of the flag manifold [29]. The affine homology Pieri rule does not appear to have a classical geometric counterpart. It can be deduced directly from the identification of the affine homology Schubert basis $\{\xi_w \mid w \in \tilde{S}_n^0\}$ as k-Schur functions in [13], together with the Pieri rule for k-Schur functions first stated in [17] and described in the notation of this article in [19].

Conjecture 5.14 asserts that the natural analogue of the affine cohomology Pieri rule holds for the cohomology $H^*(\mathcal{G}/\mathcal{B})$ of the affine flag variety. This conjecture is related to a series of combinatorial conjectures (Conjecture 5.17) concerning the strong Schur functions $\mathrm{Strong}_{v/w}(x)$, including symmetry and positivity when expressed in terms of k-Schur functions. The analogous properties for weak Schur functions were established in [12, 13].

In the last part of our paper we translate weak and strong tableaux, together with the affine insertion bijection, from permutations into the more traditional language of partitions. This is performed using a classical bijection [20, 25] between $\tilde{S}_n^0$ and the set of partitions which are n-*cores*. For weak strips and weak tableaux, the corresponding combinatorics involving cores was worked out in [16]. Our main result here (Theorem 11.5) gives a purely partition-theoretic description of marked strong covers, and hence strong strips and strong tableaux. As a consequence the affine cohomology Pieri rule acquires a form similar to that of the Pieri rule for Schur functions, with horizontal strips replaced by "strong strips built on cores". We also use the combinatorial description of strong covers to define a "spin"-statistic on strong tableaux and conjecture (Conjecture 11.11) that the original k-Schur functions (depending on a parameter t) of [14, 15] are spin-weight generating functions of strong tableaux of fixed shape.

Our work poses further challenges for both geometers and combinatorialists. The two Pieri rules beg for a more geometric proof; in the cohomology case there should be a geometric proof similar to that of Sottile [29], and alternatively a more algebraic derivation might be possible using the recursive machinery of Kostant and Kumar's nilHecke ring [10]. The "monomial positivity" of the cohomology classes $\xi^w = \mathrm{Weak}_w(x)$ can be interpreted geometrically as arising from Bott's map $(\mathbb{CP}^n)^\infty \to \mathrm{Gr}$. It would be interesting to obtain a geometric explanation of the monomial positivity of $\xi_w = \mathrm{Strong}_w(x)$.

The positivity of structure coefficients for both weak and strong Schur functions are yet to be given a combinatorial interpretation. The structure constants for the strong Schur functions yields as a special case the WZW fusion coefficients (or equivalently the structure constants of the quantum cohomology $QH^*(G/P)$ of the Grassmannian) as proved by Lapointe and Morse [18]. More generally, Peterson has shown that the structure constants of the quantum cohomology of any (partial) flag manifold can be obtained from the structure constants of the homology $H_*(\mathrm{Gr})$ of the affine Grassmannian. Obtaining a combinatorial interpretation for these structure constants is likely to be a challenging problem. Another interesting problem is to give a direct combinatorial proof of the symmetry of strong Schur functions.

Organization. This paper is roughly divided into three parts. In the first part (Sections 2–5) we give the necessary definitions and present our main theorems. Section 2 contains notation for symmetric functions and Schubert bases of the affine Grassmannian. Our two main objects, strong and weak tableaux, are introduced in Sections 3 and 4 respectively. In Section 5, we present and prove our main results modulo the proof of affine insertion.

The affine insertion algorithm is defined and studied in the second part of our paper (Sections 6–9). We reduce the construction of affine insertion to a local rule in Section 6. In Sections 7 and 8 we define the forward and reverse local rules, and show that they are well-defined. In Section 9 we prove that affine insertion is bijective.

The last part of our paper (Sections 10–12) contains translations of our combinatorial constructions into the language of partitions and cores. In Section 10, we explain a number of bijections between the coroot lattice, affine Grassmannian permutations, cores, offset sequences and k-bounded partitions. In Sections 11 and 12, we explain weak and strong tableaux and affine insertion using the combinatorial language of cores.

2. Schubert bases of Gr and symmetric functions

2.1. **Symmetric functions.** Here we introduce notation for symmetric functions, which can be found in greater detail in [24]. Let $\Lambda = \Lambda(x)$ denote the ring of symmetric functions in infinitely many variables $x_1, x_2, \ldots$ over $\mathbb{Z}$. It is generated over $\mathbb{Z}$ by the algebraically independent *homogeneous symmetric functions* $h_1, h_2, \ldots$, where $\deg h_i = i$. The ring Λ is equipped with an algebra involution $\omega : \Lambda \to \Lambda$ given by $\omega(h_i) = e_i$ where e_i denotes the *elementary symmetric functions*. For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ let $h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots$. The *Hall inner product* $\langle \, , \, \rangle_\Lambda : \Lambda \times \Lambda \to \mathbb{Z}$ is defined by $\langle h_\lambda \, , \, m_\mu \rangle_\Lambda = \delta_{\lambda\mu}$ where m_μ denotes the *monomial symmetric functions.*

The ring Λ has a coproduct $\Delta : \Lambda \to \Lambda \otimes_Z \Lambda$ given by $\Delta(h_i) = \sum_{0 \leq j \leq i} h_j \otimes h_{i-j}$ where $h_0 := 1$. Together with the Hall inner product, this gives Λ the structure of a self-dual commutative and cocommutative Hopf algebra. The antipode c is given by $c(h_i) = (-1)^i e_i$; thus if $f \in \Lambda$ is homogeneous of degree d then $c(f) = (-1)^d \omega(f)$.

Now let $\Lambda^{(n)} := \Lambda/(m_\lambda \mid \lambda_1 \geq n)$. This is a quotient Hopf algebra of Λ. Let $\Lambda_{(n)} := \mathbb{Z}[h_1, h_2, \ldots, h_{n-1}]$. This is a sub-Hopf algebra of Λ. The Hall inner product gives $\Lambda^{(n)}$ and $\Lambda_{(n)}$ the structures of dual Hopf algebras. One possible choice of dual bases is $\{m_\lambda \mid \lambda_1 \leq n-1\}$ for $\Lambda^{(n)}$ and $\{h_\lambda \mid \lambda_1 \leq n-1\}$ for $\Lambda_{(n)}$. The algebra involution ω of Λ restricts to an involution of $\Lambda_{(n)}$. By duality we also obtain an involution ω^+ of $\Lambda^{(n)}$ characterized by the property $\langle f \, , \, g \rangle_\Lambda = \langle \omega(f) \, , \, \omega^+(g) \rangle_\Lambda$ for $f \in \Lambda_{(n)}$ and $g \in \Lambda^{(n)}$. For $f \in \Lambda$ let $\overline{f} \in \Lambda^{(n)}$ denote its image in the quotient. Then $\omega^+(\overline{f}) = \overline{\omega(f)}$. If $f \in \Lambda$, when the context makes it clear we will just write f to denote its image in $\Lambda^{(n)}$.

2.2. **Schubert bases of** Gr. Let Gr denote the affine Grassmannian of $SL(n, \mathbb{C})$. It is an ind-scheme equipped with a Schubert-decomposition

$$\mathrm{Gr} = \bigsqcup_{w \in \tilde{S}_n^0} \Omega_w = \bigcup_{w \in \tilde{S}_n^0} X_w$$

where the unions are taken over the set $\tilde{S}_n^0$ of all 0-Grassmannian permutations in the affine symmetric group $\tilde{S}_n$ (see Section 3); Ω_w denotes the Schubert cell indexed by w and X_w denotes the Schubert variety. Let $\xi^w \in H^*(\mathrm{Gr})$ and $\xi_w \in H_*(\mathrm{Gr})$ denote the corresponding Schubert classes in cohomology and homology; see [7, 13, 11]. The cap product yields a pairing

$$\langle \cdot \, , \, \cdot \rangle_{\mathrm{Gr}} : H^*(\mathrm{Gr}) \times H_*(\mathrm{Gr}) \to \mathbb{Z}$$

under which the Schubert bases $\{\xi^w \mid w \in \tilde{S}_n^0\}$ and $\{\xi_w \mid w \in \tilde{S}_n^0\}$ are dual. Throughout this paper, all (co)homology rings have coefficients in $\mathbb{Z}$. For

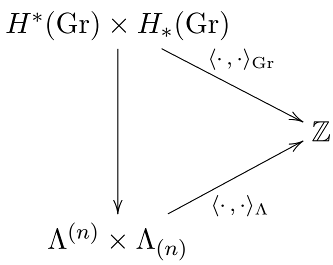
$u, v, w \in \tilde{S}_n^0$ define $c_{uv}^w \in \mathbb{Z}$ and $d_{uv}^w \in \mathbb{Z}$ by

$$\xi_u \xi_v = \sum_w d_{uv}^w \xi_w, \tag{2.1}$$

$$\xi^u \xi^v = \sum_w c_{uv}^w \xi^w. \tag{2.2}$$

The structure constants c_{uv}^w were studied in [10] using the *nilHecke ring*. It follows from work of Graham [7] and Kumar [11] that $c_{uv}^w \in \mathbb{Z}_{\geq 0}$ and from work of Peterson [26] that $d_{uv}^w \in \mathbb{Z}_{\geq 0}$. Our work yields combinatorial interpretations for some of these numbers.

The space Gr is homotopy-equivalent to the based loop space $\Omega SU(n)$; see [6, 27]. Thus $H_*(\mathrm{Gr})$ and $H^*(\mathrm{Gr})$ are endowed with the structures of dual commutative and co-commutative Hopf algebras. In [3], Bott calculated these Hopf algebras explicitly. By identifying the generators explicitly one obtains isomorphisms $H^*(\mathrm{Gr}) \cong \Lambda^{(n)}$ and $H_*(\mathrm{Gr}) \cong \Lambda_{(n)}$ such that the diagram



commutes.

A natural problem is the identification of the Schubert classes ξ_w and ξ^w as symmetric functions. Confirming a conjecture of Shimozono, in [13] Lam showed that the Schubert classes ξ_w and ξ^w are represented respectively by the *k-Schur functions* [14, 15, 17] and *affine Schur functions* (also called *dual k-Schur functions*) [12] [18].

Theorem 2.1. [13] *Under the isomorphism* $H^*(\mathrm{Gr}) \cong \Lambda^{(n)}$*, the Schubert class* ξ^w *is sent to the affine Schur function* $\tilde{F}_w \in \Lambda^{(n)}$*. Under the isomorphism* $H_*(\mathrm{Gr}) \cong \Lambda_{(n)}$*, the Schubert class* ξ_w *is sent to the k-Schur function* $s_w^{(k)} \in \Lambda_{(n)}$*, where* $k = n-1$*.*

The affine Schur functions $\tilde{F}_w$ are generating functions of combinatorial objects known as *k-tableaux* and were first introduced by Lapointe and Morse in [16]. We shall define these objects in Section 4, following the approach of [12]. It is shown in [12, 18] that $\{\tilde{F}_w \mid w \in \tilde{S}_n^0\}$ forms a basis of $\Lambda^{(n)}$. The k-Schur functions $\{s_w^{(k)} \mid w \in \tilde{S}_n^0\}$ form the dual basis of $\Lambda_{(n)}$ to the affine Schur functions. The k-Schur functions $s_\lambda^{(k)}(x)$ or $s_w^{(k)}(x)$ used here are conjecturally (see [17]) the $t = 1$ specializations of the k-Schur functions $s_\lambda^{(k)}(x;t)$ first introduced by Lascoux, Lapointe, and Morse in [14] to study

Macdonald polynomials. The k-Schur functions (and affine Schur functions) are usually indexed by partitions λ such that $\lambda_1 \leq k$. We will describe the bijection between the sets $\{\lambda \mid \lambda_1 \leq k\}$ and $\{w \in \tilde{S}_n^0\}$ in Section 10. For the first portion of this paper we will use affine permutations as indices for affine Schur and k-Schur functions.

2.3. **Schubert basis of the affine flag variety.** Let $\mathcal{G}/\mathcal{B}$ denote the flag variety for the affine Kac-Moody group $\widehat{SL}(n, \mathbb{C})$. Again we omit the construction of this ind-scheme and refer the reader to [11]. The space $\mathcal{G}/\mathcal{B}$ has a decomposition into Schubert varieties indexed by affine permutations $w \in \tilde{S}_n$. We let $\xi_B^w \in H^*(\mathcal{G}/\mathcal{B})$ for $w \in \tilde{S}_n$ denote the cohomology Schubert basis of the affine flag variety. There is a (surjective) morphism $\mathcal{G}/\mathcal{B} \to \mathrm{Gr}$ which induces an algebra inclusion $\iota : H^*(\mathrm{Gr}) \hookrightarrow H^*(\mathcal{G}/\mathcal{B})$. The Schubert classes are sent to Schubert classes under ι, so that $\iota(\xi^w) = \xi_B^w$ for $w \in \tilde{S}_n^0$. Thus we may define integers $c_{uv}^w \in \mathbb{Z}$ as the structure constants of $H^*(\mathcal{G}/\mathcal{B})$:

$$\xi_B^u \xi_B^v = \sum_w c_{uv}^w \xi_B^w \tag{2.3}$$

and when $w, u, v \in \tilde{S}_n^0$ this agrees with the definition in Section 2.2. Again, by general results of [7, 11], the integers c_{uv}^w are nonnegative.

3. Strong tableaux

We introduce some combinatorial constructions involving the affine symmetric group $\tilde{S}_n$. For $a \in \mathbb{Z}$ let $\overline{a}$ denote the coset $a + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$.

3.1. **$\tilde{S}_n$ as a Coxeter group.** The affine symmetric group $\tilde{S}_n$ is a Coxeter group, with generators $\{s_0, s_1, \ldots, s_{n-1}\}$ called *simple reflections* and relations $s_i^2 = \mathrm{id}$, $s_i s_j = s_j s_i$ if $\overline{i}$ and $\overline{j}$ are not adjacent in $\mathbb{Z}/n\mathbb{Z}$, and $s_i s_j s_i = s_j s_i s_j$ if $\overline{i}$ and $\overline{j}$ are adjacent in $\mathbb{Z}/n\mathbb{Z}$. Here pairs of elements of the form $\{\overline{i}, \overline{i+1}\}$ are adjacent in $\mathbb{Z}/n\mathbb{Z}$ and other pairs of elements are not. The length $\ell(w)$ of $w \in \tilde{S}_n$ is the number ℓ of simple reflections in a reduced decomposition of w, which by definition is a factorization $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ of w into a minimum number of simple reflections. An *inversion* of $w \in \tilde{S}_n$ is a pair $(i, j) \in \mathbb{Z}^2$ such that $\overline{i} \neq \overline{j}$, $i < j$, $1 \leq i \leq n$, and $w(i) > w(j)$. Let $\mathrm{Inv}(w)$ denote the set of inversions of w. Then we have (see [8])

$$\ell(w) = |\mathrm{Inv}(w)|. \tag{3.1}$$

For $a, b \in \mathbb{Z}$ we write $s_a = s_b$ if $\overline{a} = \overline{b}$. A *reflection* is an element that is conjugate to a simple reflection.

3.2. **$\tilde{S}_n$ as periodic permutations of $\mathbb{Z}$.** The affine symmetric group $\tilde{S}_n$ can be realized as the set of permutations w of $\mathbb{Z}$ such that $w(i+n) = w(i)+n$ for all $i \in \mathbb{Z}$ and $\sum_{i=1}^n (w(i) - i) = 0$ (see [23]). We sometimes specify an element $w \in \tilde{S}_n$ by "window" notation

$$w = [w(1), w(2), \ldots, w(n)]$$

as this uniquely determines w. Multiplication of elements $v, w \in \tilde{S}_n$ is given by function composition: $(vw)(i) = v(w(i))$ for all $i \in \mathbb{Z}$. We recall Shi's length formula [28]

$$\ell(w) = \sum_{1 \le i < j \le n} \left| \left\lfloor \frac{w(j) - w(i)}{n} \right\rfloor \right|. \tag{3.2}$$

For $r, s \in \mathbb{Z}$ with $\overline{r} \neq \overline{s}$ let $t_{r,s}$ be the unique element of $\tilde{S}_n$ defined by $t_{r,s}(r) = s$, $t_{r,s}(s) = r$, and $t_{r,s}(i) = i$ for $\overline{i} \notin \{\overline{r}, \overline{s}\}$. We have

$$w\, t_{ij} w^{-1} = t_{w(i),w(j)} \qquad \text{for all } w \in \tilde{S}_n. \tag{3.3}$$

The simple reflections are given by $s_i = t_{i,i+1}$ for $i \in \mathbb{Z}$. By (3.3) the reflections in $\tilde{S}_n$ are precisely the elements of the form $t_{r,s}$.

Example 3.1. For $n = 3$, $t_{0,4} = [-3, 2, 7]$ since $t_{0,4}(1) = t_{0,4}(4) - 3 = 0 - 3 = -3$ and $t_{0,4}(3) = t_{0,4}(0) + 3 = 4 + 3 = 7$.

One may also specify $w \in \tilde{S}_n$ in "two-line notation" by expanding the window to include all values of $w : \mathbb{Z} \to \mathbb{Z}$:

$$\begin{matrix} \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\ \cdots & w(-2) & w(-1) & w(0) & w(1) & w(2) & \cdots \end{matrix}$$

In this notation $t_{ij}\, w$ is obtained from w by the exchanging the values $i + kn$ and $j + kn$ in the lower row for all $k \in \mathbb{Z}$. By (3.3) the permutation $w\, t_{ij}$ is obtained from w by exchanging the elements in the *positions* $i + kn$ and $j + kn$ for all $k \in \mathbb{Z}$.

3.3. **Fixing a maximal parabolic.** For the duration of the paper, fix $l \in \mathbb{Z}$. Let $S_n^{\overline{l}} \subset \tilde{S}_n$ be the maximal parabolic subgroup generated by s_i for $\overline{i} \neq \overline{l}$. It is isomorphic to S_n. The definitions of strong strip and strong tableau in Subsection 3.4 depend on the choice of l. We say that an element $w \in \tilde{S}_n$ is *l-Grassmannian* if it is of minimum length in its coset $wS_n^{\overline{l}}$, that is, $w(l+1) < w(l+2) < \cdots < w(l+n)$. We say that w is *Grassmannian* to mean that it is 0-Grassmannian.

3.4. **Strong order and strong tableaux.** The strong order on $\tilde{S}_n$ is the partial order with covering relation $w \lessdot u$, which holds exactly when $\ell(u) = \ell(w) + 1$ and $wt_{ij} = u$ for some reflection t_{ij}.

For $a, b \in \mathbb{Z}$ we sometimes use the notation $[a, b] = \{c \in \mathbb{Z} \mid a \le c \le b\}$ and $(a, b) = \{c \in \mathbb{Z} \mid a < c < b\}$ for intervals of integers.

Lemma 3.2. *Let $w \in \tilde{S}_n$ and let $i < j$ be integers such that $\overline{i} \neq \overline{j}$. Then*

(1) *$w \lessdot w\, t_{ij}$ if and only if $w(i) < w(j)$ and for each $k \in (i, j)$, $w(k) \notin [w(i), w(j)]$.*
(2) *$wt_{ij} \lessdot w$ only if $w(j) < w(i)$ and for each $k \in (i, j)$, $w(k) \notin [w(j), w(i)]$.*

Moreover if the strong cover holds then either $j - i < n$ or $|w(i) - w(j)| < n$.

Proof. Suppose $w(i) < w(j)$. We have

$$t_{ij} = (s_i s_{i+1} \cdots s_{i+a-1}) s_{i+a} (s_{i+a-1} \cdots s_{i+1} s_i)$$

for $i + a = j - 1$. Let

$$\begin{aligned} w^{(r)} &= w\, s_i s_{i+1} \cdots s_{i+r-1} && \text{for } r \in [0, a+1] \\ w_{(r)} &= w^{(a+1)}\, s_{i+a-1} \cdots s_{i+a-r} && \text{for } r \in [0, a]. \end{aligned}$$

We have $\ell(w^{(r)}) = \ell(w^{(r-1)}) \pm 1$ for $r \in [1, a+1]$ and $\ell(w_{(r)}) = \ell(w_{(r-1)}) \pm 1$ for $r \in [1, a]$. Note that in passing from $w = w^{(0)}$ to $w^{(a)}$ (resp. $w^{(a+1)} = w_{(0)}$ to $w_{(a)} = wt_{ij}$), given A, B and C as defined below, we compare $w(i)$ (resp. $w(j)$) with all $w(k)$ where $k \in A \cup B$ (resp. $k \in A \cup C$) and count the number of inversions that have been gained or lost Note that $w(j) > w(i) > w(k)$ for all $k \in C$. Including the "middle" step $w^{(a)}$ to $w^{(a+1)}$ that exchanges the elements $w(i)$ and $w(j)$ in adjacent positions $j - 1$ and j respectively, and writing

$$\begin{aligned} A &= \{k \in (i, j) \mid \overline{k} \notin \{\overline{i}, \overline{j}\}\} \\ A_- &= \{k \in A \mid w(k) < w(i)\} \\ A_+ &= \{k \in A \mid w(k) > w(j)\} \\ A_= &= \{k \in A \mid w(i) < w(k) < w(j)\} \\ B &= \{k \in (i, j) \mid \overline{k} = \overline{j}\} \\ B_+ &= \{k \in B \mid w(k) > w(i)\} \\ B_- &= \{k \in B \mid w(k) < w(i)\} \\ C &= \{i - n, i - 2n, \dots, i - \lfloor (w(j) - w(i))/n \rfloor n\}, \end{aligned}$$

we have

$$\begin{aligned} \ell(w^{(a+1)}) &= \ell(w) + |A_=| + |A_+| - |A_-| + |B_+| - |B_-| + 1; \\ \ell(w\, t_{ij}) = \ell(w_{(a)}) &= \ell(w^{(a+1)}) + |A_=| - |A_+| + |A_-| + |C|. \end{aligned}$$

But $|B| = |B_+| + |B_-| = |C|$. So $\ell(w\, t_{ij}) - \ell(w) = 2|A_=| + 2|B_+| + 1$. Therefore $w \lessdot w\, t_{ij}$ if and only if $|A_=| = 0$ and $|B_+| = 0$. But the following are equivalent: B_+ is empty; $j - i < n$ or $w(j) - w(i) < n$; for every $k \in (i, j)$ with $\overline{k} \in \{\overline{i}, \overline{j}\}$, $w(k)$ is not in the interval $(w(i), w(j))$. Part (1) follows.

The statement in Part (2) follows immediately from Part (1). □

Example 3.3. Even if $w(j) < w(i)$ and for each $k \in (i, j)$ we have $w(k) \notin [w(j), w(i)]$, it is not always true that $wt_{ij} \lessdot w$. For example, let $n = 3$, $w = [10, 2, -6]$, $i = 1$, $j = 5$. Then w is not a strong cover of $wt_{ij} = [5, 7, -6]$.

A *marked strong cover* $C = (w \xrightarrow{i,j} u)$ consists of $w, u \in \tilde{S}_n$ and an ordered pair $(i, j) \in \mathbb{Z}^2$ such that

(1) $w \lessdot u$ is a strong cover with $w\, t_{ij} = u$.
(2) The reflection t_{ij} *straddles* l, that is, $i \le l < j$, where l is defined in Subsection 3.3.

We use the notation inside$(C) = w$ and outside$(C) = u$. We say that C is *marked* at the integer

$$m(C) = w(j) = u(i). \tag{3.4}$$

Remark 3.4. Let $w \lessdot u$ be a strong cover. The number of pairs (i, j) such that $w \xrightarrow{i,j} u$ is a marked strong cover, is equal to the affine Chevalley multiplicity, which by definition is the structure constant c^u_{w,s_l} for the Schubert basis of the cohomology $H^*(\mathcal{G}/\mathcal{B})$ of the affine flag variety. This is merely a translation of the Chevalley rule in [10] for a Kac-Moody flag manifold, in the special case of the affine flag variety.

Remark 3.5. Our notion of strong marked cover when restricted to the symmetric group $S_n \subset \tilde{S}_n$ essentially produces the k-Bruhat order studied by Sottile [29] and Bergeron and Sottile [1].

Proposition 3.6. *If $w \xrightarrow{i,j} u$ is a marked strong cover and w is l-Grassmannian then u is l-Grassmannian as well.*

Proof. Suppose w is l-Grassmannian so that $w(l + 1) < w(l + 2) < \cdots < w(l + n)$. We may pick i', j' so that $t_{i,j} = t_{i',j'}$ and $i' \leq l < j' \leq l + n$. By Lemma 3.2, we must have $w(i') < w(j')$ and for each k satisfying $l + 1 \leq k < j'$ we must have $w(k) < w(i')$ (since $w(k) < w(j')$). In particular, since $k > i'$, we must have $\overline{k} \neq \overline{i'}$. Thus $u(l + 1) < u(l + 2) < \cdots < u(j')$.

Now suppose $\overline{i'} = \overline{a}$ where $j' < a \leq l + n$. Then $u(a) > w(a)$ so we must have $u(a) > u(a-1) > \cdots > u(j')$. Now for each k satisfying $a < k \leq l+n$, we have $i' < k - bn < j'$ where $i' = a - bn$ for a positive integer b. Thus $u(a) = w(j')+bn$. Again by Lemma 3.2, we have either $w(k-bn) < w(i') = w(a) - bn$ or $w(k - bn) > w(j')$. Since $w(k) > w(a)$ the first situation does not occur. Thus $u(k) = w(k) > w(j') + bn = u(a)$. Combining the inequalities, we conclude that u is l-Grassmannian. □

A *strong tuple* $S = [w; (C_1, C_2, \ldots, C_r); u]$ consists of a "sequence of marked strong covers from w to u", that is, elements $w, u \in \tilde{S}_n$ and a sequence of marked strong covers C_k such that outside$(C_k) =$ inside(C_{k+1}) for each $0 \leq k \leq r$, where by convention outside$(C_0) = w$ and inside$(C_{r+1}) = u$. This is a certain kind of chain in the strong order from w to u with data attached to the covers. Sometimes w and u are suppressed in the notation. We write inside$(S) = w$ and outside$(S) = u$. The size of S is the number r of covers in S. By definition size$(S) = \ell(u) - \ell(w)$. If size$(S) > 0$, we refer to the first and last covers in S by first$(S) = C_1$ and last$(S) = C_r$. If size$(S) = 0$ then the empty strong tuple S is determined only by the element inside$(S) =$ outside(S). We also use notation C^- and C^+ to refer respectively to the cover before and after the cover C within a given strong tuple $S = (\ldots, C^-, C, C^+, \ldots)$. If $C = (w \xrightarrow{i,j} u)$ then we use the notation $C^- = (w^- \xrightarrow{i^-,j^-} u^-)$ and $C^+ = (w^+ \xrightarrow{i^+,j^+} u^+)$.

A *strong strip* is a strong tuple S with increasing markings, that is, $m(C_1) < m(C_2) < \cdots < m(C_r)$. A strong strip of size 1 is the same thing as a strong cover. If we wish to emphasize the inside and outside permutations then we use the notation $w \xrightarrow{S} u$.

Lemma 3.7. *Let $S = (C_1, \dots, C_r)$ be a strong tuple. Then $m(C_k) \neq m(C_{k+1})$ for $1 \leq k < r$.*

Proof. Let $S = (\dots, C^-, C, \dots)$ and $C = (w \xrightarrow{i,j} u)$. We have $m(C^-) = u^-(i^-) = w(i^-)$ and $m(C) = w(j)$. So $m(C^-) = m(C)$ implies that $i^- = j$, which contradicts the straddling inequalities $i^- \leq l < j$. □

Given strong strips S_1 and S_2 such that $\mathrm{outside}(S_1) = \mathrm{inside}(S_2)$ let $S_1 \cup S_2$ denote the strong tuple obtained by concatenating the sequences of strong covers defining S_1 and S_2. Then $S_1 \cup S_2$ is a strong strip if and only if one of the S_i is empty, or if both are nonempty and $m(\mathrm{last}(S_1)) < m(\mathrm{first}(S_2))$.

A *strong tableau* is a sequence $T = (S_1, S_2, \dots)$ of strong strips S_k such that $\mathrm{outside}(S_k) = \mathrm{inside}(S_{k+1})$ for all $i \in \mathbb{Z}_{>0}$ and $\mathrm{size}(S_k) = 0$ for all sufficiently large k. We define $\mathrm{inside}(T) = \mathrm{inside}(S_1)$ and $\mathrm{outside}(T) = \mathrm{outside}(S_k)$ for k large. The *weight* $\mathrm{wt}(T)$ of T is the sequence

$$\mathrm{wt}(T) = (\mathrm{size}(S_1), \mathrm{size}(S_2), \dots).$$

We say that T has *shape* u/v where $u = \mathrm{outside}(T)$ and $v = \mathrm{inside}(T)$. If T has shape u/id we simply say that T has shape u.

3.5. Strong Schur functions.

Definition 3.8. For fixed $u, v \in \tilde{S}_n$, define the *strong Schur function*

$$\mathrm{Strong}_{u/v}(x) = \sum_T x^{\mathrm{wt}(T)} \tag{3.5}$$

where T runs over the strong tableaux of shape u/v.

We will use the convention that $\mathrm{Strong}_u(x) = \mathrm{Strong}_{u/\mathrm{id}}(x)$. By Proposition 3.6, $\mathrm{Strong}_{u/v}(x) = 0$ if v is l-Grassmannian and u is not. We shall show later in Theorem 5.10 that when u is l-Grassmannian, $\mathrm{Strong}_u(x)$ is a k-Schur function and thus possess remarkable properties as shown in [17]. However, for general $u, v \in \tilde{S}_n$, the generating function $\mathrm{Strong}_{u/v}(x)$ is not well understood, especially compared to the weak Schur functions to be introduced in Section 4. See Section 5.5.

Example 3.9. Let $u = c_{l,m} := s_{l+m-1} \cdots s_{l+1} s_l$ for an integer m satisfying $0 \leq m \leq n-1$. For example, if $l = 0$ then in window notation, $u = [0, 1, \dots, m-1, m+1, m+2, \dots, n-1, n+m]$. Let us calculate $\mathrm{Strong}_u(x)$. The only strong cover $C = (w \xrightarrow{i,j} u)$ where w is l-Grassmannian, is given by $w = c_{l,m-1}$ and $(i,j) = (l, l+m)$. This strong cover C_m is marked at $m(C) = l + m$. Since $\mathrm{id} \in \tilde{S}_n$ is l-Grassmannian for any l, by Proposition 3.6 we see that a strong tableau $T = (S_1, S_2, \dots, S_r)$ with shape u/id is determined by specifying integers $0 = m_0 \le m_1 \le m_2 \le \cdots \le m_r = m$ such that $S_k = [c_{l,m_{k-1}}; C_{m_{k-1}+1}, \dots, C_{m_k}; c_{l,m_k}]$. Thus $\mathrm{Strong}_u(x) = h_m(x)$.

4. Weak tableaux

4.1. **Cyclically decreasing permutations and weak tableaux.** This section follows [12], which builds on earlier work in the special case of 0-Grassmannian elements (or n-cores) [16, 18]. The intervals $I = \{a, a+1, \dots, b-1, b\} \subsetneq \mathbb{Z}/n\mathbb{Z}$ considered in cyclical fashion will be denoted with the interval notation $[a, b]$.

The *left weak order* $\preceq$ on $\tilde{S}_n$ (sometimes also called the left weak Bruhat order) is defined by $w \preceq v$ if and only if there is a $u \in \tilde{S}_n$ such that $uw = v$ with $\ell(u) + \ell(w) = \ell(v)$.

Given a cyclic interval and proper subset $I = [a, b] \subsetneq \mathbb{Z}/n\mathbb{Z}$, let $c_I = s_b s_{b-1} \cdots s_a \in \tilde{S}_n$ be the product of the reflections indexed by I, appearing in decreasing order. Given any proper subset $A \subsetneq \mathbb{Z}/n\mathbb{Z}$ let $c_A = c_{I_1} \cdots c_{I_t}$ where $A = I_1 \cup \cdots \cup I_t$ is the decomposition of A into maximal cyclic intervals I_k which are called the *cyclic components* of A. The element c_A is well-defined since c_{I_i} and c_{I_j} commute for $i \neq j$. Say that $c \in \tilde{S}_n$ is *cyclically decreasing* if $c = c_A$ for some $A \subsetneq \mathbb{Z}/n\mathbb{Z}$. Write $A(c)$ for this subset A.

Example 4.1. Let $n = 10$ and $A = \{0, 1, 3, 4, 6, 9\}$. The cyclic components of A are $[9, 1]$, $[3, 4]$ and $[6, 6]$ and we have

$$c_A = (s_1 s_0 s_9)(s_4 s_3)(s_6) = (s_4 s_3)(s_1 s_0 s_9)(s_6) = \cdots .$$

Since $\overline{2} \notin A$, the action of c_A on $\mathbb{Z}$ acts on the "window" $[3, 12] \subset \mathbb{Z}$ by the cycles $12 \mapsto 11 \mapsto 10 \mapsto 9 \mapsto 12$, $5 \mapsto 4 \mapsto 3 \mapsto 5$, $7 \mapsto 6 \mapsto 7$, $8 \mapsto 8$, and on all of $\mathbb{Z}$ periodically.

A *weak strip* $W = (w \rightsquigarrow v)$ consists of a pair $w, v \in \tilde{S}_n$ such that $w \preceq v$ and vw^{-1} is cyclically decreasing; it is a certain kind of interval in the left weak order. If we wish to emphasize the cyclically decreasing element $c = vw^{-1}$ then we write $W = (w \overset{A}{\rightsquigarrow} v)$ where $A = A(c)$. The definition of $\preceq$ implies that $\ell(c) + \ell(w) = \ell(v)$. The *size* of W, denoted $\mathrm{size}(W)$, is the integer $\ell(v) - \ell(w) = \ell(c) = |A|$. We write $\mathrm{inside}(W) = w$ and $\mathrm{outside}(W) = v$.

A *weak tableau* U is a sequence $U = (W_1, W_2, \dots)$ of weak strips such that $\mathrm{outside}(W_k) = \mathrm{inside}(W_{k+1})$ for all $k \in \mathbb{Z}_{>0}$ and $\mathrm{size}(W_k) = 0$ for k large. Let $\mathrm{inside}(U) = \mathrm{inside}(W_1)$ and $\mathrm{outside}(U) = \mathrm{outside}(W_k)$ for k large. The tableau U gives precisely the data of a chain $(\mathrm{inside}(W_1), \mathrm{inside}(W_2), \dots)$ in the left weak order on $\tilde{S}_n$ such that consecutive elements define a weak strip. The *weight* $\mathrm{wt}(U)$ of a weak tableau U is the composition $\mathrm{wt}(U) = (\mathrm{size}(W_1), \mathrm{size}(W_2), \dots)$. We say that U is a tableau of *shape* u/v where $u = \mathrm{outside}(U)$ and $v = \mathrm{inside}(U)$. If U has shape u/id we say that U has shape u.

Note that unlike strong tableaux, weak tableaux do not depend on the choice $l \in \mathbb{Z}$ of maximal parabolic.

4.2. **Weak Schur functions.**

Definition 4.2. For $u, v \in \tilde{S}_n$ define the *weak Schur function* or *(skew) affine Stanley symmetric function* by

$$\mathrm{Weak}_{u/v}(x) = \sum_U x^{\mathrm{wt}(U)} \tag{4.1}$$

where U runs over the weak tableaux of shape u/v.

We use the shorthand $\mathrm{Weak}_u(x)$ to mean $\mathrm{Weak}_{u/\mathrm{id}}(x)$. The generating functions $\mathrm{Weak}_{u/v}(x)$ were first introduced in [12] where they were called skew affine Stanley symmetric functions, though weak tableaux were not explicitly used. It is not difficult to see that if $v \preceq u$ then $\mathrm{Weak}_{u/v}(x) = \mathrm{Weak}_{uv^{-1}}(x)$. Thus each weak tableaux generating function is in fact an *affine Stanley symmetric function* (denoted $\tilde{F}_w(x)$ in [12]).

If $u \in \tilde{S}_n$ is 0-Grassmannian then weak tableaux of shape u are the *k-tableaux* (with $k = n-1$) first defined in [16]. We shall translate strong and weak tableaux into the context of k-tableaux in Section 11. In the case that $u \in \tilde{S}_n^0$, $\mathrm{Weak}_u(x)$ is the dual k-Schur function introduced in [18] (called an affine Schur function in [12]).

The basic theorem for $\mathrm{Weak}_{u/v}(x)$ is the following.

Theorem 4.3 (Symmetry of weak Schurs [12])**.** *The generating function* $\mathrm{Weak}_{u/v}(x)$ *is a symmetric function in* $x_1, x_2, \ldots$.

We shall later also need the following properties of $\mathrm{Weak}_{u/v}(x)$. Let $w \mapsto w^*$ denote the involution of $\tilde{S}_n$ induced by $s_i \mapsto s_{l-i}$.

Theorem 4.4 (Conjugacy of weak Schurs [12])**.** *Let* $w \in \tilde{S}_n$*. Then*

$$\omega^+(\mathrm{Weak}_w(x)) = \mathrm{Weak}_{w^{-1}}(x) = \mathrm{Weak}_{w^*}(x).$$

Theorem 4.5 (Affine Schurs form a basis [12, 18])**.** *The set* $\{\mathrm{Weak}_w(x) \mid w \in \tilde{S}_n^0\}$ *forms a basis of* $\Lambda^{(n)}$.

Remark 4.6. If σ denotes the map on $\tilde{S}_n$ induced by a rotational automorphism of the affine Dynkin diagram, then $\mathrm{Weak}_w(x) = \mathrm{Weak}_{\sigma(w)}(x)$ (see also [12, Propostion 18]). Thus $\tilde{S}_n^0$ can be replaced by $\tilde{S}_n^l$ in Theorem 4.5.

Example 4.7. Let $w = c_{l,m} := s_{l+m-1}s_{l+m-2}\cdots s_l$ for $m \geq 0$ a non-negative integer. Then w has a unique reduced decomposition. Note that $c_{l,m}$ is always cyclically decreasing if $m \leq n-1$ and that $c_{l,m}c_{l,m'}^{-1} = c_{l+m',m-m'}$ if $m > m' \geq 1$. The weak tableaux U with shape w are of the form $U = (W_1, W_2, \ldots, W_r)$ where $W_k = (c_{l,m_{k-1}} \rightsquigarrow c_{l,m_k})$ and $1 = m_0 \leq m_1 \leq m_2 \leq \cdots \leq m_k = m$ satisfies $m_k - m_{k-1} \leq n-1$ for $1 \leq k \leq r$. The weight of U is $\mathrm{wt}(U) = (m_1 - m_0, m_2 - m_1, \ldots, m_r - m_{r-1})$. The weak tableau generating function, or affine Stanley symmetric function labeled by w is thus $\mathrm{Weak}_w(x) = \tilde{F}_w(x)$, which is given by the image of $h_m(x)$ in $\Lambda^{(n)}$.

4.3. **Properties of weak strips.** For later use we collect some properties of cyclically decreasing elements and weak strips. Let $A \subsetneq \mathbb{Z}/n\mathbb{Z}$. We say that $i \in \mathbb{Z}$ is *A-nice* if $\overline{i-1} \notin A$ and that i is *A-bad* if $\bar{i} \notin A$.

Given $i \in \mathbb{Z}$, let j be the smallest A-nice integer such that $j > i$. We have

$$i + n > c_A(i) = \begin{cases} j-1 & \text{if } i \text{ is } A\text{-nice} \\ i-1 & \text{otherwise.} \end{cases} \tag{4.2}$$

In other words, if $i < j$ are consecutive A-nice integers for $A \subsetneq \mathbb{Z}/n\mathbb{Z}$, then $j - i \le n$ and c_A acts on the set of integers $[i, j-1]$ by the cycle $j-1 \mapsto j-2 \mapsto \cdots \mapsto i+1 \mapsto i \mapsto j-1$ and $\bar{i}$ is the cyclic minimum of the cyclic component $[\bar{i}, \overline{j-2}]$ of A. In particular

$$\begin{aligned} c_A(i) &= i-1 && \text{if } i \text{ is not } A\text{-nice} \\ c_A(i) &= i && \text{if } i \text{ is } A\text{-nice and } A\text{-bad} \\ c_A(i) &> i && \text{if } i \text{ is } A\text{-nice and not } A\text{-bad.} \end{aligned} \tag{4.3}$$

We obtain the following Lemmata.

Lemma 4.8. *Let $A \subsetneq \mathbb{Z}/n\mathbb{Z}$. Then c_A restricts to an order-preserving bijection from the set of A-nice integers to the set of A-bad integers.*

Lemma 4.9. *Let $A \subsetneq \mathbb{Z}/n\mathbb{Z}$ and $i < j$. Then $c_A(i) > c_A(j)$ if and only if $j \le c_A(i)$, i is A-nice and not A-bad. Equivalently, $c_A(i) > c_A(j)$ if and only if i is A-nice and the next A-nice integer after i is larger than j. In this case $j - i < n$ and j is not A-nice.*

Lemma 4.10. *Let $c = c_A$ be cyclically decreasing and $v = cw$. The following are equivalent:*

(1) *$w \overset{A}{\rightsquigarrow} v$ is a weak strip.*
(2) *For every pair of consecutive A-nice integers $a < b$ we have*

$$w^{-1}(a) < \min(w^{-1}(a+1), w^{-1}(a+2), \ldots, w^{-1}(b-1)).$$

(3) *For every pair of consecutive A-bad integers $a < b$ we have*

$$v^{-1}(b) < \min(v^{-1}(a+1), \ldots, v^{-1}(b-1)).$$

Lemma 4.11. *Let $w \overset{A}{\rightsquigarrow} v$ be a weak strip. Then there do not exist integers $i < j$ such that $w(i) > w(j)$ and $v(i) < v(j)$.*

Proof. Suppose i and j exist. By Lemma 4.9 $w(j)$ is A-nice and $w(j) < w(i) < b$ where b is the smallest A-nice integer greater than $w(j)$. Applying Lemma 4.10(2) we have the contradiction $j < i$. □

Lemma 4.12. *Let $w \overset{A}{\rightsquigarrow} v$ be a weak strip. If v is l-Grassmannian, then w is l-Grassmannian as well.*

Proof. Suppose $v(l+1) < v(l+2) < \cdots < v(l+n)$. Then Lemma 4.11 implies that $w(l+1) < w(l+2) < \cdots < w(l+n)$. □

Lemma 4.13. *Let $A \subsetneq \mathbb{Z}/n\mathbb{Z}$, $q < p$ consecutive A-nice integers such that $\overline{q} \neq \overline{p}$ and $B = A \cup \{\overline{p-1}\}$. Then*

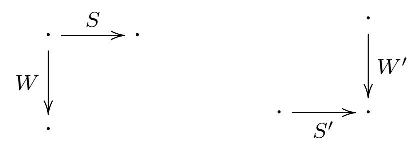
$$c_B = c_A t_{qp} \tag{4.4}$$

Proof. Let $r > p$ be the next A-nice integer. After cancelling c_I for each of the common cyclic components I of A and B, we may assume that $A = [\overline{q}, \overline{p-2}] \cup [\overline{p}, \overline{r-2}]$ and $B = [\overline{q}, \overline{r-2}]$. Then (4.4) is equivalent to

$$s_{p-1}(s_{p-2} \cdots s_q) = (s_{p-2} \cdots s_q) t_{qp}$$

which holds by (3.3). □

4.4. **Commutation of weak strips and strong covers.** An *initial pair* (W, S) consists of a weak strip W and strong strip S with inside(W) = inside(S). A special case is an initial pair (W, C) where C is a marked strong cover. A *final pair* (W', S') consists of a weak strip W' and strong strip S' such that outside(W') = outside(S'). Since a marked strong cover C' is a special case of a strong strip we can refer to a final pair (W', C').



initial pair final pair

Let $(W, C) = (w \overset{A}{\rightsquigarrow} v,\ w \xrightarrow{i,j} u)$ be an initial pair. By associativity in $\tilde{S}_n$ one always has a commutative diagram

$$\begin{array}{ccc} w & \xrightarrow{i,j} & u \\ {\scriptstyle A}\big\downarrow & & \big\downarrow{\scriptstyle A} \\ v & \xrightarrow[i,j]{} & x \end{array}$$

where $x = v\, t_{ij} = c_A u$, in which the arrows labeled by A represent left multiplication by c_A and those labeled by i, j represent right multiplication by t_{ij}. Commutativity of such a diagram means that the corresponding products of elements in $\tilde{S}_n$ result in the same element.

We say that the initial pair (W, C) *commutes* (or that W and C commute, or that c_A and C commute, and so on) if $v < x$, that is, $v(i) < v(j)$ or equivalently $x(i) > x(j)$. This should not be confused with the commutation of a diagram.

Let $(W', C') = (u \overset{A'}{\rightsquigarrow} x,\ v \xrightarrow{a,b} x)$ be a final pair. We say that (W', C') commutes if $w < u$ (that is, $w(a) < w(b)$ or equivalently $u(a) > u(b)$) where $w = u\, t_{ab} = c_{A'}^{-1} v$.

Lemma 4.14. *Let $w, u, v, x \in \tilde{S}_n$, $A \subsetneq \mathbb{Z}/n\mathbb{Z}$ and t_{ij} be such that $u = w\,t_{ij}$, $v = c_A w$, and $x = v\,t_{ij} = c_A u$. Then the following are equivalent:*

(1) *$(w \overset{A}{\rightsquigarrow} v,\ w \xrightarrow{i,j} u)$ is a commuting initial pair.*
(2) *$(u \overset{A}{\rightsquigarrow} x,\ v \xrightarrow{i,j} x)$ is a commuting final pair.*

Proof. For the forward direction we have

$$|A| + \ell(u) \geq \ell(x) \geq \ell(v) + 1 = |A| + \ell(w) + 1 = |A| + \ell(u)$$

since $c_A u = x$, $v < x$, $w \overset{A}{\rightsquigarrow} v$ is a weak strip, and $w \lessdot u$. All inequalities must be equalities, which proves that $u \overset{A}{\rightsquigarrow} x$ is a weak strip and $v \lessdot x$. This proves the forward direction. The reverse direction is similar. □

We list some Lemmata regarding noncommuting initial pairs.

Lemma 4.15. *Let $(W, C) = (w \overset{A}{\rightsquigarrow} v,\ w \xrightarrow{i,j} \cdot)$ be an initial pair and $c = c_A$.*

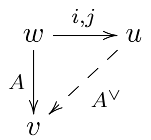
(1) *The following are equivalent: (i) (W, C) does not commute; (ii) $c(w(i)) > c(w(j))$; (iii) $w(j) \leq c(w(i))$, $\overline{w(i)} \in A$ and $w(i)$ is A-nice; (iv) $w(i)$ is A-nice and the next A-nice integer after $w(i)$ is larger than $w(j)$.*
(2) *If (W, C) does not commute then $0 < w(j) - w(i) < n$ and $w(j)$ is not A-nice.*

Proof. By definition, (W, C) commutes if and only if $v(i) < v(j)$. The lemma then follows from Lemma 4.9. □

Lemma 4.16. *Let $(w \overset{A}{\rightsquigarrow} v,\ w \xrightarrow{i,j} u)$ be a noncommuting initial pair and $A^\vee = A - \{\overline{w(j) - 1}\}$. Then $u \overset{A^\vee}{\rightsquigarrow} v$ is a weak strip.*

Proof. By a length calculation it suffices to show that $c_{A^\vee} u = v$. By Lemma 4.15(iv), $w(i)$ and $w(j)$ are consecutive $A^\vee$-nice integers. Applying Lemma 4.13 with the consecutive $A^\vee$-nice integers $w(i) < w(j)$, we get $c_{A^\vee} = c_A t_{w(i)w(j)} = c_A t_{u(i)u(j)}$ which easily gives the result. □

The picture for Lemma 4.16 is given below.



Lemma 4.17. *Let $u \overset{A^\vee}{\rightsquigarrow} v$ be a weak strip with $|A^\vee| < n - 1$, $q < p$ consecutive $A^\vee$-nice integers such that $u^{-1}(q) < u^{-1}(p)$, $A' = A^\vee \cup \{\overline{p-1}\}$, and $x = c_{A'} u$. Then*

$$x = c_{A'}\, u = v\, t_{u^{-1}(q), u^{-1}(p)}, \tag{4.5}$$

$u \overset{A'}{\rightsquigarrow} x$ is a weak strip, and $v \lessdot x$ is a strong cover. If in addition $u^{-1}(q) \leq l < u^{-1}(p)$ then $(u \overset{A'}{\rightsquigarrow} x,\ v \xrightarrow{a,b} x)$ is a noncommuting final pair where $(a,b) = (u^{-1}(q), u^{-1}(p))$.

Proof. Let r be the next $A^\vee$-nice integer after p. Using the hypothesis $u^{-1}(q) < u^{-1}(p)$ and applying Lemma 4.10 to the pairs of consecutive $A^\vee$-nice integers $q < p$ and $p < r$, and to the pair of consecutive A'-nice integers $q < r$, we conclude that $u \rightsquigarrow c_{A'}u = x$ is a weak strip. Equation (4.5) follows from Lemma 4.13. The strong cover assertion follows from (4.5) and a length computation. Noncommutativity holds since $q < p$. □

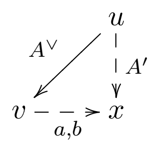
Lemma 4.18. *Let $u \overset{A^\vee}{\rightsquigarrow} v$ be a weak strip with $|A^\vee| < n-1$, $q < p$ consecutive $A^\vee$-bad integers such that $v^{-1}(q) < v^{-1}(p)$, $A' = A^\vee \cup \{\overline{q}\}$, and $x = c_{A'}u$. Then*

$$x = c_{A'}\, u = v\, t_{v^{-1}(q), v^{-1}(p)}, \tag{4.6}$$

$u \overset{A'}{\rightsquigarrow} x$ is a weak strip, and $v \lessdot x$ is a strong cover. Moreover if $v^{-1}(q) \leq l < v^{-1}(p)$ then $(u \overset{A'}{\rightsquigarrow} x,\ v \xrightarrow{a,b} x)$ is a noncommuting final pair where $(a,b) = (v^{-1}(q), v^{-1}(p))$.

Proof. By Lemma 4.8 and (4.2), $q_1 = c_{A^\vee}^{-1}(q)$ and $p_1 = c_{A^\vee}^{-1}(p)$ are consecutive $A^\vee$-nice integers with $p_1 - 1 = q$. We have $v^{-1}(q) = u^{-1}(c_{A^\vee}^{-1}(q)) = u^{-1}(q_1)$ and $v^{-1}(p) = u^{-1}(p_1)$. The Lemma follows by an application of Lemma 4.17 to the weak strip $u \overset{A^\vee}{\rightsquigarrow} v$ and the $A^\vee$-nice integers $q_1 < p_1$. □

The diagram for Lemmata 4.17 and 4.18 is given as follows.



We now give the corresponding lemmata for final pairs.

Lemma 4.19. *Let $(W', C') = (u \overset{A'}{\rightsquigarrow} x,\ v \xrightarrow{a,b} x)$ be a final pair.*

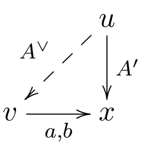
1. *(W', C') does not commute if and only if $u(a) < u(b)$.*
2. *If (W', C') does not commute then $\overline{u(a)} \in A'$, $u(a)$ is A'-nice, $u(b)$ is not A'-nice, $c_{A'}u(a) \geq u(b)$, and $0 < u(b) - u(a) < n$. In particular, the next A-nice integer after $u(a)$ is in that case larger than $u(b)$.*

Proof. The proof is similar to that of Lemma 4.15. □

Lemma 4.20. *Let $(u \overset{A'}{\rightsquigarrow} x,\ v \xrightarrow{a,b} x)$ be a noncommuting final pair and $A^\vee = A' - \{\overline{u(b)-1}\}$. Then $u \overset{A^\vee}{\rightsquigarrow} v$ is a weak strip.*

Proof. By a length calculation it suffices to show that $c_{A^\vee} u = v$. By Lemma 4.19, $u(a) < u(b)$ are consecutive $A^\vee$-nice integers. The result then follows by an application of Lemma 4.13 with the consecutive $A^\vee$-nice integers $u(a) < u(b)$. $\square$

The picture for Lemma 4.20 is given below.



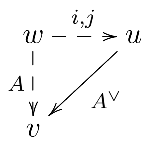
Lemma 4.21. *Let $u \overset{A^\vee}{\rightsquigarrow} v$ be a weak strip with $|A^\vee| < n-1$ and $p < q$ consecutive $A^\vee$-nice integers such that $u^{-1}(q) < u^{-1}(p)$, $A = A^\vee \cup \{\overline{q-1}\}$, and $w = c_A^{-1} v$. Then*

$$w = c_A^{-1} v = u t_{u^{-1}(q), u^{-1}(p)},$$

$w \overset{A}{\rightsquigarrow} v$ is a weak strip, and $w \xrightarrow{i,j} u$ is a strong strip where $(i,j) = (u^{-1}(q), u^{-1}(p))$. If also $u^{-1}(q) \le l < u^{-1}(p)$ then $(w \overset{A}{\rightsquigarrow} v,\ w \xrightarrow{i,j} u)$ is a noncommuting initial pair.

Proof. The proof of the weak and strong strip properties is similar to the proof of Lemma 4.17. The noncommuting property is equivalent to $c_A(p) > c_A(q)$, which holds by the assumptions on p and q. $\square$

The diagram for Lemma 4.21 is given below.



5. Affine insertion and affine Pieri

This section contains our main theorems.

5.1. **The affine insertion bijection.** Let $\mathcal{M}$ be the set of n-bounded matrices, where an n-bounded matrix is by definition a matrix $m = (m_{ij})_{i,j>0}$ with nonnegative integer entries, only finitely many of which are nonzero, all of whose row sums are strictly less than n. We write rowsums(m) and colsums(m) to indicate the sequences of integers given by the row sums and column sums of m respectively.

Fix $u, v \in \tilde{S}_n$. Let $\mathcal{I}_{u,v}$ be the set of triples (T, U, m) where T is a skew strong tableau, U is a skew weak tableau, and $m \in \mathcal{M}$, such that

$$\begin{aligned} \mathrm{inside}(T) &= \mathrm{inside}(U) \\ \mathrm{outside}(T) &= u \\ \mathrm{outside}(U) &= v \\ \mathrm{wt}(U) + \mathrm{rowsums}(m) &\leq (n-1, n-1, \ldots) \end{aligned}$$

When $u = v = \mathrm{id} \in \tilde{S}_n$ then $\mathcal{I}_{\mathrm{id},\mathrm{id}} \cong \mathcal{M}$ since T and U must respectively be the empty strong strip and weak strip from id to id.

Let $\mathcal{O}_{u,v}$ be the set of pairs (P, Q) where P is a skew strong tableau and Q a skew weak tableau with $\mathrm{inside}(P) = v$, $\mathrm{inside}(Q) = u$, and $\mathrm{outside}(P) = \mathrm{outside}(Q)$.

Our main theorem is a constructive proof of the following result.

Theorem 5.1. *There is a bijection*

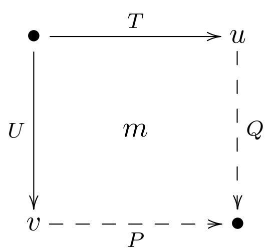
$$\begin{aligned} \Phi_{u,v} : \mathcal{I}_{u,v} &\to \mathcal{O}_{u,v} \\ (T, U, m) &\mapsto (P, Q) \end{aligned} \tag{5.1}$$

such that

$$\mathrm{wt}(T) + \mathrm{colsums}(m) = \mathrm{wt}(P) \tag{5.2}$$

$$\mathrm{wt}(U) + \mathrm{rowsums}(m) = \mathrm{wt}(Q). \tag{5.3}$$

We picture the input and output of the bijection by the diagram



where solid and dotted arrows indicate input and output conditions respectively. We shall construct such a bijection $\Phi_{u,v}$ and call it *affine insertion.*

As a special case of affine insertion (and using Proposition 3.6) we obtain an RSK bijection for the affine Grassmannian.

Theorem 5.2. *For* $l = 0$, $\Phi_{\mathrm{id},\mathrm{id}}$ *gives a bijection* $\mathcal{M} \to \mathcal{O}_{\mathrm{id},\mathrm{id}}$ *from* n*-bounded matrices to pairs* (P, Q) *of strong and weak tableaux from* id *to a common* 0*-Grassmannian element* $\mathrm{outside}(P) = \mathrm{outside}(Q)$*.*

We shall prove later (Theorem 12.3) that when $n \to \infty$ the affine insertion $\Phi_{\mathrm{id},\mathrm{id}}$ bijection coincides with the classical row insertion RSK correspondence.

5.2. **Cauchy identity and Pieri rules for strong and weak tableaux.** Define the *affine Cauchy kernel* $\Omega_n(x,y)$ by

$$\Omega_n(x,y) = \prod_i \left(1 + y_i h_1(x) + y_i^2 h_2(x) + \cdots + y_i^{n-1} h_{n-1}(x)\right)$$
$$= \sum_{\lambda\,:\,\lambda_1<n} h_\lambda(x) m_\lambda(y).$$

It is an element of a completion $\Lambda_{(n)}(x)\hat{\otimes}\Lambda^{(n)}(y)$ of $\Lambda_{(n)}(x)\otimes\Lambda^{(n)}(y)$.

The following is an immediate enumerative consequence of Theorem 5.1.

Theorem 5.3 (Generalized Affine Cauchy Identity)**.** *Let $u, v \in \tilde{S}_n$. Then the following identity holds in the quotient of the formal power series ring $\mathbb{Z}[[x_1, x_2, \ldots, y_1, y_2, \ldots]]/(y_1^n, y_2^n, \ldots)$:*

$$\Omega_n(x,y) \sum_{w\in\tilde{S}_n} \mathrm{Strong}_{u/w}(x)\mathrm{Weak}_{v/w}(y) = \sum_{z\in\tilde{S}_n} \mathrm{Strong}_{z/v}(x)\mathrm{Weak}_{z/u}(y).$$

We now deduce two combinatorial Pieri rules from Theorem 5.1.

Theorem 5.4 (Strong Pieri rule)**.** *Let $u \in \tilde{S}_n$ and $1 \le r \le n-1$. Then*

$$h_r(x)\,\mathrm{Strong}_u(x) = \sum_{u\rightsquigarrow z} \mathrm{Strong}_z(x),$$

where the summation is over weak strips $u \rightsquigarrow z$ of size r.

Note that by Proposition 3.6, $\mathrm{Strong}_u(x) = 0$ unless u is l-Grassmannian so all the permutations in the theorem can be taken to be l-Grassmannian.

Proof. In Theorem 5.1, set $v = \mathrm{id}$ and restrict the bijection to triples (T, U, m) such that m has non-zero entries only in the first row and such that the total entries in the first row of m sum to r, where $1 \le r \le n-1$. Since $v = \mathrm{id}$, we must have $\mathrm{inside}(T) = \mathrm{inside}(U) = \mathrm{id}$ as well, so that in effect $\Phi_{u,v}$ restricts to a bijection from pairs (T, m') to pairs (P, Q) where T is a strong tableau of shape u, the infinite vector m' given by $m'_j = m_{1j}$ has non-negative integer entries summing to r, and Q is a weak strip of size r with $\mathrm{inside}(Q) = u$, and finally P is a strong tableau with shape $z = \mathrm{outside}(P) = \mathrm{outside}(Q)$. Now note that the weight generating function of the vectors m' is $h_r(x)$. In view of (5.2), taking the strong tableau generating functions for the input and output of the bijection gives our theorem. □

One may define *dual weak strips* $w \tilde{\rightsquigarrow} v$ by replacing cyclically decreasing permutations with cyclically increasing permutations (defined in the obvious way).

Theorem 5.5 (Dual strong Pieri rule)**.** *Let $u \in \tilde{S}_n$ and $1 \le r \le n-1$. Then*

$$e_r(x)\,\mathrm{Strong}_u(x) = \sum_{u\tilde{\rightsquigarrow} z} \mathrm{Strong}_z(x),$$

where the sum runs over dual weak strips $u \tilde{\rightsquigarrow} z$ *of size* r.

Proof. In Theorem 5.10 we show that $\mathrm{Strong}_u(x)$ is a k-Schur function when $u \in \tilde{S}_n^l$ and by Proposition 3.6, $\mathrm{Strong}_u(x) = 0$ otherwise. It is known from [17] that k-Schur functions behave well under the involution ω of $\Lambda_{(n)}$ and in our notation we have $\omega(\mathrm{Strong}_u(x)) = \mathrm{Strong}_{u^*}(x)$ where $u \mapsto u^*$ is as defined in Section 4. The involution $u \mapsto u^*$ also interchanges cyclically increasing and cyclically decreasing permutations. Thus the dual strong Pieri rule follows from applying ω to Theorem 5.4. □

Theorem 5.6 (Weak Pieri rule)**.** *Let* $w \in \tilde{S}_n$ *and* $1 \leq r$*. Then the following identity holds in* $\Lambda^{(n)}$*:*

$$h_r(x)\,\mathrm{Weak}_w(x) = \sum_S \mathrm{Weak}_{\mathrm{outside}(S)}(x),$$

where the sum runs over strong strips S *of size* r *such that* $\mathrm{inside}(S) = w$*.*

Proof. The proof is similar to the proof of Theorem 5.4. We take $u = \mathrm{id}$, and restrict to matrices m with non-zero entries only in the first column. □

For the weak Pieri rule, we can use Theorem 4.4 to give a dual rule.

Theorem 5.7 (Dual weak Pieri rule)**.** *Let* $w \in \tilde{S}_n$ *and* $1 \leq r$*. Then the following identity holds in* $\Lambda^{(n)}$*:*

$$e_r(x)\,\mathrm{Weak}_w(x) = \sum_S \mathrm{Weak}_{\mathrm{outside}(S)^{-1}}(x),$$

where the sum runs over strong strips S *of size* r *such that* $\mathrm{inside}(S) = w^{-1}$*.*

One can replace w^{-1} and $\mathrm{outside}(S)^{-1}$ above by w^* and $\mathrm{outside}(S)^*$.

Proof. Clearly $\omega^+(h_r) = e_r$ as elements of $\Lambda^{(n)}$. The theorem follows from applying ω^+ to Theorem 5.6 and using Theorem 4.4. □

5.3. **Pieri rule for the affine Grassmannian.** For this section, we will assume that $l = 0$ and write "Grassmannian" instead of 0-Grassmannian. We will use Theorem 2.1 to deduce two Pieri rules for the affine Grassmannian Gr.

Corollary 5.8. *Let* $u, v \in \tilde{S}_n^0$*. Then the following identity holds in the quotient of the formal power series ring* $\mathbb{Z}[[x_1, x_2, \ldots, y_1, y_2, \ldots]]/(y_1^n, y_2^n, \ldots)$*:*

$$\Omega_n(x,y) \sum_{w \in \tilde{S}_n^0} \mathrm{Strong}_{u/w}(x)\mathrm{Weak}_{v/w}(y) = \sum_{z \in \tilde{S}_n^0} \mathrm{Strong}_{z/v}(x)\mathrm{Weak}_{z/u}(y).$$

Proof. In Theorem 5.3, use Proposition 3.6 and Lemma 4.12 to deduce that $\mathrm{Strong}_{z/v}(x) = 0$ unless $z \in \tilde{S}_n^0$ and $\mathrm{Weak}_{v/w}(y) = 0$ unless $w \in \tilde{S}_n^0$. □

Corollary 5.9 (Affine Cauchy Identity)**.** *The following identity holds in* $\mathbb{Z}[[x_1, x_2, \ldots, y_1, y_2, \ldots]]$*:*

$$\Omega_n(x,y) = \sum_{w \in \tilde{S}_n^0} \mathrm{Strong}_w(x)\mathrm{Weak}_w(y).$$

Proof. Put $u = v = \mathrm{id}$ in Corollary 5.8. □

Theorem 5.10 (Monomial expansion of k-Schur)**.** *Let* $u \in \tilde{S}_n^0$ *be Grassmannian. Then* $\mathrm{Strong}_u(x)$ *coincides with the k-Schur function* $s_u^{(k)}(x)$ *for* $k = n-1$.

We conjecture in Conjecture 11.11 that the k-Schur functions depending on a parameter t can also be expressed as generating functions of strong tableaux, using an additional statistic called spin.

Proof. The k-Schur functions $s_u^{(k)}(x)$ are defined in [17] as the symmetric functions satisfying a certain Pieri rule. Equivalently, one may define $\{s_u^{(k)}(x) \mid u \in \tilde{S}_n^0\}$ as the basis of $\Lambda_{(n)}$ dual to the affine Schur basis $\{\tilde{F}_u(x) \mid u \in \tilde{S}_n^0\}$ of $\Lambda^{(n)}$ see [18, 12]. By definition we have $\mathrm{Weak}_w(x) = \tilde{F}_w(x)$. By Corollary 5.9 it suffices to show that $\mathrm{Strong}_w(x) \in \Lambda_{(n)}$, for then the duality (and the fact that $\{\mathrm{Strong}_w(x) \mid w \in \tilde{S}_n^0\}$ forms a basis of $\Lambda_{(n)}$) will follow from an argument similar to [24, (4.6)].

To show that $\mathrm{Strong}_w(x) \in \Lambda$ we let σ_i be the ring-involution of the ring of formal power series in $x_1, x_2, \ldots$, which interchanges x_i and x_{i+1}. Then by Corollary 5.9,

$$\begin{aligned}
&\sum_{w \in \tilde{S}_n^0} (\sigma_i \cdot \mathrm{Strong}_w(x))\,\mathrm{Weak}_w(y) \\
&= \sigma_i \cdot \prod_i \left(1 + y_i h_1(x) + y_i^2 h_2(x) + \cdots + y_i^{n-1} h_{n-1}(x)\right) \\
&= \prod_i \left(1 + y_i h_1(x) + y_i^2 h_2(x) + \cdots + y_i^{n-1} h_{n-1}(x)\right) \\
&= \sum_{w \in \tilde{S}_n^0} \mathrm{Strong}_w(x)\mathrm{Weak}_w(y).
\end{aligned}$$

By Theorem 4.5, the $\mathrm{Weak}_w(x)$ are linearly independent elements of $\Lambda^{(n)}$. Taking the coefficient of $\mathrm{Weak}_w(y)$ in the above equation we obtain

$$\mathrm{Strong}_w(x) = \sigma_i \cdot \mathrm{Strong}_w(x).$$

Since this holds for all i, we have $\mathrm{Strong}_w(x) \in \Lambda$. Finally, $\{\mathrm{Weak}_w(x) \mid w \in \tilde{S}_n^0\}$ is independent and no terms $h_r(x)$ for $r > n-1$ occurs in the affine Cauchy kernel; so $\mathrm{Strong}_w(x) \in \Lambda_{(n)}$. □

Thus by Theorem 2.1, the generating functions $\mathrm{Strong}_w(x)$ are explicit combinatorial representatives of ξ_w.

Recall from Examples 3.9 and 4.7 that $c_{l,m} := s_{l+m-1} \cdots s_{l+1} s_l$.

Theorem 5.11 (Pieri rule for $H_*(\mathrm{Gr})$)**.** *Let $w \in \tilde{S}_n^0$ and $1 \leq m \leq n-1$. Then*

$$\xi_{c_{0,m}} \xi_w = \sum_{w \rightsquigarrow z} \xi_z,$$

where the sum runs over weak strips $w \rightsquigarrow z$ of size m.

Proof. Example 3.9 says that $\mathrm{Strong}_{c_{0,m}}(x) = h_m(x)$. The theorem then follows immediately from Theorems 5.4, 5.10 and 2.1. □

The Pieri rule for k-Schur functions was first stated in [17] and in the notation here in [19]. Combining this with the geometric identification in [13], one can obtain Theorem 5.11 directly as a corollary (though it was never stated).

Theorem 5.12 (Pieri rule for $H^*(\mathrm{Gr})$)**.** *Let $w \in \tilde{S}_n^0$ and $1 \leq m$. Then*

$$\xi^{c_{0,m}} \xi^w = \sum_S \xi^{\mathrm{outside}(S)},$$

where the sum runs over strong strips S of size m such that $\mathrm{inside}(S) = w$*.*

Proof. Example 4.7 says that $\mathrm{Weak}_{c_{0,m}}(x) = h_m(x)$ in $\Lambda^{(n)}$. The theorem then follows immediately from Theorems 5.6 and 2.1. □

It is not difficult to see that both the weak and strong Pieri rules reduce to the classical Pieri rule for the finite Grassmannian when $\ell(w) + m < n$. Theorems 5.5 and 5.7 also gives us a rule for multiplication by $\xi^{c_{0,m}^*}$ in $H_*(\mathrm{Gr})$ and $H^*(\mathrm{Gr})$ (note that if $w \in \tilde{S}_n^0$ then $w^* \in \tilde{S}_n^0$).

Proposition 5.13. *Theorems 5.11 and 5.12 determine the multiplicative structures of $H_*(\mathrm{Gr})$ and $H^*(\mathrm{Gr})$.*

Proof. It suffices to show that the Pieri rules can be inverted, so that ξ^w (or ξ_w) for any $w \in \tilde{S}_n^0$ can be written in terms of $\{\xi^{c_{0,m}}\}$ (or $\{\xi_{c_{0,m}}\}$). The theorem will then follow from the fact that $\{\xi^w \mid w \in \tilde{S}_n^0\}$ (or $\{\xi_w \mid w \in \tilde{S}_n^0\}$) forms a basis of $H^*(\mathrm{Gr})$ (or $H_*(\mathrm{Gr})$).

The transition matrix between $\{\mathrm{Weak}_w(x) \mid w \in \tilde{S}_n^0\}$ and $\{h_\lambda(x)\}$, given by Theorem 5.6 is the same as the transition matrix between $\{m_\lambda(x)\}$ and $\{\mathrm{Strong}_w(x) \mid w \in \tilde{S}_n^0\}$. Note that $\{h_\lambda(x)\}$ does not form a basis for $\Lambda^{(n)}$, so the matrix is "rectangular" (and infinite). Since $\{\mathrm{Strong}_w(x) \mid w \in \tilde{S}_n^0\}$ is linearly independent, the matrix has full rank (when restricted to submatrices of each degree) and so the Pieri rule can be inverted to write $\mathrm{Weak}_w(x)$ in terms of $h_\lambda(x)$. Applying Theorem 2.1, we may write ξ^w in terms of $\xi^{c_{0,m}}$. Similarly, the fact that $\{\mathrm{Weak}_w(x) \mid w \in \tilde{S}_n^0\}$ is linearly independent allows one to write $\{\mathrm{Strong}_w(x) \mid w \in \tilde{S}_n^0\}$ in terms of $\{h_\lambda \mid \lambda_1 < n\}$. □

5.4. **Conjectured Pieri rule for the affine flag variety.** Theorems 5.6 and 5.12 suggest that we make the following conjecture. In the following conjecture, we let l be arbitrary again.

Conjecture 5.14 (Conjectured Pieri rule for $H^*(\mathcal{G}/\mathcal{B})$)**.** *Let $w \in \tilde{S}_n$ and $1 \leq m$. Then in $H^*(\mathcal{G}/\mathcal{B})$ we have*

$$\xi_B^{c_{l,m}}\, \xi_B^w = \sum_S \xi_B^{\mathrm{outside}(S)},$$

where the sum runs over strong strips S of size m such that $\mathrm{inside}(S) = w$.

In [13] it is observed that $\mathrm{Weak}_w(x) = \tilde{F}_w(x)$ is the pullback of ξ_B^w from $H^*(\mathcal{G}/\mathcal{B})$ to $H^*(\mathrm{Gr})$ under the map $\mathrm{Gr} \to \mathcal{G}/\mathcal{B}$ induced by the map $\Omega SU(n) \to LSU(n) \to LSU(n)/T$ where $\Omega SU(n)$ denotes the based loop-space, $LSU(n)$ denotes the loop space, and T denotes the maximal torus. The pullback of Conjecture 5.14 is consistent with Theorem 5.6.

Remark 5.15. Recurrences for the structure constants of $H^*(\mathcal{G}/\mathcal{B})$ are given in [10]. It may be possible to derive Conjecture 5.14 directly from these recurrences.

Remark 5.16. Conjecture 5.14 is consistent with the Pieri rule for the classical finite-dimensional flag manifold. Indeed if $w, z \in S_n \subset \tilde{S}_n$ and $l \in [1, n]$, then the existence of a marked strong cover $w \xrightarrow{i,j} z$ is exactly the combinatorial condition appearing in Monk's rule, while strong strips agree with the "path formulation" of the Pieri rule in [29]. Note that in [29], using the language that we have introduced, a strong cover $w \xrightarrow{i,j} z$ would be "marked" at $w(i)$ rather than $w(j)$. However the Pieri rules obtained from the two different markings agree.

5.5. **Geometric interpretation of strong Schur functions.** In this section we list some conjectural properties of strong Schur functions, assuming $l = 0$ for simplicity.

Conjecture 5.17. *Let $u, v \in \tilde{S}_n$ be two affine permutations. We have the following successively stronger properties.*

1. *We have* $\mathrm{Strong}_{u/v}(x) \in \Lambda$.
2. *We have* $\mathrm{Strong}_{u/v}(x) \in \Lambda_{(n)}$.
3. *We have* $\mathrm{Strong}_{u/v}(x) = \sum_{w \in \tilde{S}_n^0} c_{vw}^u \,\mathrm{Strong}_w(x)$ *where c_{vw}^u is defined by* (2.3).

The corresponding properties of weak Schur functions are known. Symmetry (Theorem 4.3) was proven combinatorially in [12] (see also [18]) while positivity was shown in [13] using geometric work of Peterson [26].

Proposition 5.18. *Conjecture 5.17 follows from Conjecture 5.14.*

Proof. By making the identification $\Lambda^{(n)} \cong H^*(\mathrm{Gr})$, we may consider the affine Cauchy kernel Ω_n as an element of the completion $\Lambda_{(n)} \hat{\otimes} H^*(\mathrm{Gr})$. If

Conjecture 5.14 holds, then it completely determines the action of $H^*(\mathrm{Gr})$ on $H^*(\mathcal{G}/\mathcal{B})$, obtained from the inclusion $H^*(\mathrm{Gr}) \subset H^*(\mathcal{G}/\mathcal{B})$.

Let $\langle . \, , . \rangle_{\mathcal{G}/\mathcal{B}}$ denote the inner product on $H^*(\mathcal{G}/\mathcal{B})$ defined by $\langle \xi_B^w \, , \, \xi_B^v \rangle = \delta_{wv}$. By the definition of Ω_n and Conjecture 5.14, we have

$$\mathrm{Strong}_{u/v}(x) = \langle \Omega_n \cdot \xi_B^v \, , \, \xi_B^u \rangle_{\mathcal{G}/\mathcal{B}}$$

where $\Omega_n \cdot \xi^v \in \Lambda_{(n)} \hat{\otimes} H^*(\mathcal{G}/\mathcal{B})$. By Corollary 5.9 and Theorem 2.1, we may also write

$$\Omega_n = \sum_{w \in \tilde{S}_n^0} \mathrm{Strong}_w(x) \otimes \xi^w$$

so that

$$\mathrm{Strong}_{u/v}(x) = \sum_{w \in \tilde{S}_n^0} \mathrm{Strong}_w(x) \langle \xi_B^w \, \xi_B^v \, , \, \xi_B^u \rangle_{\mathcal{G}/\mathcal{B}}.$$

Thus $\mathrm{Strong}_{u/v}(x) \in \Lambda_{(n)}$. By general positivity results of [7, 11], we have $c_{vw}^u \in \mathbb{Z}_{\geq 0}$. □

6. Reduction of affine RSK to a local rule

We employ Fomin's general method of growth diagrams [4], which describes a systematic way to reduce the construction of an RSK bijection $\Phi_{u,v}$ to that of a local rule $\phi_{u,v}$. Let $\mathcal{I}_{u,v}^\circ$ be the set of triples (W, S, e) where (W, S) is an initial pair with $\mathrm{outside}(W) = v$ and $\mathrm{outside}(S) = u$, and $e \in \mathbb{Z}_{\geq 0}$ is such that $\mathrm{size}(W) + e < n$. Let $\mathcal{O}_{u,v}^\circ$ be the set of final pairs (W', S') such that $\mathrm{inside}(W') = u$ and $\mathrm{inside}(S') = v$.

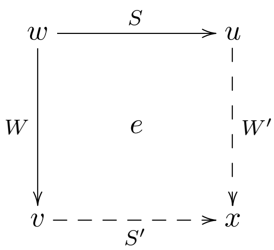
Lemma 6.1. *To define a bijection $\Phi_{u,v}$ as in Theorem 5.1 it suffices to define a bijection*

$$\phi_{u,v} : \mathcal{I}_{u,v}^\circ \to \mathcal{O}_{u,v}^\circ$$
$$(W, S, e) \mapsto (W', S')$$

such that

$$\mathrm{size}(S') = \mathrm{size}(S) + e. \tag{6.1}$$

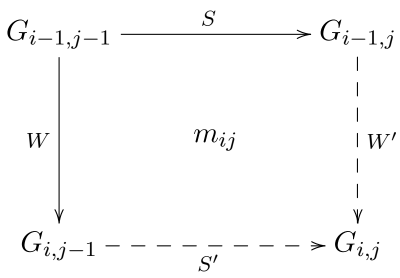
The weak and strong strips form the edges of a commutative diagram



Proof of Lemma 6.1. Let $(T, U, m) \in \mathcal{I}_{u,v}$. To this data we associate a directed graph G called the growth diagram. It shall have the following properties. It has vertices $G_{i,j}$ for $i, j \geq 0$ given by elements of $\tilde{S}_n$ indexed matrix-style. G has directed vertical edges $G_{i,j} \to G_{i+1,j}$ which must define weak strips, and directed horizontal edges $G_{i,j} \to G_{i,j+1}$, which are labeled by strong strips between the corresponding elements of $\tilde{S}_n$. Moreover G shall have the following finiteness property: there exist positive integers N_1 and N_2 such that for all $i \geq N_1$ and all $j \in \mathbb{Z}_{\geq 0}$, $G_{ij} = G_{i+1,j}$, and for all $i \in \mathbb{Z}_{\geq 0}$ and all $j \geq N_2$, $G_{ij} = G_{i,j+1}$. It follows that for all $i \in \mathbb{Z}_{\geq 0}$, the i-th row $G_{i,\bullet}$ of G is a strong tableau and for all $j \in \mathbb{Z}_{\geq 0}$ the j-th column $G_{\bullet,j}$ is a weak tableau. It also follows that for $i \geq N_1$, $G_{i,\bullet} = G_{N_1,\bullet}$; denote this limiting "south" row by $G_{\infty,\bullet}$. Similarly, for $j \geq N_2$, $G_{\bullet,j} = G_{\bullet,N_2}$; denote this limiting "east" column by $G_{\bullet,\infty}$.

We set the north and west boundaries of G to T and U respectively, that is, $G_{0,\bullet} = T$ and $G_{\bullet,0} = U$.

Suppose by induction that for a given $(i,j) \in \mathbb{Z}_{>0}$, that the part of G northwest of G_{ij} has been determined. By assumption, we may use the bijection $\phi_{u',v'}$ for $u' = G_{i,j-1}$ and $v' = G_{i-1,j}$ to complete the two-by-two subdiagram



By induction all of G is defined. We define

$$\Phi_{u,v}(T,U,m) = (P,Q) := (G_{\infty,\bullet}, G_{\bullet,\infty}).$$

The finiteness conditions on T, U, and m, combined with (6.1), imply the existence of the integers N_1 and N_2, so that the south and east tableaux $G_{\bullet,\infty}$ and $G_{\infty,\bullet}$ are well-defined. By construction $(P,Q) \in \mathcal{O}_{u,v}$ and $\Phi_{u,v}$ is well-defined.

To show that $\Phi_{u,v}$ is a bijection we define the inverse map $\Psi_{u,v}$. Given $(P,Q) \in \mathcal{O}_{u,v}$, let $x = \text{outside}(P) = \text{outside}(Q)$. Let N_1 and N_2 be sufficiently large such that for all $i \geq N_1$ and $j \geq N_2$, the i-th element of P and the j-th element of Q are equal to x. We define a growth diagram G as follows. We set $G_{i,\bullet} = P$ for $i \geq N_1$ and $G_{\bullet,j} = Q$ for $j \geq N_2$. This is consistent: these definitions overlap for $i \geq N_1$ and $j \geq N_2$, where the entries $G_{i,j}$ are all equal to x. In the middle of each two-by-two subdiagram in this area we place the integer 0. We now use the inverse $\psi_{u',v'}$ of the local rule $\phi_{u',v'}$ to fill in each two-by-two subdiagram of G (including the "excitation integers" in the middle) given its south and east borders. Then all of G may be computed, as well as a matrix m. Letting $T = G_{0,\bullet}$ and $U = G_{\bullet,0}$ be the north and west boundaries of G, we define $\Psi(P,Q) = (T,U,m)$. It is easy to show that $\Psi_{u,v} \circ \Phi_{u,v} = \text{id}_{\mathcal{I}_{u,v}}$ and $\Phi_{u,v} \circ \Psi_{u,v} = \text{id}_{\mathcal{O}_{u,v}}$. □

7. The local rule $\phi_{u,v}$

In this section we shall define a local rule as in Lemma 6.1, as a sequence of operations called *internal* and *external insertion* steps.

7.1. Internal insertion at a marked strong cover.

Let $C = (w \xrightarrow{i,j} u)$ be a marked strong cover. *Internal insertion at* C is a map that takes as input, a final pair of the form $(W, S_1') = (w \overset{A}{\rightsquigarrow} v, \cdot \xrightarrow{S_1'} v)$ and produces an output final pair[1] of the form $(W', S') = (u \overset{A'}{\rightsquigarrow} x, \cdot \xrightarrow{S'} x)$ such that $\text{inside}(W') = u = \text{outside}(C)$, $\text{size}(W') = \text{size}(W)$, $\text{inside}(S') = \text{inside}(S_1')$, and $\text{size}(S') = \text{size}(S_1') + 1$. This given, we define

$$C' = \text{last}(S') = (v' \xrightarrow{a,b} x).$$

Internal insertion has three cases named A, B, and C. In Cases A and B the output takes a particularly pleasant form: S' is obtained by appending C' to S_1' and $v' = v$. In Case C, S' is obtained from S_1' by placing a strong cover just *before* the last cover of S_1'.

If $\text{size}(S_1') > 0$ we write

$$S_1' = (\cdots \xrightarrow{a_1^-, b_1^-} y \xrightarrow{a_1, b_1} v). \tag{7.1}$$

We need only specify A', (a, b), and the rule for obtaining S' for then we set

$$x = c_{A'}(u) = \text{outside}(S')$$
$$v' = x\, t_{a,b}.$$

7.1.1. *Commuting case.*

Case A (Commuting case) Suppose (W, C) commutes. Set $A' = A$, $(a, b) = (i, j)$, and $S' = S_1' \cup C'$.

Example 7.1. Let $n = 4, l = 0$ and $C = ([3, 5, -2, 4] \xrightarrow{-2,1} [1, 7, -2, 4])$. Consider internal insertion at C of the final pair

$$(W, S_1') = ([3, 5, -2, 4] \overset{\{3\}}{\rightsquigarrow} [4, 5, -2, 3],\ [4, 5, -2, 3] \longrightarrow [4, 5, -2, 3]).$$

Since (W, C) commute, the output final pair is

$$(W', S') = ([1, 7, -2, 4] \overset{\{3\}}{\rightsquigarrow} [1, 8, -2, 3],\ [4, 5, -2, 3] \xrightarrow{S'} [1, 8, -2, 3]).$$

where $S' = ([4, 5, -2, 3] \xrightarrow{-2,1} [1, 8, -2, 3])$.

[1] This is an abuse of language: in the generality in which we define internal insertion, S' is only guaranteed to be a strong tuple, not necessarily a strong strip. However, whenever we apply internal insertion in the definition of the local rule $\phi_{u,v}$, S' will be a strong strip.

7.1.2. *Noncommuting cases.* Otherwise, we assume that (W, C) does not commute. Since $w(j) = u(i)$ is not A-nice by Lemma 4.15, we can let

$$p_0 = u(i) - 1 \tag{7.2}$$

$$A^\vee = A - \overline{p_0}. \tag{7.3}$$

Case B (Normal bumping case) Suppose that (W, C) does not commute and either $\mathrm{size}(S_1') = 0$, or $\mathrm{size}(S_1') > 0$ and $i \neq a_1$. Let $q < p$ be the unique pair of consecutive $A^\vee$-nice integers such that $q < u(j)$ and $u^{-1}(q) \leq l$ and q is maximal. We set $A' = A^\vee \cup \{\overline{p-1}\}$, $(a, b) = (u^{-1}(q), u^{-1}(p))$, and $S' = S_1' \cup C'$.

Example 7.2. Let $n = 6, l = 0$ and

$$C = ([5, 0, 1, 9, -2, 8] \xrightarrow{-2,1} [3, 0, 1, 11, -2, 8]).$$

Consider the final pair (W, S_1') given by

$$W = ([5, 0, 1, 9, -2, 8] \overset{\{3,4,5\}}{\rightsquigarrow} [4, -1, 1, 12, -3, 8])$$

and $S_1' = ([4, -1, 1, 12, -3, 8] \longrightarrow [4, -1, 1, 12, -3, 8])$. The pair (W, C) does not commute and we have $p_0 = 4$ and $A^\vee = \{3, 5\}$. Since $\mathrm{size}(S_1') = 0$, we are in Case B. One calculates that $q = 2, p = 3$, so that $A' = \{2, 3, 5\}$ and $(a, b) = (0, 1)$. Thus the output final pair is given by $W' = ([3, 0, 1, 11, -2, 8] \overset{\{2,3,5\}}{\rightsquigarrow} [2, -1, 1, 12, -3, 10])$ and $S' = ([4, -1, 1, 12, -3, 8] \xrightarrow{0,1} [2, -1, 1, 12, -3, 10])$.

Case C (Replacement Bump) Suppose that (W, C) does not commute, $\mathrm{size}(S_1') > 0$, and $i = a_1$. Let $q < p$ be the unique pair of consecutive $A^\vee$-bad integers such that $q < p_0$ and $y^{-1}(q) \leq l$ and q is maximal. Set $A' = A^\vee \cup \{\overline{q}\}$, $(a^-, b^-) = (y^{-1}(q), y^{-1}(p))$ and let S' be obtained by inserting (a^-, b^-) just *before* the last pair of indices $(a_1, b_1) = (i, b_1)$ of S_1'.

Example 7.3. Let $n = 4, l = 0$ and $C = (w = [1, 7, -2, 4] \xrightarrow{-2,4} u = [1, 8, -2, 3])$. Consider internal insertion at C of the final pair

$$(W, S_1') = ([1, 7, -2, 4] \overset{\{3\}}{\rightsquigarrow} [1, 8, -2, 3],\ [4, 5, -2, 3] \xrightarrow{S_1'} [1, 8, -2, 3]),$$

where $S_1' = (y = [4, 5, -2, 3] \xrightarrow{-2,1} v = [1, 8, -2, 3])$. Since the pair (W, C) does not commute, $\mathrm{size}(S_1') = 1 > 0$ and $i = a_1 = -2$ we are in Case C. We have $p_0 = 3$ and $A^\vee = \emptyset$ so all integers are $A^\vee$-bad. We find that $q = 1, p = 2, A' = \{1\}$ and obtain

$$S' = ([4, 5, -2, 3] \xrightarrow{-2,7} [4, 6, -3, 3] \xrightarrow{-2,1} [2, 8, -3, 3]).$$

Note that $W' = ([1, 8, -2, 3] \overset{\{1\}}{\rightsquigarrow} [2, 8, -3, 3])$ is indeed a weak strip.

7.1.3. *External Insertion.*

Case X (External Insertion) Let $W = (w \overset{A}{\rightsquigarrow} v)$ be a weak strip such that $|A| < n-1$. Let $q < p$ be the unique pair of consecutive A-bad integers such that $v^{-1}(q) \leq l$ and q is maximal. We set $A' = A \cup \{\overline{q}\}$ and $(a, b) = (v^{-1}(q), v^{-1}(p))$. By Lemma 4.18 we have a noncommuting final pair $(W', C') = (w \overset{A'}{\rightsquigarrow} x,\ v \xrightarrow{a,b} x)$.

External insertion is the map ϕ_1 that takes as input a final pair $(W, S_1') = (w \overset{A}{\rightsquigarrow} v,\ \cdot \xrightarrow{S_1'} v)$ with $\mathrm{size}(W) < n-1$, and produces the final pair (W', S'), where $W \mapsto (W', C')$ as above and $S' = S_1' \cup C'$.

Example 7.4. Let $n = 5, l = 0$ and $W = ([2, -4, 5, 8, 4] \overset{\{3,5\}}{\rightsquigarrow} [2, -5, 6, 9, 3])$. We find $q = 4$ and $p = 6$. Then external insertion of (W, S_1'), where $S_1' = ([2, -5, 6, 9, 3] \longrightarrow [2, -5, 6, 9, 3])$ produces the final pair (W', S') where

$$W' = ([2, -4, 5, 8, 4] \overset{\{3,4,5\}}{\rightsquigarrow} [2, -5, 4, 11, 3]),$$

and $S' = ([2, -5, 6, 9, 3] \xrightarrow{-1,3} [2, -5, 4, 11, 3])$.

7.2. **Definition of $\phi_{u,v}$.** Fix $u, v \in \tilde{S}_n$. We define the value of $\phi_{u,v}$ on $(W, S, e) \in \mathcal{I}^\circ_{u,v}$ as the result of a sequence of steps. Each step, which is either an internal or external insertion, takes a final pair and produces another.

We start with the final pair $(W^{(0)}, S^{(0)}) = (W, v \longrightarrow v)$ where $S^{(0)}$ is the empty strong strip from $v = \mathrm{outside}(W)$ to itself. Iteratively, for $1 \leq k \leq m = \mathrm{size}(S)$, perform the internal insertion on the final pair $(W^{(k-1)}, S^{(k-1)})$ at C_k, and let $(W^{(k)}, S^{(k)})$ be the resulting final pair. The result of this sequence of internal insertions is the final pair $(W^{(m)}, S^{(m)})$. We now perform e external insertions. For $m < k < m + e$ define

$$(W^{(k)}, S^{(k)}) = \phi_1(W^{(k-1)}, S^{(k-1)}).$$

We define $\phi_{u,v}(W, S, e) = (W', S') := (W^{(m+e)}, S^{(m+e)})$ to be the final pair produced by this process.

7.3. **Proofs for the local rule.** We now establish the well-definedness of the local rule $\phi_{u,v}$ and some of its properties.

7.3.1. *Case X.* By construction we have a commutative diagram

$$\begin{array}{ccc} w & = & u \\ A \Big\downarrow W & & W' \Big\downarrow A' \\ v & \xrightarrow[a,b]{C'} & x \end{array} \tag{7.4}$$

where $(a, b) = (v^{-1}(q), v^{-1}(p))$. By Lemma 7.5 applied to S_1', S' is a strong strip, finishing this case.

Lemma 7.5. *In Case X let $W \mapsto (W', C')$. Then for any strong strip S such that* outside$(S) = v$, $S' = S \cup C'$ *is a strong strip.*

Proof. If size$(S) = 0$ then S' is automatically a strong strip. Otherwise let last$(S) = (\cdot \xrightarrow{i,j} v)$. By the maximality of q, we have $v(i) \leq q$ since otherwise there would exist a pair of consecutive A-bad integers r and s such that $r < v(i) < s$ with $v^{-1}(s) > l \geq i$, which would contradict Lemma 4.10 for the weak strip $(w \overset{A}{\rightsquigarrow} v)$. Therefore $m(C') = p > q \geq v(i) = m(\text{last}(S))$, so S' is a strong strip. □

By Lemma 7.5, each external insertion sends a final pair to a final pair, preserves the inside permutations of both the weak and strong strip, and adds one to the sizes of the weak and strong strips. Thus to check that $\phi_{u,v}$ is well-defined, we may reduce to the case $e = 0$ where no external insertions are required.

7.3.2. *For internal insertion cases.* We want to compute $\phi_{u,v}(W, S, 0)$ with size$(S) = m$. By induction we may assume that all of the internal insertions have been performed except the last, which computes the internal insertion on $(W^{(m-1)}, S^{(m-1)})$ at C_m, resulting in $(W^{(m)}, S^{(m)})$. To avoid the proliferation of subscripts and superscripts we change notation, forgetting the global meaning of u, v, W, S. We denote this last internal insertion step as the internal insertion on (W, S_1') at C, resulting in (W', S'). We write

$$(W, C) = (w \overset{A}{\rightsquigarrow} v,\ w \xrightarrow{i,j} u)$$

$$\tag{7.5} \begin{array}{ll} S: & \cdots \xrightarrow[C^{--}]{i^{--},j^{--}} w^- \xrightarrow[C^-]{i^-,j^-} w \xrightarrow[C]{i,j} u \\ S_1': & \cdots \xrightarrow{a_1^-,b_1^-} y \xrightarrow{a_1,b_1} v \\ S': & \cdots \xrightarrow[C'^{--}]{a^{--},b^{--}} y \xrightarrow[C'^-]{a^-,b^-} v' \xrightarrow[C']{a,b} x \end{array}$$

$$(W', C') = (u \overset{A'}{\rightsquigarrow} x,\ v' \xrightarrow{a,b} x)$$

We use the following induction hypothesis.

Property 7.6.

(i) (a) $m(C') < m(C)$ in Case B and (b) $m(C') = c_{A'}(m(C)))$ in Cases A and C.

(ii) $x(b) < m(C)$.

(iii) Case C cannot be preceded by Case B, and if Case C holds then $i^- = i$.

(iv) The final pair (W', C') commutes in Cases A and C and does not commute in Case B.

7.3.3. *Case A.*

$$\begin{array}{ccccc} \cdots & \xrightarrow{i^-,j^-} & w & \xrightarrow{i,j} & u \\ & & {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A} \\ \cdots & \xrightarrow[a_1,b_1]{} & v & \dashrightarrow_{i,j} & x \end{array} \tag{7.6}$$

By Lemma 4.14, $(u \overset{A}{\rightsquigarrow} x,\ v \xrightarrow{i,j} x)$ is a commuting final pair.

Lemma 7.7. *In Case A, Property 7.6 is satisfied.*

Proof. We have $m(C') = x(i) = c_A u(i) = c_A(m(C))$, which proves (i). (ii) is equivalent to $c_A w(i) < w(j)$, but this follows by Lemma 4.15. (iv) was proved above. □

Lemma 7.8. *In Case A, S' is a strong strip.*

Proof. We use the notation in (7.6) where the top row gives S and the bottom row gives S'. By induction S'_1 is a strong strip, so we need only check that $m(C'^-) < m(C')$, that is, $v(a_1) < v(j)$.

Since S is a strong strip $w(i^-) < w(j)$. By induction, Property 7.6(i) asserts that either (a) $v(a_1) < w(i^-)$ or (b) $v(a_1) = c_A(w(i^-))$. Suppose (a) holds. By (4.3), $v(j) = c_A(w(j))) \geq w(j) - 1 \geq w(i^-) > v(a_1)$, as desired.

Suppose (b) holds. Then $i^- = a_1$, since $v(i^-) = c_A w(i^-) = v(a_1)$. In particular $w(a_1) < w(j)$.

We shall assume that $v(a_1) \geq v(j)$ and derive a contradiction. If $v(a_1) = v(j)$ then we have the contradiction $l \geq a_1 = j > l$. So we assume $v(a_1) > v(j)$. In other words $w(a_1) < w(j)$ is inverted by c_A. By Lemma 4.9, $w(a_1)$ is A-nice, $c_A w(a_1) \geq w(j)$, and $c_A w(j) = w(j) - 1$. By induction, Property 7.6(ii) gives $v(b_1) < w(i^-) = w(a_1) \leq w(j) - 1 = v(j)$. So

$$v(b_1) < v(j) < v(a_1). \tag{7.7}$$

Next we have

$$a_1 \leq l < b_1 < j. \tag{7.8}$$

This follows from $l < j$ and $j \notin (a_1, b_1)$, which follows from Lemma 3.2 with the strong cover $C'^- = (\cdot \xrightarrow{a_1,b_1} v)$ and (7.7). We have

$$v(b_1) < v(i) < v(j) < v(a_1). \tag{7.9}$$

This is obtained from (7.7) using Lemma 3.2 for the strong cover $C' = (v \xrightarrow{i,j} \cdot)$ and $i \leq l < b_1 < j$. We have

$$i < a_1 \leq l < b_1 < j \tag{7.10}$$

since $i \notin (a_1, b_1)$ by Lemma 3.2 for the strong cover C'^-. We have

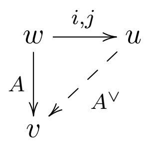
$$w(a_1) < w(i) < w(j). \tag{7.11}$$

This follows from Lemma 3.2 for the cover C, $i < a_1 \leq l < j$, and $w(a_1) < w(j)$.

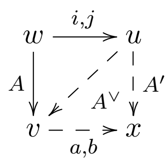
Now c_A inverts $w(a_1) < w(j)$. By Lemma 4.9 and (7.11), c_A inverts $w(a_1) < w(i)$ as well. With (7.10), Lemma 4.11 gives a contradiction. □

7.3.4. *Case B.* We first sketch the proof of Case B and fill in the proofs afterwards. Since (W, C) does not commute, by Lemma 4.16, $u \overset{A^\vee}{\rightsquigarrow} v$ is a weak strip such that the diagram commutes:

(7.12)



Recall that in Case B, $(a, b) = (u^{-1}(q), u^{-1}(p))$, $A' = A^\vee \cup \{\overline{p-1}\}$, and $x = c_{A'}u$. Using Lemma 7.10 we may apply Lemma 4.17, which says that $(u \overset{A'}{\rightsquigarrow} x,\ v \overset{a,b}{\longrightarrow} x)$ is a noncommuting final pair such that the diagram commutes.



Lemma 7.12 shows that S' is a strong strip, completing the proof sketch for Case B.

Lemma 7.9. *In Case B, let*

$$q' = u(i) - n, \tag{7.13}$$

$$p' = u(j). \tag{7.14}$$

Then

$$u^{-1}(q') \leq l < u^{-1}(p'), \tag{7.15}$$

$$0 < p' - q' < n, \tag{7.16}$$

q' is $A^\vee$-nice, p' is $A^\vee$-nice and A-nice, and $|A^\vee| = |A| - 1$.

Proof. q' is $A^\vee$-nice by definition. p' is A-nice and $A^\vee$-nice by Lemma 4.15.

We have $p' - q' = u(j) - u(i) + n = w(i) - w(j) + n$. The second inequality in (7.16) holds since C is a strong cover. Lemma 4.15 says that $w(i) < w(j)$ are inverted by c_A, and in particular, $w(j) = u(i)$ is not A-nice and $w(j) - w(i) < n$, which implies that $|A^\vee| = |A| - 1$ and the first inequality in (7.16).

By straddling, $u^{-1}(q') = i - n < i \leq l < j = u^{-1}(p')$, so (7.15) holds. □

Lemma 7.10. *In Case B, q is well-defined and*

$$q' = u(i) - n \leq q < p \leq u(j) = p' \tag{7.17}$$

$$u^{-1}(q) \leq l < u^{-1}(p) \tag{7.18}$$

Moreover $|A'| = |A|$*,* q *is* A'*-nice and* p *is* A*-nice.*

Proof. q is well-defined, is $A^\vee$-nice, and satisfies the above inequalities by Lemma 7.9, which assures that q is the maximum of a set that contains q'. p is A-nice and satisfies the above inequalities, also thanks to Lemma 7.9. q is A'-nice since it is $A^\vee$-nice and too close to p and distinct from p to be congruent to p, by (7.17) and (7.16). Since p is $A^\vee$-nice it follows that $|A'| = |A^\vee| + 1 = |A|$. □

Lemma 7.11. *In Case B Property 7.6 holds.*

Proof. Property 7.6(iv) was already proved above. Note that if $m(C') < m(C)$ then Property 7.6(ii) follows immediately: we have $x(b) < x(a) = m(C') < m(C)$ since C' is a strong cover.

To prove Property 7.6(i), we have

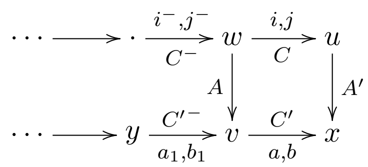
$$m(C') = x(u^{-1}(q)) = c_{A'}(q) = c_{A^\vee}(p) = \begin{cases} u(i) - 1 & \text{if } p = u(j) \\ c_A(p) & \text{otherwise.} \end{cases} \tag{7.19}$$

The first two equalities hold by definition. The third follows from (4.2). For the last equality, $u(j)$ is A-nice by Lemma 4.15. If $p = u(j)$, then by Lemma 4.15, $u(i)$ is the next $A^\vee$-nice integer after $u(j)$, and $c_{A^\vee}(p) = u(i) - 1$ by (4.2). Otherwise by (7.17) $p < u(j)$ and the cyclic component of $\overline{p}$ is the same in A and $A^\vee$. Either way the last equality holds.

For $p = u(j)$, (7.19) implies that $m(C') < u(i) = m(C)$. For $p < u(j)$, since $u(j)$ is A-nice, by (7.19) and the fact that C is a strong cover, (4.2) gives $m(C') = c_A(p) < u(j) < u(i) = m(C)$ as desired. □

Lemma 7.12. *In Case B,* S' *is a strong strip.*

Proof. We use the notation in the diagram below, where the top row is S and the bottom row is S'.



By induction it suffices to show that $m(C'^-) < m(C')$. Since $v = c_{A^\vee} u$ (see (7.12)) this is equivalent to

$$c_{A^\vee}(u(a_1)) < c_{A^\vee}(p). \tag{7.20}$$

Since p is $A^\vee$-nice it suffices to show that

$$u(a_1) < p. \tag{7.21}$$

Since S is a strong strip we have

$$w(i^-) < w(j). \tag{7.22}$$

The cases of Property 7.6(i) lead in (a) to $w(i^-) > c_A w(a_1) \geq w(a_1) - 1$ from (4.2), and in (b) to $c_A w(a_1) = c_A w(i^-)$. We thus find that

$$w(a_1) \leq w(i^-). \tag{7.23}$$

Suppose that $\overline{a_1} \notin \{\bar{i}, \bar{j}\}$. Then $u(a_1) \leq w(i^-)$. We claim that

$$u(a_1) < u(j). \tag{7.24}$$

If $i^- = i$ then since Case B has $i \neq a_1$ this implies $u(a_1) < w(i) = u(j)$ and (7.24) holds. So we assume $i^- \neq i$. Suppose $w(i^-) > w(i)$. Since C is a strong cover and $i^- < l$, by Lemma 3.2 and (7.22) we have $i^- < i \leq l < j$. By Lemma 4.15(iv) and (4.2), c_A inverts $w(i) < w(i^-)$ since $w(i) < w(i^-) < w(j)$, thus contradicting Lemma 4.11. Therefore $w(i^-) < w(i)$, which, together with (7.23) yields (7.24).

Let q_1 be the maximum $A^\vee$-nice integer such that $q_1 \leq u(a_1)$. By Lemma 4.10, $u^{-1}(q_1) \leq a_1 \leq l$. By the definition of q and (7.24), we have $q \geq q_1$. But $q < p$ and p is $A^\vee$-nice so (7.21) holds.

Next suppose $\overline{a_1} = \bar{j}$. Let $a_1 = j + kn$ for some $k \in \mathbb{Z}$. Since $a_1 \leq l < j$ we have $k < 0$ and we have $u(a_1) = u(j) + kn \leq u(j) - n < u(i) - n < p$ by (7.17), proving (7.21).

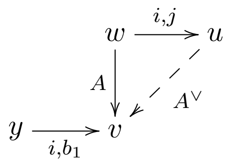
Finally suppose $\overline{a_1} = \bar{i}$ but $a_1 \neq i$. Let $a_1 = i + kn$ for some $k \neq 0$. By (7.22) and (7.23) we have $w(a_1) < w(j)$, that is, $w(i) + kn < w(j)$. Now $0 < w(j) - w(i) < n$ by Lemma 4.15, so $k < 0$. We have

$$u(a_1) = u(i) + kn \leq u(i) - n < p$$

by (7.17), proving (7.21). □

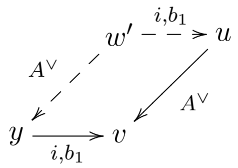
7.3.5. *Case C.* We first sketch the overall strategy for Case C using diagrams and fill in the proofs afterwards. By Lemma 4.16, $u \overset{A^\vee}{\rightsquigarrow} v$ is a weak strip such that the diagram commutes.

(7.25)

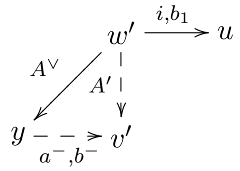


By Lemma 7.15, $(u \overset{A^\vee}{\rightsquigarrow} v,\ y \xrightarrow{i,b_1} v)$ is a commuting final pair. Let $w' = u\, t_{i,b_1} = c_{A^\vee}^{-1} y$. By Lemma 4.14, $(w' \overset{A^\vee}{\rightsquigarrow} y,\ w' \xrightarrow{i,b_1} u)$ is a commuting initial pair.

(7.26)

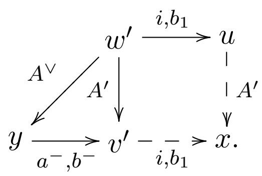


Let $(a^-, b^-) = (y^{-1}(q), y^{-1}(p))$ and $v' = t_{qp}\, y = y\, t_{a^-,b^-}$. Using Lemma 7.16 we may apply Lemma 4.18, which shows that $(w' \overset{A'}{\rightsquigarrow} v',\ y \overset{(a^-,b^-)}{\longrightarrow} v')$ is a noncommuting final pair such that the diagram commutes.



Let $x = c_{A'} u = v'\, t_{i,b_1}$. By Lemma 7.18 the initial pair $(w' \overset{A'}{\rightsquigarrow} v',\ w' \overset{i,b_1}{\longrightarrow} u)$ commutes, so that by Lemma 4.14 we have a commuting final pair $(u \overset{A'}{\rightsquigarrow} x,\ v' \overset{i,b_1}{\longrightarrow} x)$.

(7.27)



This shows that the definition of S' produces a strong tuple. Lemma 7.20 shows that S' is a strong strip, completing the argument for Case C.

Lemma 7.13. *In Case C, Property 7.6(iii) holds.*

Proof. Case B could not have occurred in the previous step for otherwise we would have $c_A w(i) = v(i) = m(C'^-) < m(C^-) < m(C) = w(j)$, contradicting Lemma 4.15.

So the previous step must be Case A or Case C. The input reflection for that step is (i^-, j^-). In Case A the output reflection is also (i^-, j^-), and in Case C it has the form $(i^-, \cdot)$. But the output reflection for this step is (i, b_1) so $i^- = i$. □

Lemma 7.14. *In Case C,*

(*i*) If $\bar{j} \neq \bar{b}_1$ then $p_0 = y(j)$.

(*ii*) Otherwise $j = b_1 + kn$ with $k > 0, p_0 = y(i) + kn$ and $b_1 - i < n$.

Furthermore, letting

$$q_0 = y(b_1) - n\,, \tag{7.28}$$

we have

$$0 < p_0 - q_0 < n \tag{7.29}$$

$$l < y^{-1}(p_0) \tag{7.30}$$

Proof. Since $\mathrm{last}(S_1')$ is a marked strong cover, we have

$$y(i) < y(b_1) \qquad \text{and} \qquad i \le l < b_1. \tag{7.31}$$

To prove (7.29), we have

$$y(b_1) = y\, t_{i,b_1}(i) = v(i) = c_A w(i) = c_A u(j) \tag{7.32}$$

so that

$$q_0 = c_A u(j) - n. \tag{7.33}$$

Then $p_0 - q_0 = (u(i)-1) - (c_A u(j) - n) = n - 1 - (c_A u(j) - u(i))$ and (7.29) is equivalent to $0 \le c_A u(j) - u(i) < n-1$. But this follows from

$$c_A u(j) - n < u(j) < u(i) \le c_A u(j), \tag{7.34}$$

which holds by (4.2), the fact that C is a marked strong cover, and Lemma 4.15. So (7.29) holds.

We have

$$\begin{aligned} p_0 &= u(i) - 1 = c_A(u(i)) = c_A(w(j)) = v(j) \\ &= y\, t_{i,b_1}(j) = \begin{cases} y(i) + kn & \text{if } j = b_1 + kn \text{ for some } k \in \mathbb{Z}. \\ y(j) & \text{otherwise.} \end{cases} \end{aligned} \tag{7.35}$$

This gives that in case (i) $p_0 = y(j)$, which yields (7.30).

Otherwise, we are in case (ii), with $j = b_1 + kn$ for some $k \in \mathbb{Z}$ and $p_0 = y(i) + kn$. We thus need to show that $k > 0$ and $b_1 - i < n$. By (7.34) and (7.32) we have $y(b_1) - n < u(i) - 1 = p_0$. Substituting for p_0 using (7.35) we have $y(b_1) - n < y(i) + kn$, which gives that $k \ge 0$ since $y(i) < y(b_1)$. Now, suppose $k = 0$ so that $j = b_1$ and $p_0 = y(i)$. By Property 7.6(iii) and Lemma 7.13, Case A must have occurred in some step and Case C must have occurred in every step thereafter. Let $(C_1, C_2, \dots, C_r = C)$ be the corresponding sequence of marked strong covers in S (starting with the above instance of Case A) with $C_k = (u_{k-1} \xrightarrow{i,j_k} u_k)$, where the first index of the reflection is always i. The Case A step had the same output reflection as its input reflection (i, j_1). In the output strong strip, the applications of Case C inserted reflections just before this reflection, which stayed at the end, so that $(i, j_1) = (i, j)$. Now, since C_1 is a strong cover we have $u_1(i) > u_1(j)$. And since each C_k for $k > 1$ is a strong cover we have $u_k(i) > u_{k-1}(i) > u_{k-1}(j) \ge u_k(j)$, using $\bar{j} \ne \bar{i}$. This contradicts $C_r = (u_{r-1} \xrightarrow{i,j} u_r)$ being a strong cover and proves that $k > 0$ in case (ii).

Since $k > 0$, we have from Lemma 4.19

$$v(b_1) \le v(j) - n < v(j) < v(i) < v(j) + n\,.$$

We thus have $|v(i) - v(b_1)| > n$, which yields $b_1 < i + n$ from Lemma 3.2. Therefore, (ii) holds.

Finally, given that $p_0 = y(i) + kn$ with $k > 0$ implies $y^{-1}(p_0) = i + kn > b_1 > l$, we have that (7.30) follows. □

Lemma 7.15. *In Case C,*

$$u(i) > u(b_1). \tag{7.36}$$

Proof. By Lemma 7.13 the previous step had to be Case A or Case C, and the final pair ($w \overset{A}{\rightsquigarrow} v$, $y \xrightarrow{i,b_1} v$) it produced was commuting, by Property 7.6(iv). By Lemma 4.19, $w(i) > w(b_1)$. By Lemma 7.14, we have either (i) $\bar{b}_1 \neq \bar{j}$ or (ii) $j = b_1 + kn$, with both implying $u(j) > u(b_1)$. Since C is a strong cover, $u(i) > u(j)$, so that $u(i) > u(b_1)$ as desired. □

Lemma 7.16. *In Case C, q and p are well-defined and satisfy*

$$q_0 \leq q < p \leq p_0 \tag{7.37}$$

$$y^{-1}(q) \leq l < y^{-1}(p). \tag{7.38}$$

Moreover q is A-bad and p is A'-bad.

Proof. Given the other assertions, it is easy to see that q is A-bad from (7.29), and that p is A'-bad.

Sublemma 7.17. *There is an $A^\vee$-bad integer q such that*

$$q_0 \leq q < p_0 \tag{7.39}$$

$$y^{-1}(q) \leq l. \tag{7.40}$$

This suffices: if q is maximal and p is the next $A^\vee$-bad integer after q, then p satisfies (7.38) by the choice of q and (7.37) holds by (7.30). □

Proof of Sublemma 7.17. Note that

$$y(i) = v(b_1) < m(C^-) = w(i^-) = w(i). \tag{7.41}$$

by Property 7.6(ii) applied to the previous step, and Lemma 7.13.

By Lemma 3.2 applied to the strong cover $y \lessdot y\, t_{i,b_1}$ we have either (a) $b_1 - i < n$ or (b) $y(b_1) - y(i) < n$. Suppose (a) holds. We shall show that $q = q_0$ satisfies Sublemma 7.17. We have $q_0 = v(i) - n = c_A w(i) - n$. By Lemma 4.15, $w(i)$ is A-nice. By Lemma 4.8, q_0 is A-bad and $A^\vee$-bad. Clearly $q = q_0$ satisfies (7.39). We have $y^{-1}(q_0) = b_1 - n < i \leq l$ using (a) and (7.31), so that $q = q_0$ satisfies (7.40).

Now suppose (b) holds. Let q' be the minimum $A^\vee$-bad integer such that $q' \geq y(i)$. We shall show that $q = q'$ satisfies Sublemma 7.17. We have

$$q_0 = y(b_1) - n < y(i) \leq q' \leq w(i) - 1 < p_0 < y(b_1) \tag{7.42}$$

by assumption (b), the definition of q', the fact that $w(i) - 1$ is $A^\vee$-bad and greater or equal to $y(i)$ by (7.41), and (7.29). It suffices to show

$$y^{-1}(q') \leq l. \tag{7.43}$$

If $y(i)$ is $A^\vee$-bad, then $q' = y(i)$ and (7.43) follows. So we may assume that $y(i)$ is not $A^\vee$-bad and $y(i) < q'$. Since $w' \overset{A^\vee}{\rightsquigarrow} y$ is a weak strip (see (7.26)), applying Lemma 4.10 to y, the $A^\vee$-bad integer q' and the next smaller one, we obtain $y^{-1}(q') < i \leq l$ and (7.43) follows. □

Lemma 7.18. *In Case C, $v'(i) < v'(b_1)$.*

Proof. It is equivalent to show that

$$t_{q,p}\, y(i) < t_{q,p}\, y(b_1). \tag{7.44}$$

We have

$$y(b_1) - n = q_0 \le q < p \le p_0 < y(b_1). \tag{7.45}$$

by (7.37) and (7.29). Using this one may deduce that

$$y(b_1) \le t_{q,p}(y(b_1)). \tag{7.46}$$

If $\overline{y(i)} \notin \{\overline{q}, \overline{p}\}$ then $t_{q,p}\, y(i) = y(i) < y(b_1)$ using (7.31). If $\overline{y(i)} \in \{\overline{q}, \overline{p}\}$ then $t_{q,p}\, y(i) \le p < y(b_1)$ by (7.45). Either way (7.44) holds by (7.46). □

Lemma 7.19. *In Case C, Property 7.6 holds.*

Proof. Property 7.6(iii) is Lemma 7.13.

We have $m(C') = x(i) = c_{A'}(u(i)) = c_{A'}(m(C))$, proving Property 7.6(i). For Property 7.6(ii), we must show that $t_{q,p}\, y(i) \le w(j) - 1 = p_0$. By the proof of Lemma 7.18, either $t_{q,p}y(i) = y(i)$ or $t_{q,p}y(i) \le p$. Using either (7.41) or (7.37) we have $t_{q,p}y(i) \le p_0$ as desired.

□

Lemma 7.20. *In Case C, S' is a strong strip.*

Proof. We use the following notation:

$$S: \qquad \cdots \xrightarrow[C^{--}]{} \cdot \xrightarrow[C^{-}]{i,j^-} w \xrightarrow[C]{i,j} u$$

$$S_1': \qquad \cdots \xrightarrow{a_1^-,b_1^-} y \xrightarrow{i,b_1} v$$

$$S': \qquad \cdots \xrightarrow[C'^{--}]{a_1^-,b_1^-} y \xrightarrow[C'^{-}]{a^-,b^-} v' \xrightarrow[C']{i,b_1} x$$

where $(a^-, b^-) = (y^{-1}(q), y^{-1}(p))$. We have $i^- = i$ by Lemma 7.19.

By induction and the definition of S', we need only check its last two pairs of consecutive marks. We have $m(C') = x(i) = c_{A'}\, u(i) > u(i) - 1 = p_0 \ge p = m(C'^-)$, using the fact that $u(i)$ is A'-nice (since $\bar{p}_0 \ne \bar{q}$ from (7.29) and Lemma 7.16) and (7.37). It remains to verify that $m(C'^-) > m(C'^{--})$, that is,

$$p > y(a_1^-). \tag{7.47}$$

Suppose that $y(a_1^-) \le p_0$. Since $y^{-1}(p_0) > l$ and $a_1^- \le l$ we have $y(a_1^-) < p_0$. Let $r < r'$ be consecutive $A^\vee$-bad integers such that $r \le y(a_1^-) < r'$. Since p_0 is $A^\vee$-bad we have $r' \le p_0$. If $r = y(a_1^-)$, then $y^{-1}(r) = a_1^- \le l$, and thus $p > q \ge r = y(a_1^-)$ by the maximality of q. Otherwise, by Lemma 4.10 for the weak strip $w' \overset{A^\vee}{\rightsquigarrow} y$ we have $y^{-1}(r') < a_1^- \le l$. As before $r' < p_0$. Therefore by the definition of p and q, $p > q \ge r' > y(a_1^-)$ as desired.

Otherwise $p_0 < y(a_1^-)$. Since S_1' is a strong strip, its last two marks satisfy $y(a_1^-) < y(b_1)$. By (7.41),

$$y(i) < w(i) \leq p_0 < y(a_1^-) < y(b_1). \tag{7.48}$$

By Lemma 3.2 applied to the strong cover $y \lessdot y\, t_{i,b_1}$, we have

$$a_1^- < i \tag{7.49}$$

since $a_1^- \leq l < b_1$. We claim that

$$v(j) < v(a_1^-) < v(i). \tag{7.50}$$

According to Lemma 7.14, we have two cases to consider. First, suppose that Lemma 7.14(i) holds. We have $y(b_1) = vt_{i,b_1}(b_1) = v(i)$. Since t_{ij} is a reflection and $\bar{b}_1 \neq \bar{j}$, $y(j) = v(j)$. Therefore (7.48) gives $v(j) < y(a_1^-) < v(i)$. We claim that $\overline{a_1^-} \not\in \{\bar{i}, \overline{b_1}\}$. Suppose $\overline{a_1^-} = \bar{i}$. By (7.48) $y(i) < y(a_1^-)$, which implies that $i < a_1^-$, contradicting (7.49). Suppose $\overline{a_1^-} = \overline{b_1}$. Write $a_1^- = b_1 + kn$. By (7.48) $y(a_1^-) < y(b_1)$ so $k < 0$. Then since $\bar{b}_1 \neq \bar{j}$, we have

$$v(j) = y(j) < y(a_1^-) = y(b_1) + kn \leq y(b_1) - n = v(i) - n.$$

But $w \overset{A}{\rightsquigarrow} v$, $w \xrightarrow{i,j} u$ is a noncommuting pair so by Lemma 4.15 $w(j) - w(i) < n$, which leads to the contradiction $v(i) - v(j) < n$. Therefore our claim holds. Consequently $y(a_1^-) = v(a_1^-)$ and (7.50) holds in this case.

Now suppose that Lemma 7.14(ii) holds. We have $y(b_1) = v(i)$ and $p_0 = v(j)$ from (7.35). Since we still have $v(i) - v(j) < n$, we obtain $y(b_1) - p_0 < n$, and so $\overline{a_1^-} \not\in \{\bar{i}, \overline{b_1}\}$ holds. This leads again to (7.50).

Finally, by Lemma 4.15, we know that there are no A-nice integer in the interval $(w(i), w(j)]$ and thus by Lemma 4.8, that there are no A-bad integer in the interval $[v(j), v(i))$. Letting r be the A-bad integer immediately before $v(i)$, we have from (7.50) that $r < v(a_1^-) < v(i)$. Hence, from Lemma 4.10 applied to the weak strip $w \overset{A}{\rightsquigarrow} v$ and the consecutive A-bad integers r and $v(i)$, that $i < a_1^-$, contradicting (7.49). ☐

8. Reverse local rule

We describe an algorithm to compute the inverse $\psi = \psi_{u,v} : \mathcal{O}^\circ_{u,v} \to \mathcal{I}^\circ_{u,v}$ of the local rule $\phi_{u,v}$ defined in the previous section. With the proof of Lemma 6.1 it defines reverse affine insertion, the inverse of the map $\Phi_{u,v}$ of Theorem 5.1.

8.1. **Reverse insertion at a cover.** Let $C' = (v \xrightarrow{a,b} x)$ be a marked strong cover. *Reverse insertion at* C' is a map that takes as input an initial pair $(W', S_1) = (u \overset{A'}{\rightsquigarrow} x,\ u \xrightarrow{S_1} \cdot)$ such that $\text{outside}(W') = \text{outside}(C')$, and produces an initial pair of the form $(W, S) = (w \overset{A}{\rightsquigarrow} v,\ w \xrightarrow{S} \cdot)$, such that $\text{outside}(S) = \text{outside}(S_1)$ and $\text{outside}(W) = \text{inside}(C')$.

There are four cases, RA, RB, RC, and RX, which denote the inverses of cases A, B, C, and X in the forward insertion algorithm.

We only need to specify A, (i,j), and S, because $w = c_A^{-1} v$ and $C = \mathrm{first}(S) = (w \xrightarrow{i,j} \cdot)$. If $\mathrm{size}(S_1) > 0$ let

$$S_1 : \qquad u \xrightarrow{i_1,j_1} z \xrightarrow{i_1^+,j_1^+} \cdots \tag{8.1}$$

8.1.1. *Commuting case.*

Case RA (Commuting Case) If (W', C') commutes, then we set $A = A'$, $(i,j) = (a,b)$, and $S = C \cup S_1$.

Example 8.1. Let $n = 4, l = 0$ and $C' = ([4,6,-3,3] \xrightarrow{-2,1} [2,8,-3,3])$. Consider the initial pair (W', S_1) where $W' = ([1,8,-2,3] \overset{\{1\}}{\rightsquigarrow} [2,8,-3,3])$ and $S_1 = ([1,8,-2,3] \longrightarrow [1,8,-2,3])$ has size 0. Since the pair (W', C') commutes, we obtain the output initial pair (W, S) where

$$W = ([4,5,-2,3] \overset{\{1\}}{\rightsquigarrow} [4,6,-3,3])$$

and $S = ([4,5,-2,3] \xrightarrow{-2,1} [1,8,-2,3])$.

8.1.2. *Noncommuting cases.* In the rest of the cases we assume that (W', C') does not commute. By Lemma 4.19, $\overline{u(b)-1} \in A'$ and we set

$$A^\vee = A' - \{\overline{u(b)-1}\}. \tag{8.2}$$

We say that condition B (not to be confused with case B) holds if there exists an $A^\vee$-nice integer q such that $u(b) < q$ and $u^{-1}(q) \le l$. If condition B holds let q_B be the minimal such q.

Say that condition C holds if $\mathrm{size}(S_1) > 0$ (so that (i_1, j_1) is defined) and $u(j_1)$ is $A^\vee$-nice. If condition C holds let $q_C = u(j_1)$; in this case it follows that $u(b) < u(j_1)$ (see Lemma 8.11).

Note that if conditions B and C both hold then $q_B \neq q_C$ since $j_1 > l$ by the straddling condition.

Case RX Suppose condition B does not hold and $\mathrm{size}(S_1) = 0$ (and so in particular condition C does not hold). We set $w = u$, $A = A^\vee$, and $S = (w \longrightarrow w)$.

Example 8.2. Let $n = 5, l = 0$ and $C' = ([2,-5,6,9,3] \xrightarrow{-1,3} [2,-5,4,11,3])$. Consider the initial pair (W', S_1) where

$$W' = ([2,-4,5,8,4] \overset{\{3,4,5\}}{\rightsquigarrow} [2,-5,4,11,3])$$

and $S = ([2,-4,5,8,4] \longrightarrow [2,-4,5,8,4])$ is empty. Then (W', C') does not commute and we have $A^\vee = \{3,5\}$. Neither condition B nor condition C holds and we have the output initial pair is (W, S) where $W = ([2,-4,5,8,4] \overset{\{3,5\}}{\rightsquigarrow} [2,-5,6,9,3])$ and $S = S_1$.

Case RB Suppose condition B holds, and, in addition, either condition C does not hold or $q_B < q_C$. Set $q = q_B$ and let p be the maximum $A^\vee$-nice integer such that $p < q$. Let $A = A^\vee \cup \{\overline{q-1}\}$, $(i,j) = (u^{-1}(q), u^{-1}(p))$, and $S = C \cup S_1$.

Example 8.3. Let $n = 6, l = 0$ and

$$C' = ([4,-1,1,12,-3,8] \xrightarrow{\{0,1\}} [2,-1,1,12,-3,10]).$$

Consider the initial pair (W', S_1) given by $W' = ([3,0,1,11,-2,8] \overset{\{2,3,5\}}{\rightsquigarrow} [2,-1,1,12,-3,10])$ and $S_1 = ([3,0,1,11,-2,8] \longrightarrow [3,0,1,11,-2,8])$. The pair (W', C') does not commute and we have $A^\vee = \{3,5\}$. Condition B holds with $q = q_B = 5$ and $p = 3$. Thus $A = \{3,4,5\}$, $(i,j) = (-2,1)$ and the output initial pair (W,S) is given by $W = ([5,0,1,9,-2,8] \overset{\{3,4,5\}}{\rightsquigarrow} [4,-1,1,12,-3,8])$ and $S = ([5,0,1,9,-2,8] \xrightarrow{-2,1} [3,0,1,11,-2,8])$.

Case RC Suppose condition C holds, and, in addition, either condition B does not hold or $q_C < q_B$. Set $q = q_C$ and let p be the maximum $A^\vee$-nice integer such that $p < q$. We set $A = A^\vee \cup \{\overline{q-1}\}$, $(i,j) = (i_1, j_1)$, and S is obtained by inserting $(i_1, z^{-1}(p))$ into S_1 *after* the first reflection.

Example 8.4. Let $n = 4, l = 0$ and $C' = (v = [4,5,-2,3] \xrightarrow{-2,7} [4,6,-3,3])$. Consider the initial pair (W', S_1) given by $W' = (u = [4,5,-2,3] \overset{\{1\}}{\rightsquigarrow} [4,6,-3,3])$ and $S_1 = ([4,5,-2,3] \xrightarrow{-2,1} z = [1,8,-2,3])$. The pair (W', C') does not commute. We have $u(b) = 2$ and $A^\vee = \emptyset$. Condition B does not hold but Condition C holds with $q_C = u(1) = 4$. We have $p = 3$, $A = \{3\}$ and $z^{-1}(3) = 4$ giving us the output initial pair (W,S) where $W = ([3,5,-2,4] \overset{\{3\}}{\rightsquigarrow} [4,5,-2,3])$ and

$$S = ([3,5,-2,4] \xrightarrow{-2,1} [1,7,-2,4] \xrightarrow{-2,4} [1,8,-2,3]).$$

8.2. **The reverse local rule.** The reverse algorithm applied to a final pair $(W', S') \in \mathcal{O}^\circ_{u,v}$ consists of $m' := \mathrm{size}(S')$ steps, one for each cover in S'. The each step of the reverse algorithm is called a reverse insertion. Each reverse insertion takes as its input an *initial* pair and produces another as output. Write $S' = (C'_1, C'_2, \ldots, C'_{m'})$. We initialize $(W_{(m')}, S_{(m')}) = (W', u \longrightarrow u)$, where $u = \mathrm{inside}(W')$. For k going from m' down to 1, we compute the reverse insertion at C'_k on the initial pair $(W_{(k)}, S_{(k)})$ (which has the property that $(W_{(k)}, C'_k)$ is a final pair), which produces an initial pair $(W_{(k-1)}, S_{(k-1)})$ such that $\mathrm{outside}(S_{(k-1)}) = u$. Let $\psi_{u,v}(W', S') = (W, S, e)$ where $(W,S) = (W_{(0)}, S_{(0)})$ and $e = \mathrm{size}(S') - \mathrm{size}(S)$ is the number of times Case RX occurred.

8.3. **Proofs for the reverse insertion.** We want to compute $\psi_{u,v}(W', S')$ with $\mathrm{size}(S') = m'$. By induction we may assume that all of the reverse insertions have been performed except the last step, which computes the reverse insertion on $(W_{(m'-1)}, S_{(m'-1)})$ at $C'_{m'}$, resulting in $(W_{(m')}, S_{(m')})$. Again we change notation, forgetting the global meaning of u, v, W', S'. We denote this last reverse insertion step as the reverse insertion on (W', S_1) at C', resulting in (W, S). We write

$$
\begin{array}{llc}
 & (W', C') = (u \overset{A'}{\rightsquigarrow} x,\ v \xrightarrow{a,b} x) & \\
 & S' : & v \xrightarrow[C']{a,b} x \xrightarrow[C'^{+}]{a^+,b^+} x^+ \xrightarrow[C'^{++}]{a^{++},b^{++}} \cdots \\
(8.3) & S_1 : & u \xrightarrow{i_1,j_1} z \xrightarrow{i_1^+,j_1^+} \cdots \\
 & S : & w \xrightarrow[C]{i,j} u' \xrightarrow[C^+]{i^+,j^+} z \xrightarrow[C^{++}]{i^{++},j^{++}} \cdots \\
 & (W, C) = (w \overset{A}{\rightsquigarrow} v,\ w \xrightarrow{i,j} u') &
\end{array}
$$

In Cases RA, RB, and RC we have $\text{size}(S) > 0$ so that $C = \text{first}(S)$ is well-defined. In Case RX we make the convention that $u' = w$ and write $(W, \varnothing)$ instead of (W, C) where $\varnothing = (w \longrightarrow w)$ is the empty strong strip going from w to itself.

The following inductive hypothesis will be useful. In each case it must be re-established.

Property 8.5.

(i) In Case RA, $m(C) = c_{A'}^{-1}(m(C'))$. In Case RB, $m(C) > m(C')$. In Case RC, $m(C) \geq c_{A^\vee}^{-1}(m(C'))$.
(ii) $m(C) > x(b)$.
(iii) Case RC cannot be preceded by Case RB.
(iv) The initial pair (W, C) commutes in Cases RA and RC and does not in Case RB.

8.3.1. *Case RA.* By Lemma 4.14 $(w \overset{a,b}{\rightsquigarrow} u,\ w \xrightarrow{A'} v)$ is a commuting initial pair such that the diagram commutes.

$$
\begin{array}{ccc}
w & \overset{a,b}{\dashrightarrow} & u \\
{\scriptstyle A'}\downarrow & & \downarrow{\scriptstyle A'} \\
v & \underset{a,b}{\longrightarrow} & x
\end{array}
$$

By Lemma 8.7 S is a strong strip.

For the proofs below, for Case RA we specialize the general notation of (8.3) as follows.

$$
\begin{array}{llc}
(8.4) & S : & w \xrightarrow[C]{a,b} u \xrightarrow[C^+]{i_1,j_1} z \longrightarrow \cdots \\
(8.5) & S' : & v \xrightarrow[C']{a,b} x \xrightarrow[C'^{+}]{a^+,b^+} x^+ \longrightarrow \cdots
\end{array}
$$

Lemma 8.6. *In Case RA, Property 8.5 holds.*

Proof. (iv) was proved above. For (i) we have $m(C) = u(a) = c_{A'}^{-1}(x(a)) = c_{A'}^{-1}(m(C'))$. For (ii) we need to show $x(b) < u(a)$. Since $v \xrightarrow{a,b} x$ and $w \xrightarrow{a,b} u$ are strong covers, we have $x(b) < x(a)$ and $u(b) < u(a)$. If $u(b)$ is not A'-nice or is A'-bad then by (4.3), $x(b) = c_{A'}(u(b)) \leq u(b) < u(a)$ as desired. Otherwise by Lemma 4.9, $x(b) < u(a)$ as desired. □

Lemma 8.7. *In Case RA, S is a strong strip.*

Proof. We use the notation (8.4). Since S_1 is a strong strip by induction and $S = C \cup S_1$ we need only show that

$$u(a) = m(C) < m(C^+) = u(j_1). \tag{8.6}$$

It suffices to show that

$$u(j_1) > x(a), \tag{8.7}$$

since $x(a) = c_{A'}(u(a)) \geq u(a) - 1$ and $a \leq l < j_1$. Since S' is a strong strip,

$$x(a) = m(C') < m(C'^+) = x(b^+). \tag{8.8}$$

We apply Property 8.5(i) to the previous step. Suppose the previous step was Case RA. Then $j_1 = b^+$ and $x(a) < x(j_1)$. Since $a \leq l < j_1$, (8.6) holds by Lemma 4.11 applied to the weak strip $u \rightsquigarrow x$. Suppose the previous step was Case RB. Then $m(C^+) > m(C'^+) > m(C')$ gives (8.7). Suppose the previous step was Case RC. We have

$$x(b^+) - x(a^+) < n \tag{8.9}$$

by Lemma 4.19 applied to the noncommutative final pair ($z \rightsquigarrow x^+$, $x \xrightarrow{a^+,b^+} x^+$) at the beginning of the previous step.

Suppose first that $\overline{a} = \overline{a^+}$. Let $a = a^+ + kn$. By (8.8) and (8.9) it follows that $k \leq 0$. Then $x(a) = x(a^+) + kn = x^+(b^+) + kn \leq x^+(b^+) < m(C^+)$, the last step holding by Property 8.5(ii) applied to the previous step. This gives (8.7).

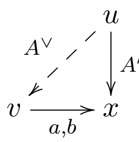
Suppose next that $\overline{a} = \overline{b^+}$. Since $a \leq l < b^+$ it follows that $a = b^+ + kn$ for some $k < 0$. Then, since (8.9) gives $x^+(a^+) - x^+(b^+) < n$, we have

$$x(a) = x(b^+) + kn = x^+(a^+) + kn \leq x^+(b^+) + (k+1)n \leq x^+(b^+) < m(C^+),$$

by Property 8.5(ii).

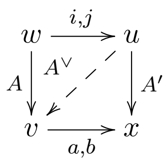
Finally suppose that $\overline{a} \notin \{\overline{a^+}, \overline{b^+}\}$. Then $m(C') = x(a) = x^+(a)$. Suppose $x^+(a) \geq x^+(b^+)$. Equality cannot hold since $a \leq l < b^+$ so $x^+(a) > x^+(b^+)$. By (8.8) $x^+(a^+) > x^+(a)$. By Lemma 4.10 for the weak strip $z \rightsquigarrow x^+$ and the noncommutativity of the final pair ($z \rightsquigarrow x^+$, $x \xrightarrow{a^+,b^+} x^+$), we have $a^+ < a$. But this contradicts Lemma 3.2 for the strong cover $x \xrightarrow{a^+,b^+} x^+$. □

8.3.2. *Reverse noncommuting cases.* By Lemma 4.20, $u \overset{A^\vee}{\rightsquigarrow} v$ is a weak strip such that the diagram commutes.



8.3.3. *Case RX.* In this case we have $w = u$ and $A = A^\vee$ so that $W = (w \overset{A}{\rightsquigarrow} v) = (u \overset{A^\vee}{\rightsquigarrow} v)$ is a weak strip.

8.3.4. *Case RB.* Thanks to Lemma 8.8, we may apply Lemma 4.21, which says that $(w \overset{A}{\rightsquigarrow} v,\ w \xrightarrow{i,j} u)$ is a noncommuting initial pair such that the diagram commutes.



This case is finished by Lemma 8.10 which shows that S is a strong strip.

Lemma 8.8. *Suppose condition B holds. Let $q = q_B$ and p be the maximum $A^\vee$-nice integer such that $p < q$. Then*

$$p_0 := u(b) \le p < q \le u(a) + n \tag{8.10}$$

$$u^{-1}(q) \le l < u^{-1}(p). \tag{8.11}$$

Proof. p_0 is $A^\vee$-nice by construction. Also $u^{-1}(p_0) = b > l$. Since $q > p_0$ it follows that $p \ge p_0$ and that (8.11) holds. For the upper bound on q, suppose $q > u(a) + n$. Then $q' := q - n > u(a)$ is $A^\vee$-nice. Since (W', C') does not commute, $u(a) < u(b)$ and by Lemma 4.19 these are consecutive $A^\vee$-nice integers. Therefore $q' \ge u(b)$. Also $u^{-1}(q') = u^{-1}(q) - n \le l$, contradicting the minimality of q. This proves the upper bound in (8.10). ☐

For the proofs below, for Case RB we specialize the general notation of (8.3) as follows.

$$S: \qquad w \xrightarrow[C]{i,j} u \xrightarrow[C^+]{i_1,j_1} \cdots$$

$$S': \qquad v \xrightarrow[C']{a,b} x \xrightarrow[C'^+]{a^+,b^+} \cdots$$

where $(i, j) = (u^{-1}(q), u^{-1}(p))$.

Lemma 8.9. *In Case RB, Property 8.5 holds.*

Proof. (iv) was already shown. (ii) follows from (i) since $m(C) > m(C') = x(a) > x(b)$, the last inequality holding because $v \xrightarrow{a,b} x$ is a strong cover. For (i) it is equivalent to show that $q > c_{A'}(u(a))$. By (8.10) the only difference between $A^\vee$-niceness and A'-niceness for integers in the interval $[u(a), u(a)+n]$, is that $u(b)$ is $A^\vee$-nice but not A'-nice. Since (W', C') does not commute, by Lemma 4.19, $u(a)$ is A'-nice and the minimum A'-nice integer r such that $r > u(a)$, satisfies $r > u(b)$. By (8.10) we see that q is A'-nice and $q > u(b)$. Therefore $r \le q$ and from (4.2), $c_{A'}(u(a)) = r - 1 < q$ as desired. □

Lemma 8.10. *In Case RB, S is a strong strip.*

Proof. By induction and the construction of S, it suffices to show that $m(C) < m(C^+)$.

We have $m(C) = q < u(j_1) = m(C^+)$, which follows from Lemma 8.12, as $u(i_1) < u(j_1)$ holds since $C^+ = (u \xrightarrow{i_1,j_1} \cdot)$ is a strong cover. □

Lemma 8.11. *Suppose* $\mathrm{size}(S_1) > 0$ *so that (i_1, j_1) is defined. Then*

$$u(b) \le x(a) < u(b^+) \le u(j_1). \tag{8.12}$$

In particular $j_1 \ne b$.

Proof. By Lemma 4.19, $u(a)$ is A'-nice and the first inequality in (8.12) holds. Let S' be as in (8.3). Since S' is a strong strip we have

$$c_{A'}u(a) = x(a) = m(C') < m(C'^+) = x(b^+) = c_{A'}u(b^+).$$

Since $u(a)$ is A'-nice and $c_{A'}$ is cyclically decreasing, it follows that the second inequality in (8.12) holds. By Property 8.5 (ii) for the previous step we have $u(j_1) > x(b^+) \ge u(b^+) - 1$, proving the last inequality in (8.12). □

Lemma 8.12. *Suppose that S_1 is nonempty so that (i_1, j_1) is defined. Let $q = q_B$ if Case RB holds and $q = q_C$ if Case RC holds. Let p be the maximum $A^\vee$-nice integer with $p < q$. Then*

(1) *If $u(b) < u(i_1)$ then Case RB holds and*

$$u(b) \le p < q \le u(i_1). \tag{8.13}$$

(2) *Otherwise we have $u(i_1) < u(b) < u(j_1)$ and*

$$u(b) \le p < q \le u(j_1). \tag{8.14}$$

Moreover, whenever Case RB holds we have $q \le u(a) + n$.

Proof of Lemma 8.12. Since $\mathrm{first}(S_1) = (u \xrightarrow{i_1,j_1} \cdot)$ is a strong cover, $u(i_1) < u(j_1)$. Lemma 8.11 implies that either $u(b) < u(i_1)$ or $u(i_1) < u(b) < u(j_1)$.

For either (8.13) or (8.14) when Case RB holds, the argument that $q \le u(a) + n$ is the same as in the proof of Lemma 8.8.

Suppose $u(b) < u(i_1)$. Suppose there is no $A^\vee$-nice integer q' such that $u(b) < q' \le u(i_1)$. Applying Lemma 4.10 to the weak strip $u \overset{A^\vee}{\rightsquigarrow} v$ and the $A^\vee$-nice integer $u(b)$ and the next larger one (which is greater than $u(i_1)$), we have $b < i_1$ which contradicts the straddling inequality $i_1 \le l < b$. So let q' be the maximum $A^\vee$-nice integer such that $u(b) < q' \le u(i_1)$. If $u(i_1)$ is $A^\vee$-nice then $q' = u(i_1)$ and $u^{-1}(q') = i_1 \le l$. Otherwise $u(i_1)$ is not $A^\vee$-nice and $q' < u(i_1)$. Let p' be the maximum $A^\vee$-nice integer with $p' < q'$. Applying Lemma 4.10 to the weak strip $u \overset{A^\vee}{\rightsquigarrow} v$ and consecutive $A^\vee$-nice integers $p' < q'$, we have $u^{-1}(q') \le i_1 \le l$. Therefore Case B holds.

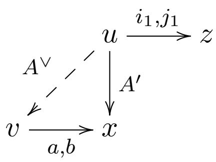
Suppose $u(i_1) < u(b) < u(j_1)$. Let q' be the maximum $A^\vee$-nice integer with $q' \le u(j_1)$; it exists and satisfies $u(i_1) < q'$ since $u(b)$ is $A^\vee$-nice.

If $q' = u(j_1)$ then either Case RB with $q < q'$ will hold or Case RC will hold for $q = q'$. (8.14) follows.

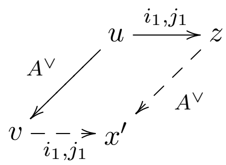
Now suppose $q' < u(j_1)$. By Lemma 4.10 applied to W', we have $u^{-1}(q') < j_1$. But by Lemma 3.2 applied to $\mathrm{first}(S_1)$ we must have $u^{-1}(q') < i_1 \le l$. Therefore q' is a witness that Condition B holds. (2) follows. □

8.3.5. *Case RC.* Again we sketch the proof and then fill in the Lemmata which prove the details.

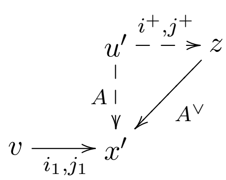
By Lemma 4.20, $u \overset{A^\vee}{\rightsquigarrow} v$ is a weak strip such that the diagram commutes:



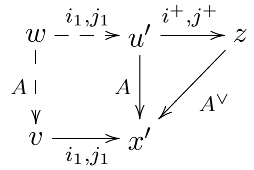
By Lemma 8.13, $(u \overset{A^\vee}{\rightsquigarrow} v,\ u \xrightarrow{i_1,j_1} z)$ is a commuting initial pair. Let $x' = v\, t_{i_1,j_1} = c_{A^\vee} z$. By Lemma 4.14 $(z \overset{A^\vee}{\rightsquigarrow} x',\ v \xrightarrow{i_1,j_1} x')$ is a commuting final pair such that the diagram commutes.



Recall that $q_C = u(j_1) = z(i_1)$. Let $(i^+, j^+) = (i_1, z^{-1}(p))$ and $u' = z\, t_{i^+,j^+} = t_{p,q_C} z$. By Lemma 8.14 we may apply Lemma 4.21 to the weak strip $z \overset{A^\vee}{\rightsquigarrow} v$ and pair $p < q_C$ of consecutive $A^\vee$-nice integers, so that $(u' \overset{A}{\rightsquigarrow} x',\ u' \xrightarrow{i^+,j^+} z)$ is a noncommuting initial pair such that the diagram commutes.



By Lemma 8.15 the final pair $(u' \overset{A}{\rightsquigarrow} x,\ v \xrightarrow{i_1,j_1} x)$ commutes. Define $w = u'\,t_{i_1,j_1} = c_A^{-1}(v)$. By Lemma 4.14, $(w \overset{A}{\rightsquigarrow} v,\ w \xrightarrow{i_1,j_1} u')$ is a commuting initial pair such that the diagram commutes.



In particular, as defined, S is a strong tuple and W is a weak strip. Lemma 8.17 shows that S is a strong strip.

Lemma 8.13. *In Case RC,* $v(i_1) < v(j_1)$*.*

Proof. Since $u \xrightarrow{i_1,j_1} z$ is a strong cover, $u(i_1) < u(j_1)$. Now $v = c_{A^\vee} u$ and $u(j_1)$ is $A^\vee$-nice since Condition C holds. It follows that $v(i_1) < v(j_1)$. ☐

Lemma 8.14. *In Case RC,* $z^{-1}(p) > l$*.*

Proof. Suppose not, that is, $z^{-1}(p) \le l$. By Lemma 8.12, $u(b) < u(j_1) = q_C$. Now $p < q_C$ are consecutive $A^\vee$-nice integers. Since $u(b)$ is also $A^\vee$-nice it follows that $u(b) \le p$. Suppose $u(b) = p$. Since $z^{-1}(p) = t_{i_1 j_1} u^{-1}(p) = u^{-1}(p)$ given that $u(j_1) - u(i_1) < n$ and $u(i_1) < p < u(j_1)$ from Lemma 4.15 on the non-commuting initial pair $(u \overset{A'}{\rightsquigarrow} x,\ u \xrightarrow{i_1,j_1} z)$, we have the contradiction $l < b = u^{-1}(u(b)) = u^{-1}(p) = z^{-1}(p) \le l$. So $u(b) < p$. But then condition B is satisfied by the integer p which is less than q_C, meaning that Case RB holds, which is a contradiction. ☐

Lemma 8.15. *In Case RC,* $u'(i_1) > u'(j_1)$*.*

Proof. Since $u'(i_1) = t_{p,q_C} z(i_1) = t_{p,q_C}(q_C) = p$, and $u'(j_1) = t_{p,q_C} z(j_1) = t_{p,q_C} u(i_1)$, this is equivalent to

$$p > t_{p,q_C} u(i_1). \tag{8.15}$$

Since we are in Case RC, by Lemma 8.12 we have $u(i_1) < u(b) \le p < q_C = u(j_1)$. If $\overline{u(i_1)} \notin \{\overline{p}, \overline{q_C}\}$ then (8.15) clearly holds. Otherwise $u(i_1) \in \{p+kn, q_C+kn\}$ for some integer $k < 0$. Then $t_{p,q_C} u(i_1) \in \{p+kn, q_C+kn\}$. Then (8.15) follows from $p+kn < q_C+kn \le q_C - n < p$, the last inequality holding since p and q_C are consecutive $A^\vee$-nice integers. ☐

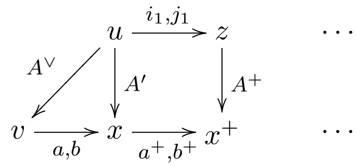
For the rest of the proofs we use the following notation for Case RC.

$$\begin{array}{lll} S' : & v \xrightarrow[C']{a,b} x \xrightarrow[C'^+]{a^+,b^+} x^+ \xrightarrow[C'^{++}]{a^{++},b^{++}} \cdots \\ S_1 : & u \xrightarrow{i_1,j_1} z \xrightarrow{i_1^+,j_1^+} \\ S : & w \xrightarrow[C]{i_1,j_1} u' \xrightarrow[C^+]{i_1,j^+} z \xrightarrow[C^{++}]{i_1^+,j_1^+} \end{array} \tag{8.16}$$

where $j^+ = z^{-1}(p)$.

Lemma 8.16. *Property 8.5 holds in Case RC.*

Proof. We use the notation (8.16). Since $m(C) = u'(i_1) = p$, as seen above equation (8.15), (i) is equivalent to $p \geq u(b)$ which holds by Lemma 8.12. (ii) then follows since the non-commutativity of the final pair $(u \overset{A'}{\rightsquigarrow} v,\ u \xrightarrow{a,b} x)$ implies from Lemma 4.19 that $u(b)$ is not A'-nice, and thus that $x(b) = c_{A'}u(b) = u(b) - 1 < p$ from (4.2). (iv) was shown above. For (iii), suppose Case RC was preceded by Case RB. We use the following diagram.



By assumption the final pairs $(W', C') = (u \overset{A'}{\rightsquigarrow} x,\ v \xrightarrow{a,b} x)$ and $(W^+, C'^+) = (z \overset{A^+}{\rightsquigarrow} x^+,\ x \xrightarrow{a^+,b^+} x^+)$ both do not commute.

We will see that it suffices to show that

(8.17) There is an integer $i' \leq l$ such that $u(b) < u(i') < u(j_1)$.

Let i' be as in (8.17). Then $u(i')$ is not $A^\vee$-nice; otherwise Condition B would hold for $u(i')$. Let $r < r'$ be the pair of consecutive $A^\vee$-nice integers such that $r < u(i') < r'$. Since $u(b)$ and $u(j_1)$ are $A^\vee$-nice (the latter by the Case RC assumption) we have $u(b) \leq r < u(i') < r' \leq u(j_1)$. By Lemma 4.10 for the weak strip $u \overset{A^\vee}{\rightsquigarrow} v$ we have $u^{-1}(r) < i' \leq l$. This gives the contradiction that Condition B holds for r if $u(b) < r$. But if $u(b) = r$ then $l < b = u^{-1}(u(b)) = u^{-1}(r) \leq l$, a contradiction.

We now prove (8.17). Since S' is a strong strip,

$$x(b) < x(a) < x(b^+). \tag{8.18}$$

Suppose $a \neq a^+$. We claim that (8.17) holds for $i' = a^+$. By Property 8.5(ii) for the previous step, we have $u(j_1) > x^+(b^+) = x(a^+) \geq u(a^+) - 1$, that is, $u(j_1) \geq u(a^+)$. But $a^+ \leq l < j_1$ so $u(j_1) > u(a^+)$, giving the right hand inequality in (8.17) for $i' = a^+$. We are done if $u(b) < u(a^+)$. Suppose not. Since $a^+ \leq l < b$ we must have $u(b) > u(a^+)$. We claim that

$$x(a^+) < x(b) < x(a) < x(b^+). \tag{8.19}$$

Since (W', C') is a noncommuting final pair, by Lemma 4.19, $u(a)$ is A'-nice, $u(b)$ is not A'-nice, and $x(a) \geq u(b) > u(b) - 1 = x(b)$. If $u(a) < u(a^+)$ then $u(a^+)$ is not A'-nice and $x(a^+) = u(a^+) - 1 < u(b) - 1 = x(b)$ so that (8.19) holds. If $u(a^+) < u(a)$ then since $u(a)$ is A'-nice, $x(a^+) < u(a) \leq x(b)$ and again (8.19) holds. By Lemma 3.2 applied to the strong cover C'^+ we have $a < a^+$. Now $0 < x(b^+) - x(a^+) < n$ by Lemma 4.19 for the noncommuting final pair (W^+, C'^+). Therefore (8.19) gives

$x^+(b^+) < x^+(b) < x^+(a) < x^+(a^+)$. By Lemma 4.10 for the weak strip W^+ we have $a^+ < a$, a contradiction.

Suppose $a = a^+$. We claim that (8.17) holds for $i' = i_1$. Since $u \xrightarrow{i_1,j_1} z$ is a strong cover we have $u(i_1) < u(j_1)$, the right hand inequality in (8.17) for $i' = i_1$. It suffices to show that

$$u(i_1) = z(j_1) \geq z(b^+) > z(b^+) - 1 = x^+(b^+) = x(a^+) = x(a) \geq u(b).$$

The first inequality holds by (8.13) for the previous (Case RB) step. The second equality holds by Lemma 4.19 for the final pair (W^+, C'^+). The last inequality holds by Lemma 4.19 for the noncommutative final pair (W', C'). □

Lemma 8.17. *In Case RC, S is a strong strip.*

Proof. By the construction of S and the fact that S_1 is a strong strip by induction, we need only show that

$$m(C) < m(C^+) < m(C^{++}).$$

The second inequality holds because $m(C^+) = z(i^+) = z(i_1) = q_C = m(\mathrm{first}(S_1)) < m(C^{++})$, the inequality holding by the strong strip condition of S_1. Given that $m(C) = p$, the first inequality holds since $p < q_C$ follows from Lemma 8.12. □

9. Bijectivity

Our main theorem is the following.

Theorem 9.1. *The maps ψ and ϕ are inverses to each other. Thus the map ϕ is a bijection.*

Our approach to proving bijectivity is to reduce to the case of at most two steps, exploiting the fact that the maps ϕ and ψ can be "factorized" into "smaller" instances of ϕ and ψ, to which induction may be applied.

Consider the sequence of steps involved in computing $\phi(W, S, e)$. With respect to this sequence, we call a subsequence of consecutive steps *irreducible* if it consists of:

(1) a Case X step.
(2) a Case A step followed by some maximum number m (possibly zero) of consecutive Case C steps.
(3) a Case B step.

Mnemonically we denote such irreducible sequences by $\mathcal{X}$, $\mathcal{A}(m)$, and $\mathcal{B}$ respectively.

Dually, consider the sequence of steps involved in computing $\psi(W', S')$. With respect to this sequence, we call a subsequence of consecutive steps *irreducible* if it consists of:

(1) a Case RX step.

(2) a Case RA step followed by some maximum number m (possibly zero) of Case RC steps.
(3) a Case RB step.

Denote these mnemonically by $\mathcal{X}^{-1}$, $\mathcal{A}(m)^{-1}$ and $\mathcal{B}^{-1}$ respectively. As a warning, note that, for example, when $m = 2$ the inverse of Case A followed by two Case Cs, is Case RA followed by two Case RCs.

It is clear that every sequence has a unique factorization into irreducible subsequences.

We shall show in Subsection 9.1 that if the forward algorithm ends with $\mathcal{X}$ then $\psi \circ \phi = \mathrm{id}$ and if the reverse algorithm begins with $\mathcal{X}^{-1}$ then $\phi \circ \psi = \mathrm{id}$.

In Subsection 9.2 we show that if the forward algorithm starts with $\mathcal{A}(0)$ or the reverse algorithm ends with $\mathcal{A}^{-1}(0)$ then we have bijectivity.

In Subsection 9.3 we show bijectivity when the forward algorithm begins with $\mathcal{B}$ and when the reverse algorithm ends with $\mathcal{B}^{-1}$.

In Subsection 9.4 we show bijectivity when the forward algorithm begins with $\mathcal{A}(m)$ and when the reverse algorithm ends with $\mathcal{A}(m)^{-1}$ for $m > 0$. This is accomplished by reducing to the case that $m = 1$.

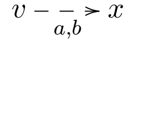
This covers all cases, so Theorem 9.1 follows.

In each of the following subsections, for the proof of $\psi \circ \phi = \mathrm{id}$, we suppose that $(W, S, e) \in \mathcal{I}^{\circ}_{u,v}$ and denote $(W', S') = \phi(W, S, e)$ and $\mathcal{S}$ for the sequence of steps in this computation. For the proof of $\phi \circ \psi = \mathrm{id}$ we suppose $(W', S') \in \mathcal{O}^{\circ}_{u,v}$, write $(W, S, e) = \psi(W', S')$ and by abuse of notation write $\mathcal{S}^{-1}$ to denote the sequence of reverse insertion steps in the computation of (W, S, e).

9.1. **External insertion.** For $\psi \circ \phi$, suppose $\mathcal{S}$ has the form $\mathcal{S} = \mathcal{S}_1 \mathcal{X}$, that is, it ends with $\mathcal{X}$. The last (Case X) step depends only on the weak strip entering that step, and $\mathcal{S}_1$ is itself a "smaller" instance of ϕ. By induction we may reduce to the case that $\mathcal{S} = \mathcal{X}$ consists of a single Case X step.

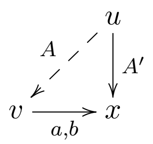
For $\phi \circ \psi$, suppose $\mathcal{S}^{-1}$ has the form $\mathcal{S}^{-1} = \mathcal{X}^{-1} \mathcal{S}_1^{-1}$. The first step, which is Case RX, depends only on the last cover in S'. The second step will not be Case RC, so $\mathcal{S}_1^{-1}$ is a smaller instance of ψ. By induction we may assume that $\mathcal{S}^{-1} = \mathcal{X}^{-1}$, so that S' consists of a single cover C'.

So for $\psi \circ \phi$, let (W, S) be a final pair such that $W = (w \overset{A}{\rightsquigarrow} v)$ with $|A| < n - 1$ and $S = (v \longrightarrow v) = \varnothing$ the empty strong strip. Let $W \mapsto (W', C')$ be the result of external insertion with $(W', C') = (u \overset{A'}{\rightsquigarrow} x,\ v \xrightarrow{a,b} x)$, $A' = A \cup \{\overline{q}\}$ and $(a, b) = (v^{-1}(q), v^{-1}(p))$ where $q < p$ are the pair of consecutive A-bad integers such that $v^{-1}(q) \le l$ and q is maximal. By construction the final pair (W', C') is noncommuting.



Applying the first step of ψ to (W', C'), we cannot be in Cases RA or RC. By Lemma 4.15, $u(b)$ is not A'-nice and thus $u(b) - 1 = c_{A'}(u(b)) = x(b) = v(a) = q$. Therefore $A^\vee$ as defined in (8.2), equals our A. By Lemma 4.8, $u(q')$ is A-nice if and only if $v(q') = c_A(u(q'))$ is A-bad. So condition B fails due to the maximality of q in the external insertion. Therefore Case RX occurs and it produces the original weak strip W as desired.

For $\phi \circ \psi$, let $S' = C'$ be a single strong cover and $(W', C') = (u \overset{A'}{\rightsquigarrow} x,\ v \xrightarrow{a,b} x)$ a final pair such that Case RX occurs. In particular (W', C') does not commute. We define $A = A^\vee$ as in (8.2) and have the commutative diagram with weak strip $W = (u \overset{A}{\rightsquigarrow} v)$.



By Lemma 4.19 $u(a) < u(b)$ are consecutive $A^\vee$-nice integers and $u(b)$ is not A'-nice. We now apply external insertion. Let $q < p$ be consecutive $A^\vee$-bad integers such that $v^{-1}(q) \leq l$ with q maximum. We have $u(b) - 1 = x(b) = v(a)$ so it suffices to show that $q = v(a)$ and $p = v(b)$. By Lemma 4.8 $v(a) < v(b)$ are consecutive $A = A^\vee$-bad integers and $v^{-1}(v(a)) = a \leq l$. If there was an $A^\vee$-bad integer $q' > v(a)$ with $v^{-1}(q') \leq l$, then $q' > v(b)$ and $c_{A^\vee}^{-1}(q')$ is an $A^\vee$-nice integer greater than $u(b)$ with $u^{-1}(c_{A^\vee}^{-1}(q')) = v^{-1}(q') \leq l$, so that condition B holds, contradicting the assumption that Case RX occurs. Therefore $q = v(a)$ and $p = v(b)$ as desired.

9.2. **Case A (commuting case).** In the case that $\mathcal{S}$ (resp. $\mathcal{S}^{-1}$) consists of a single Case A (resp. RA) step, bijectivity holds by Lemma 4.14.

In general, for $\psi \circ \phi$, suppose $\mathcal{S}$ starts with a Case A step which is not followed by a Case C step. Write $\mathcal{S} = \mathcal{A}(0)\mathcal{S}_1$. Since the first step of $\mathcal{S}_1$ is not Case C by assumption, $\mathcal{S}_1$ is an instance of ϕ involving fewer steps. By induction we may assume that $\mathcal{S}$ is a single Case A step, since a single Case RA step is unaffected by the strong covers that were produced previously.

For $\phi \circ \psi$ we may similarly reduce to the single RA step case.

9.3. **Case B (bumping case):** For $\psi \circ \phi$ we first suppose that the entire sequence $\mathcal{S}$ is a single Case B step. Let $(W, C) = (w \overset{A}{\rightsquigarrow} v,\ w \xrightarrow{i,j} u)$ be a noncommuting initial pair. As in Case B we define $A^\vee = A - \{\overline{u(i) - 1}\}$ and let $q < p$ be the pair of consecutive $A^\vee$-nice integers such that $q < u(j)$ and $u^{-1}(q) \leq l$, with q maximum. Defining $A' = A^\vee \cup \{p - 1\}$, $(a, b) = (u^{-1}(q), u^{-1}(p))$ and $x = v\, t_{a,b} = c_{A'} u$, we have that $u \overset{A^\vee}{\rightsquigarrow} v$ is a weak strip and $(W', C') = (u \overset{A'}{\rightsquigarrow} x,\ v \xrightarrow{a,b} x)$ is a noncommuting final pair such that the diagram commutes:

$$\begin{array}{ccc} w & \xrightarrow{i,j} & u \\ {\scriptstyle A}\downarrow & {\scriptstyle A^\vee}\swarrow & \downarrow{\scriptstyle A'} \\ v & \xrightarrow[a,b]{} & x \end{array}$$

We now apply the reverse algorithm to (W', C'). In the single cover context Case RC does not occur, and (W', C') does not commute. By Lemma 4.19 $u(b) = p$ is not A'-nice. Then $A' - \{\overline{u(b) - 1}\} = A' - \{\overline{p - 1}\} = A^\vee$ so that the above definition of $A^\vee$ agrees with the one in the noncommutative case of the reverse algorithm. Now $q = u(a)$ satisfies $q < u(j)$ by its definition, so $u(a) < u(j)$. Since (W', C') is a noncommuting final pair, by Lemma 4.19 $u(a) < u(b)$ are consecutive $A^\vee$-nice integers. Since (W, C) is a noncommuting initial pair, by Lemma 4.15 $u(j) < u(i)$ are consecutive $A^\vee$-nice integers. Therefore we have $u(a) < u(b) \le u(j) < u(i)$. In particular condition B holds with the $A^\vee$-nice integer $u(i)$. So Case RB holds, and say it selects the consecutive $A^\vee$-nice integers $u(b) \le p_B < q_B \le u(i)$. If $q_B < u(i)$ then since $u(j) < u(i)$ are consecutive $A^\vee$-nice integers and $j > l$, it follows that $u(b) < q_B < u(j)$, contradicting the maximality of q in Case B. Therefore $q_B = u(i)$ and $p_B = u(j)$. It is now clear that $\psi \circ \phi = \mathrm{id}$.

We now return to the general case of $\psi\circ\phi$. Let $S = (\ w \xrightarrow{i,j} u \xrightarrow{i,j_1} \cdots\ )$. We know that the output of the first (Case B) step is a noncommuting final pair. By Property 7.6(iii) the second step of ϕ is not Case C. By induction we suppose that performing the rest of ϕ and then all of ψ except for the last step, is the identity. We now consider the last step of the reverse algorithm ψ; it is the noncommutative case. It suffices to show that Case RB occurs in this last step. By the single cover case we know that condition B holds with $q_B = u(i)$, so that Case RX cannot occur. Since S is a strong strip we have $u(i) < u(j_1)$ so that Case RC cannot hold. Therefore Case RB holds and we have reduced to the single cover case.

The reasoning for $\phi \circ \psi$ in the single cover case, is entirely similar to that for $\psi \circ \phi$ in the single cover case. Suppose now that the last step of ψ on $(W', S') \in \mathcal{O}^\circ$ is Case RB. After applying ψ to (W', S') we apply ϕ. Its first step is Case B and undoes the last step of ψ by the single cover case. The second step of ϕ cannot be Case C, and by induction the rest of ϕ is the inverse of the rest of ψ, and we are done.

9.4. **Case C (replacement bump).**

9.4.1. *Reduction to* $m = 1$. For $\psi \circ \phi$ suppose $\mathcal{S} = \mathcal{A}(m)\mathcal{S}_1$ with $m > 0$ maximal. Then $\mathcal{S}_1$ does not start with Case RC and the output strong cover of the last (Case C) step of $\mathcal{A}(m)$ is involved in a commuting final pair by Property 7.6(iv). It follows that the usual reduction works and we may assume that $\mathcal{S} = \mathcal{A}(m)$.

For $\phi \circ \psi$ suppose $\mathcal{S}^{-1} = \mathcal{S}_1^{-1}\mathcal{A}(m)^{-1}$ for $m > 0$, that is, ψ produces a sequence of steps that ends with Case RA followed by some positive number of Case RC steps. Since $\mathcal{S}_1^{-1}$ is followed by a Case RA step, it is a smaller instance of ψ. In particular its inverse cannot begin with Case C. Since Case RA steps are unaffected by previously produced strong covers, again by induction we may assume that $\mathcal{S} = \mathcal{A}(m)$.

Let us consider $\psi \circ \phi$ on $(W, S, 0)$ such that ϕ on $(W, S, 0)$ consists of a single Case A step followed by m Case C steps for some $m > 0$. By Subsection 9.1 the output (W', S') is such that ψ on (W', S') does not start with RX. We shall reduce to the case $m = 1$ by showing that ϕ can be achieved by splicing together two operations: the first two steps of ϕ, which consists of Case A followed by a Case C, and another application of ϕ which consists of a Case A step (the "second half" of the first Case C step) followed by $m - 1$ Case C steps.

Suppose $m > 1$. Let

$$S = (\ w \xrightarrow{i,j} u_0 \xrightarrow{i,j_1} u_1 \xrightarrow{i,j_2} \cdots \xrightarrow{i,j_m} u_m = u\ ).$$

Since S is a strong strip we have

$$u_0(i) < u_0(j_1) = u_1(i) < u_1(j_2). \tag{9.1}$$

Let $W = (w \overset{A}{\rightsquigarrow} v)$ and $\phi(W, S, 0) = (W', S')$; since this is a Case A step followed by m Case C steps, S' has the form

$$S' = [v; ((a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m), (i, j)); x].$$

The computation of $\phi(W, S, 0)$ starts with a two step computation

$$\phi(W, [w; ((i, j), (i, j_1)); u_1], 0) = (W_1, [v; ((a_1, b_1), (i, j)); x_1]),$$

where $W_1 = (u_1 \overset{A_1}{\rightsquigarrow} x_1)$. By Property 7.6(iv) applied to the Case C step, the final pair $(W_1, x_1\, t_{ij} \xrightarrow{i,j} x_1)$ commutes. By Lemma 4.14, $(u_1\, t_{ij} \overset{A^*}{\rightsquigarrow} x_1\, t_{ij},\ u_1\, t_{ij} \xrightarrow{i,j} u_1)$ is a commuting initial pair such that the diagram commutes.

$$\begin{array}{ccc} u_1\, t_{ij} & \xrightarrow{i,j} & u_1 \\ A^* \big\downarrow W^* & & \big\downarrow A_1 \\ x_1\, t_{ij} & \xrightarrow[i,j]{} & x_1 \end{array}$$

Let

$$\begin{aligned} W^* &= (u_1\, t_{ij} \overset{A^*}{\rightsquigarrow} x_1\, t_{ij}) \\ S^* &= [u_1\, t_{i,j}; ((i, j), (i, j_2), \ldots, (i, j_m)); u]. \end{aligned}$$

Due to (9.1) we see that (W^*, S^*) is an initial pair. It is clear from the definitions that

$$\phi(W^*, S^*) = (W', [x_1\, t_{ij}; ((a_2, b_2), \ldots, (a_m, b_m), (i, j)); x])$$

which is a Case A step followed by $m-1$ Case C steps. By induction we have $\psi \circ \phi(W^*, S^*) = (W^*, S^*)$. Thus we have reduced to the case that $m = 1$.

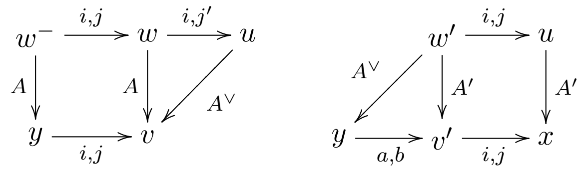
In the case that ψ on $(W', S') \in \mathcal{O}^\circ_{u,v}$ consists of an RA step followed by $m > 1$ RC steps, we may apply a similar reduction to the $m = 1$ case.

9.4.2. $m = 1$. So we assume $m = 1$. For $\psi \circ \phi$, we start with the initial pair (W, S) where

$$W = (w^- \overset{A}{\rightsquigarrow} y)$$

$$S = (\ w^- \xrightarrow{i,j} w \xrightarrow{i,j'} u\ ).$$

We give diagrams before and after the Case C step.



Here $A^\vee$ is defined by (7.3) and $A' = A^\vee \cup \{\overline{q}\}$ where $(a, b) = (y^{-1}(q), y^{-1}(p))$ and $q < p$ is the consecutive pair of $A^\vee$-bad integers such that $q < u(i) - 1$ and $u^{-1}(q) \leq l$ with q maximum. We have $S' = (\ y \xrightarrow{a,b} v' \xrightarrow{i,j} x\ )$ and $W' = (u \overset{A'}{\rightsquigarrow} x)$. By Property 7.6(iv), the final pair $(u \overset{A'}{\rightsquigarrow} x,\ v' \xrightarrow{i,j} x)$ commutes. We now apply ψ. The first step is the commuting step RA, which produces the commuting initial pair $(w' \overset{A'}{\rightsquigarrow} v',\ w' \xrightarrow{i,j} u)$. We now apply the next reverse insertion step. By the Case C construction, the final pair $(w' \overset{A'}{\rightsquigarrow} v',\ y \xrightarrow{a,b} v')$ is noncommuting. In particular by Lemma 4.19, $w'(b)$ is not A'-nice and

$$w'(b) - 1 = c_{A'} w'(b) = v'(b) = y(a) = q\,. \tag{9.2}$$

In this situation the reverse insertion algorithm defines

$$A^\vee = A' - \{\overline{w'(b) - 1}\} = A' - \{\overline{q}\}$$

which agrees with $A^\vee$ as defined above.

Condition C holds since $w'(j) = u(i)$ is $A^\vee$-nice by definition. If Case C holds, then it is easy to verify bijectivity: we have $q_C = w'(j) = u(i)$, the previous $A^\vee$-nice integer is indeed $p_C = u(j')$ by Lemma 4.15 for the initial pair $(w \overset{A}{\rightsquigarrow} v,\ w \xrightarrow{i,j'} u)$ for the forward direction, and the definition of A for the current RC step agrees with the above definition of A, as both are obtained from $A^\vee$ by adding the element $q_C - 1 = u(i) - 1$.

So it suffices to show that Case C holds, that is, there is no $A^\vee$-nice integer q_B such that $w'(b) < q_B < w'(j)$ and $g := w'^{-1}(q_B) \leq l$. Suppose such an integer exists. We must derive a contradiction.

We have

$$p_0 = u(i) - 1 = w'(j) - 1\,. \tag{9.3}$$

We claim that $q' = y(g)$ contradicts the definition of q in Case C. By definition $y^{-1}(q') = g \le l$. We have that $y(g) = c_{A^\vee}(w'(g)) = c_{A^\vee}(q_B)$. Since $q < q_B - 1 < u(i) - 1 = p_0$ are all $A^\vee$-bad integers, it follows that $q_B - 1 < y(g)$ are consecutive $A^\vee$-bad integers and that $y(g) \le p_0$. Furthermore $y(g) < p_0 = u(i) - 1$ since $g \le l < y^{-1}(p_0)$ by (7.30). It only remains to show that $y(g) > q$. We have $\overline{y(g)} \notin \{\overline{q}, \overline{p}\}$, by Lemma 7.16 and the fact that $g \le l < y^{-1}(p)$. Therefore

$$y(g) = t_{qp} y(g) = t_{qp} c_{A^\vee}(q_B) = c_{A'}(q_B) \ge q + 1 > q$$

and we arrive at the desired contradiction.

For $\phi \circ \psi$, the argument is nearly the same. By the definition of Case RC, it is clear that in the subsequent calculation of ϕ, Case C will be invoked at the second step. The only thing which needs to be checked is the maximality of q in Case C, which follows from the fact that Case B does not hold in the second step of ψ.

10. Grassmannian elements, cores, and bounded partitions

Let $k = n - 1$ from now on. The k-Schur functions, denoted $s_u^{(k)}(x)$ for $u \in \tilde{S}_n^0$ in Theorem 5.10, are traditionally written $s_\lambda^{(k)}(x)$ where λ is a partition such that $\lambda_1 \le k$. Weak tableaux for Grassmannian elements were first introduced [16] as k-tableaux, which are defined in terms of $(k+1)$-cores. In this section we recall bijections between Grassmannian elements, offset sequences, cores, and bounded partitions.

Let $l = 0$ in this section. The set $\tilde{S}_n^0 = \tilde{S}_n^l$ of Grassmannian elements is defined in Subsection 3.3.

10.1. Translation elements.

Let Q be the coroot lattice of $\mathfrak{sl}_n$, realized as the set of n-tuples of integers with sum zero. S_n acts on Q by permuting coordinates. Given $\beta = (\beta_1, \dots, \beta_n) \in Q$, the translation element $\tau_\beta \in \tilde{S}_n$ is uniquely defined by $\tau_\beta(i) = i + n\beta_i$ for $1 \le i \le n$. We have $\tau_\beta \tau_\gamma = \tau_{\beta+\gamma}$, so that $T(Q) = \{\tau_\beta \mid \beta \in Q\}$ forms an abelian subgroup of $\tilde{S}_n$ isomorphic to Q. By (3.2) we have

$$\ell(\tau_\beta) = \sum_{1 \le i < j \le n} |\beta_j - \beta_i| \tag{10.1}$$

from which it follows that

$$\ell(\tau_\beta) = \ell(\tau_{u\beta}) \qquad \text{for } \beta \in Q \text{ and } u \in S_n \tag{10.2}$$

$$\ell(\tau_\beta) = 2 \sum_{i=1}^n i\beta_i = 2\langle \rho' \,,\, \beta \rangle \qquad \text{for } \beta \text{ antidominant,} \tag{10.3}$$

where $\rho' = (1, 2, \dots, n)$ and an antidominant element is one that is weakly increasing. The inner product $\langle .\,,\,. \rangle$ is the standard on on $\mathbb{R}^n$.

$T(Q)$ acts on Q by translations: $\tau_\beta(\gamma) = \beta + \gamma$ for all $\gamma \in Q$. S_n acts on $T(Q)$ by conjugation: $s_i \tau_\beta s_i = \tau_{s_i\beta}$ for $1 \le i \le n-1$ and $\beta \in Q$. There is a well-known isomorphism ([9, Prop. 6.5]) $\tilde{S}_n \cong S_n \ltimes T(Q)$ under which $s_0 \mapsto s_\theta \tau_{-\theta}$ where $\theta = (1, 0, \dots, 0, -1) \in Q$ is the highest coroot and s_θ is the associated reflection, which satisfies $s_\theta = t_{1,n} = s_1 s_2 \dots s_{n-2} s_{n-1} s_{n-2} \cdots s_2 s_1$ and acts on $\mathbb{Z}^n$ by $s_\theta(a_1, \dots, a_n) = (a_n, a_2, \dots, a_{n-1}, a_1)$.

Again using (3.2), for $v \in S_n$ and $\gamma \in Q$ we have

$$\ell(v\tau_\gamma) = \ell(\tau_\gamma) + \sum_{1 \le i < j \le n} \chi(v(i) > v(j))(-1)^{\chi(\gamma_i < \gamma_j)} \tag{10.4}$$

where $\chi(S) = 1$ if S is true and $\chi(S) = 0$ if S is false.

Example 10.1. Let $n = 4$ and

$$w = s_1 s_2 s_3 s_0 s_3 s_2 s_1 s_0 s_3 s_2 s_0 s_3 s_1 s_0 = [-7, -1, 4, 14].$$

Then $w s_3 s_2 = [-7, 14, -1, 4]$. In $\mathbb{Z}^4$ we have the equality $(-7, 14, -1, 4) = (1, 2, 3, 4) + 4(-2, 3, -1, 0)$, so $w s_3 s_2 = t_{-2,3,1,0}$ and $w = t_{-2,3,-1,0} s_2 s_3 = s_2 s_3 t_{-2,-1,0,3}$. By (10.4) we have $\ell(w) = -2 + \ell(t_{-2,-1,0,3}) = -2 + 2(1(-2) + 2(-1) + 3(0) + 4(3)) = 14$, matching the length of the above reduced word.

For $\beta \in Q$ let $S_n^\beta \subset S_n$ denote its stabiliser.

Proposition 10.2. (1) *If $\beta \in Q$ is antidominant and $u \in S_n$ is of minimum length in its coset uS_n^β then*

$$\ell(u\tau_\beta) = \ell(\tau_\beta) - \ell(u). \tag{10.5}$$

(2) *Let u, β be as in (1). If $\gamma \in Q$ and $v \in S_n$ are such that $v\gamma = u\beta$ then $\ell(v\tau_\gamma) \ge \ell(u\tau_\beta)$ with equality if and only if $v = u$ and $\gamma = \beta$.*

Proof. Equation (10.4) says that $\ell(v\tau_\gamma) - \ell(\tau_\gamma)$ is the number of inversions (i, j) of v such that $\gamma_i \ge \gamma_j$, minus the number of inversions of v such that $\gamma_i < \gamma_j$. For u and β in the hypotheses, we have $\beta_i \le \beta_j$ and $\beta_j = \beta_i$ implies $u(i) < u(j)$. This is precisely the condition that the first set of inversions is empty and the second is the set of all inversions of u. This proves (1).

For (2) we must show that for $(v, \gamma) \in S_n \times (S_n \cdot \beta)$ such that $v\gamma = u\beta$, $\ell(v\tau_\gamma)$ is uniquely minimized by the pair (u, β).

First let $\gamma \in S_n \cdot \beta$ be fixed. Suppose for some $1 \le p < q \le n$, $\gamma_p = \gamma_q$ and $v(p) < v(q)$. Let $v' = vt_{p,q}$. We shall show that $\ell(v\tau_\gamma) < \ell(v'\tau_\gamma)$, which implies that the desired minimum element (v, γ) must have the property that v is of minimum length in the coset vS_n^γ. For $1 \le i < j \le n$ let

$$f(i, j) = (-1)^{\chi(\gamma_i < \gamma_j)}(\chi(v'(i) > v'(j)) - \chi(v(i) > v(j))) \tag{10.6}$$

so that using (10.4)

$$\ell(v'\tau_\gamma) - \ell(v\tau_\gamma) = \sum_{1 \le i < j \le n} f(i, j). \tag{10.7}$$

For (i,j) such that $\{i,j\} \cap \{p,q\} = \varnothing$, $f(i,j) = 0$. For (i,j) such that $i = p$ and $j \neq q$ we have $p < j$ and

$$\begin{aligned} f(p,j) &= (-1)^{\chi(\gamma_p < \gamma_j)}(\chi(v(q) > v(j)) - \chi(v(p) > v(j))) \\ &= (-1)^{\chi(\gamma_p < \gamma_j)}\chi(v(p) < v(j) < v(q)) \end{aligned} \tag{10.8}$$

For (i,j) such that $i = q$ we have $q < j$ and

$$\begin{aligned} f(q,j) &= (-1)^{\chi(\gamma_q < \gamma_j)}(\chi(v(p) > v(j)) - \chi(v(q) > v(j))) \\ &= -(-1)^{\chi(\gamma_p < \gamma_j)}\chi(v(p) < v(j) < v(q)) \end{aligned} \tag{10.9}$$

using $\gamma_q = \gamma_p$. We have

$$\begin{aligned} \sum_{\substack{j>p \\ j\neq q}} f(p,j) + \sum_{j>q} f(q,j) &= \sum_{p<j<q} f(p,j) \\ &= \sum_{p<j<q} (-1)^{\chi(\gamma_p < \gamma_j)}\chi(v(p) < v(j) < v(q)). \end{aligned} \tag{10.10}$$

Similarly

$$\sum_{i<p} f(i,p) + \sum_{\substack{i<q \\ i\neq p}} f(i,q) = \sum_{p<i<q} (-1)^{\chi(\gamma_i < \gamma_p)}\chi(v(p) < v(i) < v(q)). \tag{10.11}$$

The remaining term is $(i,j) = (p,q)$:

$$f(p,q) = (-1)^{\chi(\gamma_p < \gamma_q)}(\chi(v(q) > v(p)) - \chi(v(p) > v(q))) = 1. \tag{10.12}$$

Combining (10.10), (10.11), and (10.12) we have

$$\ell(v'\tau_\gamma) - \ell(v\tau_\gamma) = 1 + 2\sum_{p<i<q} \chi(\gamma_p = \gamma_i)\chi(v(p) < v(i) < v(q)) \geq 1. \tag{10.13}$$

We may therefore assume that v is of minimum length in its coset vS_n^γ. Suppose next that $\gamma \neq \beta$, so that there is an index r such that $\gamma_r > \gamma_{r+1}$. Let $\gamma' = s_r\gamma$ and $v' = vs_r$ so that $v'\gamma' = v\gamma$. It suffices to show that $\ell(v'\tau_{\gamma'}) < \ell(v\tau_\gamma)$. Let

$$g(i,j) = (-1)^{\chi(\gamma'_i < \gamma'_j)}\chi(v'(i) > v'(j)) - (-1)^{\chi(\gamma_i < \gamma_j)}\chi(v(i) > v(j)) \tag{10.14}$$

Due to (10.2) and (10.4) we have

$$\ell(v'\tau_{\gamma'}) - \ell(v\tau_\gamma) = \sum_{1\leq i<j\leq n} g(i,j). \tag{10.15}$$

If $\{i,j\} \cap \{r, r+1\} = \varnothing$ then $g(i,j) = 0$. For $i = r$ and $j > r+1$ we have

$$g(r,j) = (-1)^{\chi(\gamma_{r+1} < \gamma_j)}\chi(v(r+1) > v(j)) - (-1)^{\chi(\gamma_r < \gamma_j)}\chi(v(r) > v(j))) \tag{10.16}$$

For $i = r+1$ and $j > r+1$ we have

$$g(r+1,j) = (-1)^{\chi(\gamma_r < \gamma_j)}\chi(v(r) > v(j)) - (-1)^{\chi(\gamma_{r+1} < \gamma_j)}\chi(v(r+1) > v(j))) \tag{10.17}$$

These cancel: for $j > r+1$ we have $g(r,j) + g(r+1,j) = 0$. Similarly for $i < r$ we have $g(i,r) + g(i,r+1) = 0$. For $(i,j) = (r,r+1)$ we have

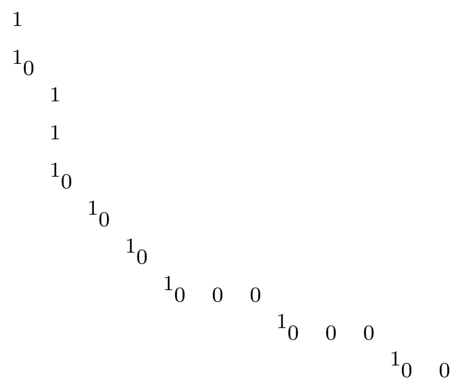
$$\begin{aligned} g(r,r+1) &= (-1)^{\chi(\gamma_{r+1}<\gamma_r)}\chi(v(r+1) > v(r)) \\ &\quad - (-1)^{\chi(\gamma_r<\gamma_{r+1})}\chi(v(r) > v(r+1)) \\ &= -\chi(v(r+1) > v(r)) - \chi(v(r) > v(r+1)) \\ &= -1 \end{aligned} \tag{10.18}$$

so that $\ell(v'\tau_{\gamma'}) = \ell(v\tau_\gamma) - 1$, which suffices. □

10.2. **The action of $\tilde{S}_n$ on partitions.** Consider the positive quadrant $\mathbb{Z}^2_{>0}$ in the plane, where an element (i,j) is depicted as a *cell* (square) in the plane, indexed using standard Cartesian coordinates. The *diagram* of the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$ is the set of cells $\{(i,j) \mid 1 \le i \le p, 1 \le j \le \lambda_i\}$ with λ_i left-justified cells in row i for all i. Define the *diagonal index* of a cell by $\mathrm{diag}(i,j) = j - i$ and the *residue* of a cell by $\mathrm{res}(i,j) = \overline{j-i} \in \mathbb{Z}/n\mathbb{Z}$.

Given a partition λ, one may associate a bi-infinite binary word $p(\lambda) = p = \cdots p_{-1}p_0p_1\cdots$ called its *edge sequence*. The edge sequence $p(\lambda)$ traces the border of the diagram of λ, going from northwest to southeast, such that every letter 0 (resp. 1) represents a south (resp. east) step, such that some cell in the i-th diagonal is touched by the steps p_{i-1} and p_i.

Example 10.3. The diagram of $\lambda = (10,7,4,3,2,1,1,1)$ is pictured below.



The edge sequence is $p(\lambda) = \cdots 11|0111|0101 \bullet 0100|0100|0100|\cdots$ where $\bullet$ indicates the 0 diagonal, which separates the bits p_{-1} and p_0.

The affine symmetric group $\tilde{S}_n$ acts on partitions in an obvious way, if we identify elements of $\tilde{S}_n$ with functions $\mathbb{Z} \to \mathbb{Z}$ and partitions with their edge sequences, which are certain functions $\mathbb{Z} \to \{0,1\}$. Say that the cell x is λ-*addable* (resp. λ-*removable*) if adding (resp. removing) the cell x to (resp. from) the diagram of λ results in the diagram of a partition. Then for $i \in \mathbb{Z}/n\mathbb{Z}$, $s_i\lambda$ is obtained by removing from λ every λ-removable cell of residue i, and adding to λ every λ-addable cell of residue i.

Example 10.4. For $n = 4$, applying the reflections of the reduced word of w in Example 10.1 to the empty partition from right to left, we obtain the sequence of partitions $() \subset (1) \subset (2) \subset (2,1) \subset (2,2) \subset (3,2,1) \subset (4,2,2) \subset (5,2,2) \subset (6,3,2,1) \subset (7,4,2,2) \subset (8,5,2,2) \subset (9,6,3,2,1) \subset (9,6,3,3,1,1) \subset (9,6,3,3,1,1,1) \subset (10,7,4,3,2,1,1,1)$.

In fact, the action of $\tilde{S}_n$ on partitions coincides with the action of the Kashiwara reflection operators on the crystal graph of Fock space [25].

10.3. **Cores and the coroot lattice.** An *n-ribbon* is a skew partition diagram λ/μ (the difference of the diagrams of the partitions λ and μ) consisting of n rookwise connected cells, all with distinct residues. We say that this ribbon is *λ-removable* and *μ-addable.* An *n-core* is a partition that admits no removable n-ribbon. Henceforth when we say "core" we mean "n-core". Since the removal of an n-ribbon is the same thing as exchanging bits $p_i = 0$ and $p_{i+n} = 1$ in the edge sequence for some i, it follows that λ is a core if and only if for every i, the sequence $p^{(i)}(\lambda) := \cdots p_{i-2n}p_{i-n}p_ip_{i+n}p_{i+2n}\cdots$ consisting of the subsequence of bits indexed by $i \bmod n$, has the form $\cdots 1111100000 \cdots$. Thus the core is specified by the positions where this sequence changes from 1 to 0 for various i. For a core λ and $i \in \mathbb{Z}$ let $d_i(\lambda) = d_i \in \mathbb{Z}$ be the integer such that $p_{i-1+n(d_i-1)} = 1$ and $p_{i-1+nd_i} = 0$.[2] The sequence $(d_i)_{i\in\mathbb{Z}}$ is called the *extended offset sequence* of λ, and $d(\lambda) = (d_1, d_2, \ldots, d_n)$ the *offset sequence* of λ. Observe that

$$d_{i-n} = d_i + 1 \qquad \text{for all } i \in \mathbb{Z} \tag{10.19}$$

and that $\sum_{i=1}^n d_i = 0$.

Example 10.5. For $n = 4$ and $\lambda = (10,7,4,3,2,1,1,1)$, we have $d(\lambda) = (-2,3,-1,0)$. These are the heights "above sea level" that the 1s attain if we draw the edge sequence $p(\lambda)$ in an $\infty \times n$ array with the r-th row given by the binary word with bits p_{i+nr} for $i = 0,1,2,\ldots,n-1$, and "sea level" is the division between rows -1 and 0. Below we depict the bit sequence $p(\lambda)$, whose columns, read from bottom to top, are $p^{(0)}(\lambda)$ through $p^{(n-1)}(\lambda)$.

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\hline
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

The following results are well known; see for example [20, 22].

[2]Using $i-1$ rather than i, allows an elegant statement of Theorem 11.5.

Lemma 10.6. *For any core λ and $i \in \mathbb{Z}/n\mathbb{Z}$, either there are no λ-addable cells of residue i or there are no λ-removable cells of residue i.*

Proposition 10.7. *There is a bijection from the set $\mathcal{C}_n$ of n-cores to the root lattice Q of $\mathfrak{sl}_n$ given by $\lambda \mapsto d(\lambda)$.*

Q already has an action of $\tilde{S}_n$ defined in Subsection 10.1, which is transitive since the subgroup $T(Q)$ of $\tilde{S}_n$ acts transitively on Q. Via the bijection $\lambda \mapsto d(\lambda)$, $\tilde{S}_n$ acts transitively on the set $\mathcal{C}_n$ of cores. This induced action coincides with the action of $\tilde{S}_n$ on partitions defined above.

Proposition 10.8. *The action of $\tilde{S}_n$ on partitions restricts to an action on $\mathcal{C}_n$. Moreover, the bijection of Proposition 10.7 is an isomorphism of sets with $\tilde{S}_n$-action:*

$$d(w \cdot \lambda) = w \cdot d(\lambda) \qquad \text{for every } w \in \tilde{S}_n. \tag{10.20}$$

Proof. It suffices to consider the case $w = s_i$. This is easily verified, the most interesting case being $i = 0$, where one uses (10.19). □

Proposition 10.9. $\mathcal{C}_n = \tilde{S}_n \cdot \varnothing$. *In particular there is a bijection*

$$c : \tilde{S}_n^0 \to \tilde{S}_n/S_n \to \mathcal{C}_n \tag{10.21}$$

$$w \mapsto wS_n \mapsto c(w). \tag{10.22}$$

Proof. We already know that $\tilde{S}_n$ acts transitively on $\mathcal{C}_n$ and $\varnothing \in \mathcal{C}_n$, so that $\mathcal{C}_n = \tilde{S}_n \cdot \varnothing$. Since S_n is the stabilizer of $\varnothing$ the result follows. □

10.4. **Grassmannian elements and the coroot lattice.** The bijection $d \circ c : \tilde{S}_n^0 \to Q$ has the following affine Lie-theoretic interpretation.

Proposition 10.10. *Write $w \in \tilde{S}_n^0$ as $w = u\tau_\beta$ where $u \in S_n$ and $\beta \in Q$.*

(1) $d(c(w)) = u\beta \in Q$.
(2) *β is antidominant and u is of minimum length in uS_n^β where S_n^β is defined before Proposition 10.2.*
(3) $\ell(w) = \ell(t_\beta) - \ell(u)$.

Proof. Let $\lambda = c(w)$ (that is, $\lambda = w \cdot vn$) and $d = d(\lambda)$. We have $d(c(w)) = d = d(w \cdot \varnothing) = w \cdot d(\varnothing) = w \cdot (0^n) = u\tau_\beta(0^n) = u\beta$ using Proposition 10.8. In fact this works for any $w \in \tilde{S}_n$ such that $w \cdot \varnothing = \lambda$ and $w = u\tau_\beta$.

Suppose $w' = v\tau_\gamma$ is such that $w' \cdot \varnothing = \lambda$. Then $v\gamma = d$ as well. We have

$$\begin{aligned} w' = v\tau_\gamma &= v\tau_{v^{-1}u\beta} \\ &= vv^{-1}u\tau_\beta u^{-1}v \\ &= u\tau_\beta(u^{-1}v) = w(u^{-1}v). \end{aligned}$$

The result follows from Proposition 10.2. □

Example 10.11. For $n = 4$ and $w = [-7, -1, 4, 14]$, by Example 10.1 we have $u = s_2s_3$ and $\beta = (-2, -1, 0, 3)$. So $d = u\beta = (-2, 3, -1, 0)$, which agrees with Example 10.5.

10.5. **Bijection from cores to bounded partitions.** Say that a partition λ is n-bounded[3] if $\lambda_1 < n$. Denote by $\mathcal{B}_n$ the set of n-bounded partitions.

The *hook length* of a cell (i, j) in a skew shape λ/μ is the number of cells in λ/μ in the i-th row to the right of (i, j), plus the number of cells in λ/μ in the j-th column above (i, j), plus one (for the cell (i, j) itself).

Proposition 10.12. [16] *Let $\lambda \in \mathcal{C}_n$. Let $\mu \subset \lambda$ be the smallest partition such that the hook length of every cell in λ/μ is less than n. Then $\lambda \mapsto b(\lambda) := (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \dots)$ defines a bijection $\mathcal{C}_n \to \mathcal{B}_n$.*

Example 10.13. For $n = 4$ and λ as in Example 10.5, we have $\mu = (7, 4, 2, 1, 1)$ so that $\lambda \mapsto (3, 3, 2, 2, 1, 1, 1, 1)$, which is obtained by reading the sizes of the rows of the skew diagram λ/μ, which is pictured below.

10.6. k**-conjugate.** Transposition defines an involution ω on the set of partitions, which is easily seen to restrict to an involution on the set $\mathcal{C}_n$ of n-cores. Denote by $\omega^{(n)} = b \circ \omega \circ b^{-1}$ the induced involution on $\mathcal{B}_n$. For $k = n - 1$, $\omega^{(n)}$ is the k-conjugate map of [16].

Example 10.14. Reading the column sizes of the skew shape in Example 10.13 we have $\omega^{(4)}(3, 3, 2, 2, 1, 1, 1, 1) = (3, 2, 2, 1, 1, 1, 1, 1, 1, 1)$.

10.7. **From Grassmannian elements to bounded partitions.** Recall from Subsection 3.1 the definition of an inversion of $w \in \tilde{S}_n$. Let the *code* of w be the sequence $\mathrm{code}(w) = (c_1, c_2, \dots, c_n)$ where c_i is the number of inversions of w of the form (i, j) for some j; see also [2] where the code is called the inversion table. For a Grassmannian permutation it is easy to see that the code is a weakly increasing sequence of nonnegative integers that starts with 0; reversing this sequence yields a partition with fewer than n parts. Applying the transpose ω, we obtain an $(n-1)$-bounded partition.

Proposition 10.15. *The composite bijection $b \circ c : \tilde{S}_n^0 \to \mathcal{B}_n$ is given by $w \mapsto \omega^{(n)}(\omega(c_n, \dots, c_2, c_1))$ where $\mathrm{code}(w) = (c_1, c_2, \dots, c_n)$. This bijection satisfies $\ell(w) = |b \circ c(w)|$.*

Proof. Björner and Brenti [2, Theorem 4.4] recursively construct a bijection from bounded partitions to Grassmannian affine permutations w, from which one obtains the code $\mathrm{code}(w)$. This recursive construction is compatible with the construction of a core from a Grassmannian permutation as in Section 10.2. □

[3]Our language differs slightly from that of [16].

Example 10.16. Let $n = 4$ and $w = [-7, -1, 4, 14]$. Then w has inversions $(2, 5)$, $(3, 5)$, $(3, 6)$, $(3, 9)$, $(4, 5)$, $(4, 6)$, $(4, 7)$, $(4, 9)$, $(4, 10)$, $(4, 11)$, $(4, 13)$, $(4, 14)$, $(4, 17)$, $(4, 21)$ so that $\mathrm{code}(w) = (0, 1, 3, 10)$. Now $\omega(\mathrm{code}(w)) = (3, 2, 2, 1, 1, 1, 1, 1, 1, 1)$ and by Example 10.14 we have $\omega^{(4)}(\omega(\mathrm{code}(w)) = (3, 3, 2, 2, 1, 1, 1, 1)$, which agrees with Example 10.13.

11. Strong and weak tableaux using cores

We now specialize all constructions to the special case of Grassmannian elements. By the previous section we may work instead with cores.

11.1. **Weak tableaux on cores are k-tableaux.** Consider a weak tableau $U = (W_1, W_2, \dots)$ where $\mathrm{inside}(W_i) = u_{i-1}$ and $\mathrm{outside}(W_i) = u_i$ for all i, and $u_i \in \tilde{S}_n^0$ for all $i \geq 0$. Let $\mu = c(\mathrm{inside}(U))$ and $\lambda = c(\mathrm{outside}(U))$. The weak tableau U can be depicted as a usual tableau of shape λ/μ in which all cells of the skew shape $c(u_i)/c(u_{i-1})$ have been filled with the letter i for all i. Such a tableau is called a *k-tableau* [16]. The following characterization of k-tableaux is given by Lapointe, Morse and Wachs in [19].

Lemma 11.1. *k-tableaux are semistandard (increasing in rows and strictly increasing in columns). Conversely, if a chain of cores in (left) weak order defines a semistandard tableau then it comes from a weak tableau.*

Proof. If s_σ is a cyclically decreasing permutation, then s_j is never applied after s_{j+1} in $s_\sigma(\gamma)$. This means that two cells are never added on top of each other in the same column and thus $s_\sigma(\gamma)/\gamma$ is a horizontal strip. This shows the first statement of the lemma.

Now suppose that $\delta = c(w)$ and $\gamma = c(v)$ where $w \preceq v \in \tilde{S}_n^0$ are such that δ/γ is a horizontal strip of size m (meaning that the difference $\ell(w) - \ell(v)$ is m). Theorem 56 of [16] then says that $\delta = s_{i_1} \cdots s_{i_m}(\gamma)$, for some $i_1, \dots, i_m$ that are all distinct. Furthermore, in the proof of Theorem 56, it is shown that i_m can be chosen as the residue of the southeasternmost cell c_{SE} in δ/γ. By induction on the size of δ/γ, it suffices to show that s_{i_m+1} is never applied at some point after s_{i_m} so that $s_{i_1} \cdots s_{i_m}$ is cyclically decreasing. Suppose this is the case. Then, since δ/γ is a horizontal strip, this means that there is a cell of residue $i_m + 1$ to the northwest of c_{SE} without a cell above it, and thus that the extremal cell of residue i_m to its left is not at the end of its row. This is a contradiction to [16, Proposition 15(1)] which says that all extremal cells of residue i_m to the northwest of c_{SE} are at the end of their row. ☐

Example 11.2. Let $n = 4$ and $l = 0$ and consider the weak tableau defined by the length-additive factorization of the Grassmannian element

$$w = (s_1)(s_2)(s_3)(s_0)(s_3 s_2 s_1)(s_0 s_3 s_2)(s_0 s_3)(s_1 s_0)$$

of Example 10.1 into cyclically decreasing elements. The corresponding sequence of cores is $() \subset (2) \subset (2, 2) \subset (5, 2, 2) \subset (8, 5, 2, 2) \subset (9, 6, 3, 2, 1) \subset$

$(9,6,3,3,1,1) \subset (9,6,3,3,1,1,1) \subset (10,7,4,3,2,1,1,1)$, which gives the k-tableau

8								
7								
6								
5	8							
4	4	6						
3	3	5	8					
2	2	4	4	4	5	8		
1	1	3	3	4	4	4	5	8

11.2. **Strong tableaux on cores.** We assume that $l = 0$ with l as in Subsection 3.3. Since id is Grassmannian, by Proposition 3.6 any strong tableau with inner shape id, involves only Grassmannian elements of $\tilde{S}_n$.

The strong order on Grassmannian permutations corresponds to containment of cores.

Proposition 11.3 ([20, 25])**.** *For $v, w \in \tilde{S}_n^0$, $v \le w$ if and only if $c(v) \subset c(w)$.*

Lemma 11.4. *Let μ be a core and $t_{r,s} \in \tilde{S}_n$ a reflection with $r < s$. Then*

(1) *$t_{r,s}\mu \ge \mu$ if and only if $d_r \ge d_s$ using the extended offset sequence d of μ.*
(2) *$t_{r,s}\mu \gtrdot \mu$ if and only if $d_r > d_s$ and for all $r < i < s$, $d_i \notin [d_s, d_r]$.*

Proof. (2) follows from (1). To prove (1), for cores μ and λ, by Proposition 11.3, $\mu \le \lambda$ if and only if for all $i \in \mathbb{Z}$, $\sum_{j \le i}(p_j(\mu) - p_j(\lambda)) \ge 0$. Therefore $t_{r,s}\mu \ge \mu$ if and only if $p_{r+qn}(\mu) \ge p_{s+qn}(\mu)$ for all $q \in \mathbb{Z}$. $d_r \ge d_s$ implies that this holds for $q = 0$ and (10.19) implies it for all other q. $\square$

The *head* of a ribbon is its southeastmost cell.

Theorem 11.5. *Let $\mu \lessdot \lambda$ be cores with $t_{r,s}\mu = \lambda$ and $0 < r < s$. Then*

(1) *$s - r < n$.*
(2) *Each connected component of λ/μ is a ribbon with $s - r$ cells in diagonals of reside $\overline{r}, \overline{r+1}, \dots, \overline{s-1}$.*
(3) *The components are translates of each other and their heads lie on "consecutive" diagonals of residue $s - 1$.*
(4) *The skew shape λ/μ has $d_r - d_s$ components.*

Example 11.6. Let $n = 4$ and $l = 0$ and consider the strong marked cover $C = (w \xrightarrow{i,j} u)$ where $w = [-8, -3, 6, 15]$, $u = [-8, -6, 9, 15]$, and $(i, j) = (-1, 10)$. We have $m(C) = w(j) = 5$. w and u have associated cores $\mu = (11, 8, 5, 5, 3, 3, 1, 1, 1)$ and $\lambda = (11, 8, 7, 6, 5, 4, 3, 2, 1)$ and offsets $d(\mu) = (-1, 1, 3, -3)$ and $d(\lambda) = (2, -2, 3, -3)$ respectively. Letting (r, s) be such that $t_{r,s}\mu = \lambda$, we have $t_{r,s}w = u = wt_{i,j}$, so that $t_{r,s} = wt_{i,j}w^{-1} = t_{w(i),w(j)}$. Thus we may take $r = w(i) = 2$ and $s = w(j) = 5$. Letting $d = d(\mu)$, the skew shape λ/μ has $d_2 - d_5 = 1 - (-2)$ components, each of size $5 - 2$.

$*$

Proof of Theorem 11.5. All of the assertions follow from the first and Lemma 11.4. The first assertion follows from Lemma 11.4(2) and (10.19). □

Proposition 11.7. *Let $C = (w \xrightarrow{i,j} u)$ be a marked strong cover with $w, u \in \tilde{S}_n^0$ and write $c(w) = \mu$ and $c(u) = \lambda$. Then the head of one of the ribbons forming the connected components of λ/μ, is on the diagonal $m(C) - 1$.*

Proof. We apply Theorem 11.5 with $r = w(i)$ and $s = w(j) = m(C)$. Since the r-th bit 0 is being exchanged with the s-th bit 1 in the strong cover, the head of one of the ribbons must lie on the previous diagonal $s - 1$. □

Example 11.8. In Example 11.6 the head of one of the ribbons in λ/μ is marked with a $*$; it is in the diagonal $m(C) - 1 = 5 - 1$.

In light of Proposition 11.7, a marked strong cover of cores $\mu \xrightarrow{i,j} \lambda$ is given by the disjoint union of ribbons comprising the skew shape λ/μ, together with a marking on the head of one of the ribbons. For a strong strip, the sequence of marked cells must have strictly increasing diagonal indices. Since each strong cover on cores is a skew partition shape, this is equivalent to saying that the marked heads must proceed weakly to the south and strictly to the east.

We make the following convention for strong tableaux on cores. Let $T = (S_1, S_2, \dots)$ be a strong tableau going between the elements $u_0 \le u_1 \le \dots$ of $\tilde{S}_n^0$. Let $\lambda^{(i)} = c(u_i)$, $\mu = c(\text{inside}(T))$, and $\lambda = c(\text{outside}(T))$. We depict T as a tableau of shape λ/μ. The strong strip S_i is depicted by the skew subtableau of shape $\lambda^{(i)}/\lambda^{(i-1)}$. We distinguish the cells of the strong covers in S_i by placing the subscripted letter i_j in a cell if it occurs in the skew shape corresponding to the j-th strong cover in S_i. Finally, a mark * must be placed on the head of some ribbon in each strong cover, such that the diagonals of the heads increase in each strong strip.

Example 11.9. Here is a strong tableau for $n = 3$.

$$P = \begin{array}{|c|c|c|c|c|} \hline 3_1 \\ \cline{1-3} 2_1^* & 3_3 & 3_3 \\ \hline 1_1^* & 3_1^* & 3_2^* & 3_3 & 3_3^* \\ \hline \end{array} \, .$$

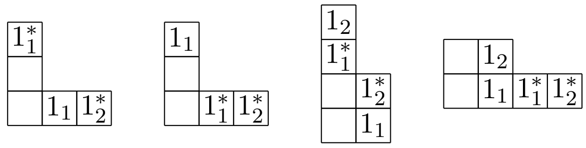
It has three nonempty strong strips S_1, S_2, S_3. S_1 consists of a single marked strong cover $\varnothing \xrightarrow{0,1} (1)$. S_2 also consists of a single marked strong cover

$(1) \xrightarrow{-1,1} (1,1)$. S_3 is given by $(1,1) \xrightarrow{0,4} (2,1,1) \xrightarrow{0,2} (3,1,1) \xrightarrow{0,5} (5,3,1)$. To understand the markings, we convert the 3-cores in S_3 into elements of $\tilde{S}_n$, giving the sequence $[-1,3,4], [-2,3,5], [-2,2,6], [-2,0,8]$ in window notation. Call these u_0, u_1, u_2, u_3. Then the marks of the three covers in S_3 are given by $u_0(4) = 2$, $u_1(2) = 3$, and $u_2(5) = 5$. This means that the heads of the marked ribbons in S_3 occur in diagonals $2-1, 3-1, 5-1$, which indeed is the case.

Example 11.10. We compute an example of Theorem 5.6 using cores. Let $n = 3$, $r = 2$ and $w = s_2 s_0$. Then we have the equality (in $\Lambda^{(3)}$)

$$h_2(x)\mathrm{Weak}_{s_2s_0}(x) = 2\mathrm{Weak}_{s_2s_1s_2s_0}(x) + \mathrm{Weak}_{s_0s_1s_2s_0}(x) + \mathrm{Weak}_{s_0s_2s_1s_0}(x).$$

The right hand side corresponds to the following four strong strips on cores.



11.3. **Monomial expansion of t-dependent k-Schur functions.** Theorem 11.5 allows us to give a conjectural monomial expansion for the k-Schur functions of [14, 15] which depend on a parameter t. Suppose $C = \mu \xrightarrow{i,j} \lambda$ is a marked strong cover of $k+1$-cores. By Theorem 11.5 the skew shape λ/μ consists of m identical ribbons each of height h, where the height of a ribbon is the number of rows it occupies. Define the *spin* of C to be $\mathrm{spin}(C) = m(h-1) + (l-1)$ where the marked ribbon in C is the l-th one from the top. The spin $\mathrm{spin}(T)$ of a strong tableau T is the sum of the spins of its marked strong covers. For example, the strong tableau of Example 11.9 has spin $4 = 0 + 0 + 1 + 0 + 3$. Our use of the name "spin" is due to the similarities between this statistic and Lascoux, Leclerc and Thibon's spin statistic for ribbon tableaux [21].

Conjecture 11.11. *Let $\lambda \in \mathcal{B}_n$ where $n = k+1$. The k-Schur function $s_\lambda^{(k)}(x;t)$ of* [14] *and of* [15] *is given by*

$$s_\lambda^{(k)}(x;t) = \sum_Q x^{\mathrm{wt}(Q)}\, t^{\mathrm{spin}(Q)}$$

where the summation runs over strong tableaux Q of shape given by the n-core $b^{-1}(\lambda)$.

12. Affine insertion in terms of cores

We now translate the affine insertion algorithm for the special case of Grassmannian elements, into the language of cores. It suffices to describe the local rule of Section 7 and its reverse in Section 8.

12.1. **Internal insertion for cores.** Let C be a strong cover $\mu \lessdot t_{r,s} \cdot \mu = \lambda$ with $m(C) = s$. By Proposition 11.7 this is equivalent to marking the cell of λ/μ on the diagonal $s-1$. We shall use this equivalence freely without further mention.

Let $W = (\mu \overset{A}{\rightsquigarrow} \nu)$ be a weak strip and (W, S'_1) the input final pair for the internal insertion at C, and let (W', S') be the output final pair, with $\text{outside}(W') = \gamma$.

In every case we shall define $W' = (\lambda \overset{A'}{\rightsquigarrow} \gamma)$ in terms of a set $A' \subsetneq \mathbb{Z}/n\mathbb{Z}$.

Lemma 12.1. (W, C) *does not commute if and only if the marked component of* λ/μ *is contained in* ν/μ*. Moreover, if this occurs then each component of* λ/μ *is contained within a single row.*

Proof. By Lemma 4.15, (W, C) does not commute if and only if the marked component of C is contained in the skew shape ν/μ and the tail (northwestmost cell) of the marked component has residue equal to the minimum of some cyclic interval I in A. Since ν/μ corresponds to a weak strip, by Lemma 11.1 it is a horizontal strip in the usual sense. This implies that each of the ribbon components of λ/μ, lies in a single row. In particular the above condition regarding the tail, automatically holds. □

Using Lemma 12.1 we rephrase the commutation conditions.

12.1.1. *Commuting case for cores.*

Case A (Commuting case) Suppose the marked component of λ/μ is not contained in ν/μ. Then let $A' = A$ and $S' = S'_1 \cup C'$ where C' is the strong cover $\nu \lessdot \gamma = t_{c_A(r), c_A(s)} \cdot \nu$, with $m(C') = c_A(s)$.

12.1.2. *Noncommuting cases for cores.* Let p_0 be the diagonal of the head of the marked component of C and $A^\vee = A \backslash \{\overline{p_0}\}$.

Case B (Normal bumping case) Suppose the marked component of λ/μ is contained in λ/ν and that either $\text{size}(S'_1) = 0$, or $\text{size}(S'_1) > 0$ and $m(C) \neq m(\text{last}(S'_1))$. Let q_0 be the diagonal of the tail of the marked component of λ/μ. Let $q < p$ be the unique pair of consecutive $A^\vee$-nice integers such that $q < q_0$, λ has an addable cell in the diagonal q, and q is maximal. Let $A' = A^\vee \cup \{\overline{p-1}\}$ and $S' = S'_1 \cup C'$ where C' is the strong cover $\nu \lessdot \gamma := t_{c_A(q), c_A(p)} \cdot \nu$ with $m(C') = c_A(p)$.

Case C (Replacement Bump) Suppose the marked component of λ/μ is contained in λ/ν, $\text{size}(S'_1) > 0$, and $m(C) = m(\text{last}(S'_1))$. Let ν^- be the core obtained by removing $\text{last}(S'_1)$ from ν. Let $q < p$ be the unique pair of consecutive $A^\vee$-bad integers such that $q < p_0$, ν^- has an addable cell in the diagonal q, and q is maximal. Set $A' = A^\vee \cup \{\overline{q}\}$ and $S' = (S'_1 \backslash \text{last}(S'_1)) \cup C'^- \cup C'$ where C'^- is the strong cover $\nu^- \lessdot \nu' := t_{q,p} \cdot \nu^-$ with $m(C'^-) = p$, and C' is the strong cover $\nu' \lessdot \gamma$, with $m(C') = t_{q,p}\, m(C)$.

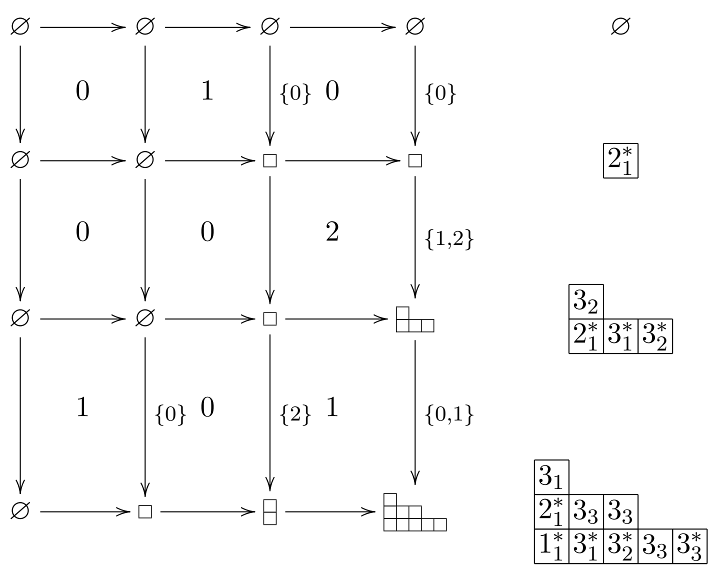


FIGURE 1. Growth diagram

12.2. **External Insertion for cores (Case X).** Let $W = (\mu \overset{A}{\rightsquigarrow} \nu)$ be a weak strip. Let $q = \nu_1$ and $p > q$ the next larger A-bad integer. Let $A' = A \cup \{\overline{q}\}$ and $W' = (\mu \overset{A'}{\rightsquigarrow} \gamma)$. Let C' be the strong cover $\nu \lessdot \gamma := t_{q,p}\,\nu$ with $m(C') = p$. This marks the component of γ/ν in the first row.

External insertion on (W, S_1') is given by computing (W', C') as above and setting $S' = S_1' \cup C'$.

12.3. **An example.**

Example 12.2. Let $n = 3$ and let

$$m = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}.$$

The growth diagram to compute the image (P, Q) of m is given by Figure 12.2. Each row of the diagram defines a strong tableau, which is depicted to the right of the row. The tableau Q is given by

$$Q = \begin{array}{|c|c|c|c|c|} \hline 3 \\ \cline{1-1} 2 & 3 & 3 \\ \cline{1-3} 1 & 2 & 2 & 3 & 3 \\ \hline \end{array}.$$

We explain the computation of the last row of 2×2 squares. The first square consists of a single external insertion where the input data consists of empty weak and strong strips, all going from $\varnothing$ to itself. This external insertion just adds the residue 0, giving the new shape (1).

The second square has input strong strip S that consists of a single marked strong cover $C = (\varnothing \lessdot (1) = t_{0,1}(\varnothing))$ with $m(C) = 1$, and input weak strip $W = (\varnothing \overset{A}{\rightsquigarrow} (1))$ with $A = \{\overline{0}\}$. Since the marked component of C consists of the cell $(1,1)$ which is contained in W, we are in the noncommuting case. Since the old output strip S'_1 is empty, Case B holds. We have $p_0 = q_0 = 0$, $q = -1$, and $p = 0$, so that $A' = \{\overline{-1}\} = \{\overline{2}\}$ with output weak strip $(1) \overset{A'}{\rightsquigarrow} (1,1)$ and the output strong strip consists of a single cover $C' = ((1) \lessdot (1,1))$ with $m(C') = 0$.

The third square has the input strong strip $S = ((1) \lessdot (2) \lessdot (3,1)) = C_1 \cup C_2$ with $m(C_1) = 2$ and $m(C_2) = 3$. It has input weak strip $W = ((1) \overset{A}{\rightsquigarrow} (1,1))$ where $A = \{\overline{2}\}$. The internal insertion at C_1 is in Case A: the marked component consists of the cell $(1,2)$, which is not contained in W, which consists of the cell $(2,1)$. This insertion produces the output strong strip $(1,1) \lessdot (3,1)$ with mark 3 and the same set A, which now is associated with the weak strip $(2) \overset{A}{\rightsquigarrow} (3,1)$.

Next we perform the internal insertion at C_2. We reindex with input weak strip $W = ((2) \overset{A}{\rightsquigarrow} (3,1))$, and $A = \{ba2\}$, old output strong strip $S'_1 = ((1,1) \lessdot (3,1))$, and $C = C_2 = ((2) \lessdot (3,1))$ with $m(C) = 3$. The marked component of C is the single cell $(1,3)$ which is contained in W. This is the noncommuting case. Now $m(\mathrm{last}(S'_1)) = 3 = m(C)$ so we are in Case C. We have $p_0 = 2$, $A^\vee = \{\}$, $\nu^- = (1,1)$, $q = 1$, $p = 2$. Thus $A' = \{\overline{-2}\} = \{\overline{1}\}$, with output weak strip $((3,1) \overset{A'}{\rightsquigarrow} (3,1,1)$ and output strong strip $S' = ((1,1) \lessdot (2,1,1) \lessdot (3,1,1)) = C'_1 \cup C'_2$, where $m(C'_1) = 2$ and $m(C'_2) = 3$.

To finish up we apply a single external insertion to the output final pair from the previous step, which are reindexed as $W = ((3,1) \overset{A}{\rightsquigarrow} (3,1,1))$ with $A = \{\overline{1}\}$ and $S = ((1,1) \lessdot (2,1,1) \lessdot (3,1,1)) = C'_1 \cup C'_2$ with $m(C'_1) = 2$ and $m(C'_2) = 3$. We have $q = 3$, $p = 5$, so that $A' = \{0,1\}$ and $W' = ((3,1) \overset{A'}{\rightsquigarrow} (5,3,1))$ and $C' = ((3,1,1) \lessdot (5,3,1))$ with $m(C') = 5$ and $S' = S \cup C'$.

12.4. **Coincidence with RSK as $n \to \infty$.**

Theorem 12.3. *As $n \to \infty$ the bijection of Theorem 5.2 converges to RSK row insertion.*

Proof. In the limit the set of n-bounded matrices becomes the set of all matrices with finitely many nonzero entries. All residues of corner cells in a partition will be distinct. Thus a weak strip becomes an ordinary horizontal strip with one cell per residue and a weak tableau becomes a semistandard one. A marked strong cover becomes a skew shape consisting of a single cell, a strong strip becomes a horizontal strip, and a strong tableau becomes a semistandard one.

We now consider the local rule. The data consists of a horizontal strip $S = \lambda/\mu$, a horizontal strip $W = \nu/\mu$, and some $e \in \mathbb{Z}_{\geq 0}$. The algorithm consists of performing internal insertions on W at the cells of S from left to right. Case A occurs when the active cell C (the one at which the internal insertion occurs) is not in W, so no bumping takes place. Case B occurs when the active cell is in W, in which case the active cell C is removed from the "weak strip" and another cell C' is added, namely, the next addable one of smaller residue. This is equivalent to bumping the cell to the end of the next row. Case C never occurs because cells are always bumped to strictly earlier diagonals.

Finally, each of the e external insertions is given by adding a cell to the end of the first row (and putting this cell in the new weak and strong strips).

Since this coincides with the local rule for ordinary RSK row insertion, the Theorem follows. □

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