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ON THE COMBINATORICS OF THE WALDSPURGER DECOMPOSITION


By

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This thesis is comprised of an introduction, three chapters, and a brief appendix of figures. The introduction develops the theory of root systems and finite groups generated by reflections. For brevity and readability we occasionally argue by fiat, providing references for omitted proofs and further compensating with ample examples. We establish conventions for ordering and coordinatizing root and weight vectors for the classical types and conclude with two central theorems from Waldspurger and Meinrenken.

Chapter 1 introduces a combinatorial algorithm for studying the Waldspurger and Meinrenken theorems in the type A setting where the underlying reflection group is the symmetric group, $\mathfrak{S}_n$. Our algorithm associates $\pi \in \mathfrak{S}_n$ with an $(n-1) \times (n-1)$ matrix denoted $\mathbf{WT}(\pi)$. We characterize the column, row, and diagonal vectors of $\mathbf{WT}(\pi)$ in terms of certain lattice paths. Because componentwise order on $\mathbf{WT}(\mathfrak{S}_n)$ is isomorphic to Bruhat order, we extend the domain of $\mathbf{WT}$ to the set of alternating sign matrices to obtain a new combinatorial model of the classical ASM lattice.

Chapter 2 uses the map $\mathbf{WT}$ and folding techniques to study Waldspurger and Meinrenken's theorems in types B and C. In particular, we characterize the set of join-irreducible elements of the Dedekind-MacNeille completion of Bruhat order.

Chapter 3 uses symmetries from Meinrenken's theorem to compare three notions of dimension for permutations, the most novel of which relates to SIF permutations. We conclude by considering a dual graph structure on $n$-cycles, study its degree sequence and presenting a number of conjectures relating to its recursive structure.

*To Micah, may you grow in Truth and Love.*

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# Contents

# Introduction

Reflective symmetries abound in the world around us. A pretty face, a still body of water, and many architectural designs demonstrate that reflection is somehow fundamental to our human experience. In geometry the consideration of reflective symmetries predates the abstract notion of group.

Today, groups generated by reflections are central objects found at the intersection of Lie theory, physics, geometry, algebra, and combinatorics and many central theorems in mathematics are built upon the Coxeter diagram classification in Figure 1. Unfortunately, the literature surrounding this intersection quickly becomes reminiscent of the parable of the blind men and the elephant; notations are never quite as standard as one might hope, and the number of definitions can be overwhelming.

This introduction aims to give a succinct overview, providing only necessary definitions and motivating examples, and ending with a list of abbreviations and symbols. All but the last section may be regarded as classical, and the familiar reader is invited to skip to Section 0.7 or 0.8. We conclude the introduction with an organizational summary beginning on page 32.

## 0.1  Finite Real Reflection Groups and Coxeter Groups

For our purposes, a **reflection** is an element of the orthogonal group, $O_n(\mathbb{R})$ whose action on $\mathbb{R}^n$ sends some non-zero vector $\alpha$ to its negative and fixes $\alpha^\perp$, the hyperplane orthogonal to $\alpha$, pointwise. If $W$ is a finite group with a generating set of involutions,

$\Pi$, and a faithful representation $\rho : W \hookrightarrow GL(\mathbb{R}^n)$ such that $\rho(\alpha)$ is a reflection for all $\alpha \in \Pi$, then, we call the pair $(W, \rho)$ a **finite real reflection group**.

The formal study of finite real reflection groups was motivated by Lie theory and developed throughout the nineteenth century — notably in the works of Möbius, Jordan, Schläfli, Killing, Cartan, and Weyl. In 1934, Coxeter studied a larger class of groups (later called "Coxeter groups" by Tits [43]) given by presentations of the form

$$\langle r_1, r_2, \ldots, r_n \mid (r_i r_j)^{m_{ij}} = 1 \rangle$$

where $m_{ij} \in \{1, 2, \ldots, \infty\}$ and $m_{ij} = m_{ji}$ and $m_{ij} = 1 \Leftrightarrow i = j$. Coxeter showed that if such a group is finite, then it can be faithfully represented as a group generated by real reflections, i.e., that finite Coxeter groups and finite real reflection groups are the same class. This equivalence allowed him to completely classify such groups [14] in 1935, using Coxeter diagrams. A Coxeter diagram is simple edge labeled graph that encodes a Coxeter group in the following way:

- Vertices are the generators, $r_i$ for $i \in \{1, 2, \ldots, n\}$.

- There is no edge between vertex $r_i$ and $r_j$, iff $m_{i,j} = 2$.

- There is an unlabeled edge between vertex $r_i$ and $r_j$ iff $m_{i,j} = 3$.

- An edge between vertex $r_i$ and $r_j$ has label $m_{i,j}$, meaning that $(r_i r_j)^{m_{i,j}} = 1$. (While a priori the order of element $r_i r_j$ must *divide* $m_{i,j}$, one can show that the order of $r_i r_j$ actually *equals* $m_{i,j}$.)

Given two finite reflection groups $(W_1, \rho_1)$, and $(W_2, \rho_2)$ one can always cook up a third by taking the direct product of the groups and the orthogonal direct sum of the representations, $(W_1 \times W_2, \rho_1 \oplus \rho_2)$. We say that a finite reflection group $(W, \rho)$ is **reducible** if $(W, \rho) \cong (W_1 \times W_2, \rho_1 \oplus \rho_2)$ up to orthogonal change of basis for some non-trivial groups $W_1$ and $W_2$. $W$ is irreducible if it is not reducible. $W$ is irreducible
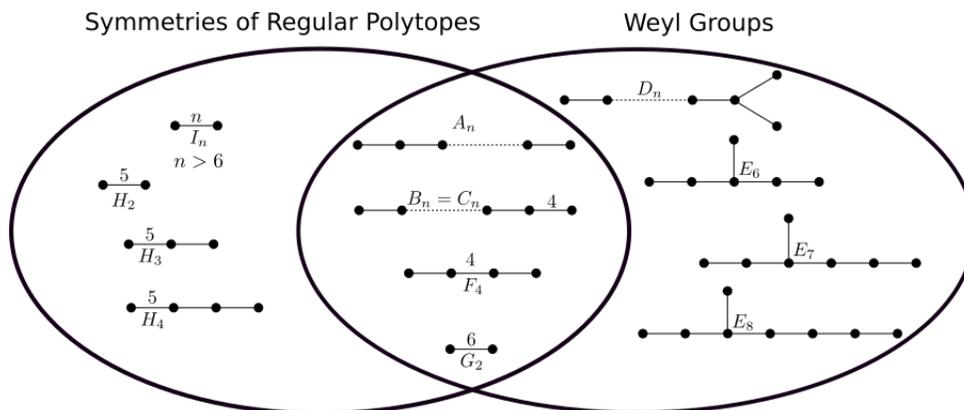
Figure 1: A subscript in the notation indicates the rank of the reflection group, which is the dimension of the representation. ($I_n$ is an exception. As the group is the symmetries of a regular $n$-gon, and is always rank 2.)

if and only if its Coxeter diagram is connected, and so it is sufficient to classify the irreducibles.

It is worth noting that while the term "Coxeter group" is quite standard, it can be ambiguous if the generating set $S$ is not specified. For example, the dihedral group of order twelve has two realizations as a Coxeter group:

$$\langle a, b, c : a^2 = b^2 = c^2 = (ab)^3 = (ac)^2 = (bc)^2 = 1 \rangle,$$

$$\langle r, t : r^2 = t^2 = (rt)^6 = 1 \rangle.$$

The first is the *reducible* Coxeter group $A_1 \times A_2$ of rank 3. The second is the irreducible Coxeter group $G_2$ of rank 2. To be precise, one must specify a **Coxeter system** $(W, S)$ giving both the group, and the generating set of reflections.

## 0.2   Geometric Realization of Coxeter Groups

Recall that finite real reflection groups came with three pieces of information: a group $W$, a generating set of involutions $S$, and finally, a faithful representation $\rho$

such that for all $s \in S$, $\rho(s)$ is a reflection. A Coxeter system $(W, S)$, in contrast, has no representation, and consequently, no geometry. One may attempt to fix this by constructing a linear representation called the **geometric realization** (as defined in Section 5.3 of Humphreys [19]):

Let $V$ be a vector space over $\mathbb{R}$ spanned by the abstract symbols $\{\alpha_s : s \in S\}$ and define a symmetric bilinear form $B$ on $V$ pairwise on the basis:

$$B(\alpha_s, \alpha_t) := -\cos \frac{\pi}{m(s,t)}.$$

The motivating idea is that this form will, under sufficiently nice conditions, encode the dihedral angles between generating hyperplanes. Next, for each $s \in S$, define the "reflection" across the "hyperplane" corresponding to involution $s$ by

$$\rho_s(\lambda) := \lambda - 2B(\alpha_s, \lambda)\alpha_s.$$

The **geometric realization** is the linear map $\rho : W \hookrightarrow GL(V)$ defined on the generators by $s \mapsto \rho_s$. This representation preserves $B$, the bilinear form $B(\rho(\alpha_s), \rho(\alpha_t)) = B(\alpha_s, \alpha_t)$ and in so doing satisfies the relations from the group presentation.

We call the elements $\{\rho_s \, s \in S\}$ **simple reflections**. The geometric realization of the Coxeter system $(W, S)$, is well defined for *any* Coxeter system (with the convention that $B(\alpha_s, \alpha_t) = -1$ if $m(s,t) = \infty$) but while the realization is always faithful, $B$ will not, in general, be positive definite, and the "reflections" need not be orthogonal.

**Theorem 0.1.** *(Section 6.4 in Humphreys [19]) The form $B$ is positive definite if and only if $W$ is finite.*

In this case, it turns out that the geometric realization also **essential**, meaning that no nontrivial subspace of $V$ is fixed pointwise by $W$ i.e. the trivial representation is not a factor.

If $B$ is positive definite, then $(V, B)$ is an inner product space, so for a finite Coxeter group $W$ we may identify the inner product space $(V, B)$ with $\mathbb{R}^n$ where $n$ is the rank of $(W, S)$. Because the inner product is invariant under the group action, the geometric realization is orthogonal and the generating "reflections" are actually *Euclidean* reflections. It then turns out that the dihedral angle between the fixed hyperplanes of the simple reflections $\rho_s$ and $\rho_t$ is $\pi/m(s,t)$ for all $s, t \in S$.

For any finite Coxeter system $(W, S)$ then, there is a naturally associated hyperplane arrangement $\mathcal{A}$, called the **Coxeter arrangement**, consisting of the simple reflections and their conjugates. Hyperplane arrangements are of general interest (see Stanley's introduction to the subject [36]) with the most basic question being the enumeration of regions, i.e. connected components of $V \smallsetminus (\bigcup_{H in \mathcal{A}} H)$. For Coxeter arrangements, regions are also called **chambers** and one has the following (see Hall [18] chapter 8 for details):

**Proposition 0.2.** *The group $W$ acts freely and transitively on the chambers. Thus, the order of $W$ is equal to the number of chambers.*

**Proposition 0.3.** *Fix a chamber $\mathcal{C}_\Omega$. Then for all $v \in V$, the $W$-orbit of $v$ contains exactly one point in the closure $\overline{\mathcal{C}_\Omega}$ of $\mathcal{C}_\Omega$.*

The finite irreducible Coxeter groups fall into two overlapping classes: groups of symmetries of regular polytopes, and groups which stabilize a lattice in $\mathbb{R}^n$ (called Weyl groups). This second class is closely related to the classification of semisimple Lie algebras and may be endowed with the additional structure of a "crystallographic root system" (see Section 0.3). The irreducible real finite reflection groups are exactly those with Coxeter diagrams given in Figure 1.

Throughout this thesis, we will always start with type $A_n$, where the finite real reflection group is the symmetric group on $n + 1$ letters $\mathfrak{S}_{n+1}$ generated by adjacent transpositions $S = \{r_i = (i, i+1)\}$ along with the **reflection representation** defined as follows:
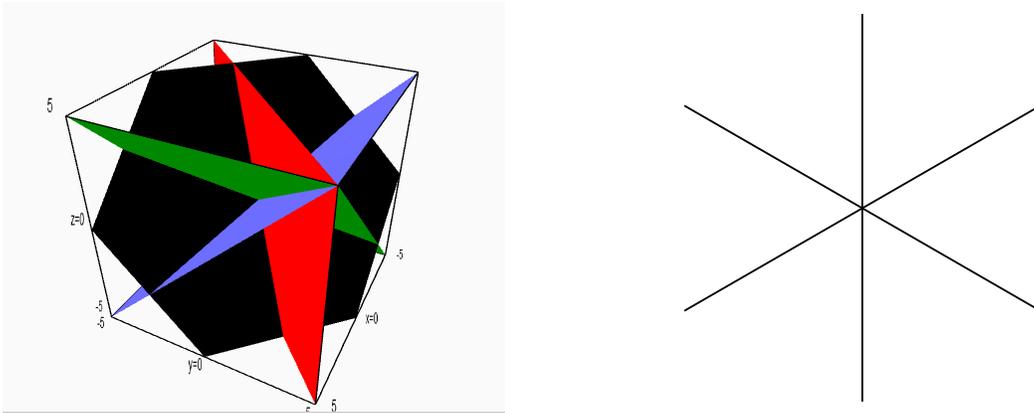
Figure 2: The Weyl group for $A_2$ is a finite real reflection group of rank 2, but its defining representation is of rank 3, generated by the hyperplanes $x - y = 0$ and $y - z = 0$. Restriction to the hyperplane $(1, 1, 1)^\perp$ yields the reflection representation.

Geometrically, $\mathfrak{S}_{n+1}$ acts on $\mathbb{R}^{n+1}$ by permuting coordinates and the generating adjacent transpositions $(i, i + 1)$ are represented by generating reflections across the hyperplanes $(e_i - e_{i+1})^\perp$. This representation is sometimes called the **defining representation** of $\mathfrak{S}_{n+1}$. The defining representation is not essential since the one-dimensional subspace generated by the vector of all ones is acted upon trivially. We obtain the reflection representation from the defining representation by restricting to the hyperplane where the coordinates sum to zero (the black hexagon in Figure 2) which we denote $\mathbb{R}_0^{n+1}$. The reflection representation is isomorphic to the geometric realization of the Coxeter system $(A_n, S)$.

## 0.3 Root Systems

Reflections correspond geometrically to hyperplanes– objects of *codimension* one. It is often more convenient to work with objects of *dimension* one, such as the vectors normal to the reflecting hyperplanes. This inspires the axiomatic definition of a root system.

A finite set $\Phi$ of vectors (which we will call roots) in a real vector space $V$ with inner product $(\cdot, \cdot)$ forms a **root system** if it satisfies the following conditions:

1. The roots span $V$:

$$\mathbb{R}\Phi = V.$$

2. The only scalar multiples of a root $\alpha \in \Phi$ that belong to $\Phi$ are $\alpha$ itself and $-\alpha$:

$$\forall \alpha \in \Phi, \quad \mathbb{R}\alpha = \{\pm\alpha\}.$$

3. For every root $\alpha \in \Phi$, the set $\Phi$ is closed under reflection through the hyperplane perpendicular to $\alpha$:

$$\forall \alpha, \beta \in \Phi, \quad t_\alpha(\beta) := \beta - 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}\alpha \in \Phi.$$

The root system is called **crystallographic** if a fourth condition is met:

4. (Integrality) If $\alpha$ and $\beta$ are roots in $\Phi$, then the projection of $\beta$ onto the line through $\alpha$ is an integer or half-integer multiple of $\alpha$. Equivalently:

$$\forall \alpha, \beta \in \Phi, \qquad 2\frac{(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}.$$

The group generated by the $t_\alpha$ we will call $W(\Phi)$. It is a finite real reflection group (finiteness follows from the fact that it injects into the group of permutations of $\Phi$). In fact, every finite real reflection group arises from a root system in this way. On the other hand, given a Coxeter system $(W, S)$, assuming $B(\alpha_s, \alpha_s) = 1$ for all $\alpha \in \Phi$, and following the notation conventions from Section 0.2, one can show that

$$\Phi(W) := \{w(\alpha_s) : w \in W, s \in S\}$$

is a root system. One may verify that $W(\Phi(W)) \cong W$ and if we define isomorphism of root systems up to orthogonal transformations and length of roots, then $\Phi(W(\Phi)) \cong \Phi$.

Because root systems are highly symmetric, it helps to make some arbitrary choices to establish a frame of reference. If $\Phi$ is a root system in the real vector space $V$, any generic hyperplane will partition $\Phi$ into two sets,

$$\Phi = \Phi^+ \sqcup \Phi^-$$

which we call a **positive system** and **negative system** respectively. The entire root system is contained in the cone generated by $\Phi^+$, which consists of a **positive cone** $\mathcal{C} = \left\{ \sum_{\alpha \in \Phi^+} c_\alpha \alpha : c_\alpha \in \mathbb{R}_{\geq 0} \right\}$ and a **negative cone** $-\mathcal{C} = \left\{ \sum_{\alpha \in \Phi^-} c_\alpha \alpha : c_\alpha \in \mathbb{R}_{\geq 0} \right\}$. Let $\Pi$ denote the set of roots generating the extremal rays of the positive cone. Every root $\alpha \in \Phi$ may be expressed as a linear combination (over $\mathbb{R}$ in general, $\mathbb{Z}$ if the root system is crystallographic) of elements of $\Pi$ with either all coefficients positive (if $\alpha$ is in the positive cone) or all coefficients negative (if $\alpha$ is in the negative cone). It is less obvious (but true) that $\Pi$ is a vector space basis for $V$. We say that $\Pi$ is a **simple system** because it possesses these two properties. In general, simple systems and positive systems determine each other uniquely (Section 1.3 of [19]). The simple vectors $\alpha$ of $\Phi$ have a natural interpretation in terms of the Coxeter group $W(\Phi)$ because their associated reflections $t_\alpha$ form a Coxeter generating set for $W$. Moreover, given a Coxeter system $(W, S)$ there exists some hyperplane for which the roots $\{\alpha_s : s \in S\}$ are a simple system for $\Phi(W)$. We will conflate these notions by always picking such a hyperplane as our "generic" one, thus identifying $\Pi$ and $\{\alpha_s : s \in S\}$, the set of **simple roots**.
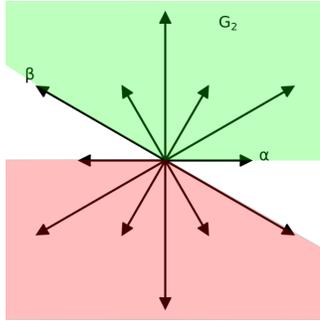
Figure 3: The $G_2$ root system with a choice of $\alpha$ and $\beta$ simple roots. The positive cone is in green, and the negative cone is in red.

## 0.4 The Crystallographic Restriction: Coroots and Weights

A subgroup $G \subset GL(V)$ is said to be **crystallographic** if it stabilizes a lattice (discrete additive subgroup) $L$ in $V$. That is, $gL \subset L$ for all $g \in G$. In Figure 1, notice that $H_3$, $H_4$, and most of the dihedral groups are not crystallographic. The following theorem explains why:

**Theorem 0.4.** *(Section 2.8 of Humphreys [19]) If $(W, S)$ is a Coxeter group whose geometric realization is crystallographic, then for all $\alpha, \beta \in S$, the integer $m(\alpha, \beta)$ must be either 2,3,4, or 6.*

*Proof.* If $\alpha \neq \beta$ we know that $\rho_\alpha \rho_\beta \neq 1$ acts on the plane spanned by $\alpha$ and $\beta$ as a rotation through the angle $\theta := 2\pi/m(\alpha, \beta)$, while fixing the orthogonal complement pointwise. Thus its trace, relative to a compatible choice of basis for $V$, is $(n - 2) + 2\cos\theta$ where $n = \dim(V)$. On the other hand, the matrix of $\rho_\alpha \rho_\beta$ with respect to the basis of simple roots, is an integral matrix so it has an integral trace. Thus, $\cos\theta$ must be a half-integer, while $0 < \theta \leq \pi$. The only possibilities are $\cos\theta = -1, -1/2, 0, 1/2$, corresponding to the cases $m(\alpha, \beta) = 2, 3, 4, 6$. $\square$

Crystallographic information may be conveniently encoded via the $n \times n$ **Cartan matrix**: $C_\Phi := [a_{ij}]$ where

$$a_{ij} := \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.$$

In the previous section we mentioned the classification of root systems up to orthogonal transformation and length of roots; ignoring crystallographic structure. The crystallographic restriction on the root system $\Phi$ is equivalent to $W(\Phi)$ being a crystallographic group. In this setting, the integer span of the roots $\mathbb{Z}\Pi = \mathbb{Z}\Phi \subset V$ is discrete and it is called the **root lattice**, $\Lambda_\Phi$.

**Proposition 0.5.** *(corollary of proposition 8.6 in Hall [18]) Crystallographic root systems can have at most two distinct lengths for roots, which are naturally called **long roots** and **short roots**.*

If one replaces $\alpha \in \Phi$ with $\alpha^\vee := \frac{2}{(\alpha,\alpha)} \alpha$ the resulting set of vectors is again a root system, called the **dual root system** of $\Phi$. We say that the roots of the dual system $\Phi^\vee$ are **coroots** of $\Phi$. The root systems for types A, D, $F_4$ and E only have one length of root and are self dual (sometimes called **simply laced**). $G_2$ has two root lengths, but happens to be isomorphic to its dual via 30° rotation (see Figure 3). $B_2$ and $C_2$ are duals which also happen to be isomorphic via 45° rotation and scaling by a factor of two. For larger $n$, root systems $B_n$ and $C_n$ are dual and not isomorphic (see Figure 4).

It turns out that there will always be a unique **highest root**; that is a positive root whose inner product with the positive normal vector to the generic hyperplane used to partition $\Phi = \Phi^+ \bigsqcup \Phi^-$ is as large as possible. We call this highest root $\alpha_{max}$ and we will need it later.

The crystallographic restriction gives rise to a second notion of duality for roots: vectors called **weights**. Historically weights predate roots, and are defined in Lie theoretic terms as a sort of generalized eigenvalue. We will give an equivalent com-
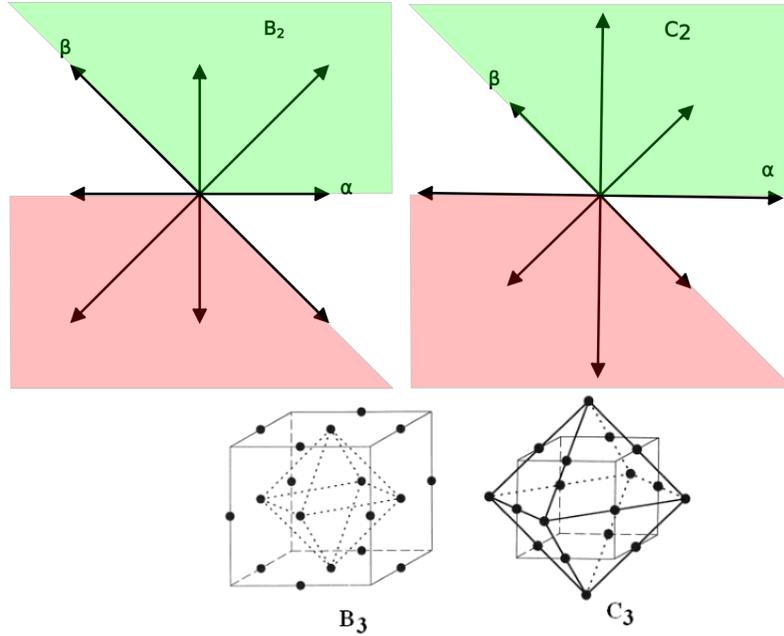
Figure 4: The regular hypercube and regular cross polytope have the same group of symmetries, but they give rise to non-isomorphic crystallographic root systems in all dimensions greater than two.

binatorial definition here, but refer the reader to Bourbaki [10] or Hall [18] for Lie theoretic definitions and exposition. Define the set of weights

$$\Lambda_\Omega := \left\{ \omega \in V : \frac{2(\alpha, \omega)}{(\alpha, \alpha)} \in \mathbb{Z} \ \forall \alpha \in \Phi \right\}.$$

Because the inner product is a continuous and bilinear function, $\Lambda_\Omega$ is discrete and forms an additive subgroup, and is accordingly called the **weight lattice**. The crystallographic restriction further implies that the weight lattice includes the root lattice as a sublattice (see Figure 5). The order of the group $\Lambda_\Omega / \Lambda_\Phi$ is called the **index of connection**. (In the general theory, this determines the number of nonisomorphic Lie groups possessing the same Lie algebra structure).

Given a roots system $\Phi$ with positive simple roots $\{\alpha_1, \ldots, \alpha_n\} = \Pi$, We define **fundamental weights** $\omega_j$ such that

$$\frac{2(\alpha_i, \omega_j)}{(\alpha_i, \alpha_i)} = \delta_{i,j}$$

for all $i, j \in [n]$. Because the simple roots $\{\alpha_i\}$ form a basis for the vector space $V$, so do the fundamental weights $\{\omega_j\} =: \Omega$.

**Proposition 0.6.** *Fundamental weights are, in fact, weights, and they generate the weight lattice.*

$$\mathbb{Z}\Omega = \Lambda_\Omega.$$

**Proposition 0.7.** *The columns of the **Cartan matrix** give the coordinates of the simple roots in the basis of the fundamental weights. Moreover, the determinant of the Cartan matrix is the index of connection.*

For example, the root system $G_2$ has Cartan matrix $\left[\begin{smallmatrix} 2 & -3 \\ -1 & 2 \end{smallmatrix}\right]$ meaning that $\alpha_1 = 2\omega_1 - 1\omega_2$ and $\alpha_2 = -3\omega_1 + 2\omega_2$ as one can verify in Figure 5. Notice that intrinsic to the definition of the Cartan matrix, is an ordering of the simple roots, $\alpha_1, \alpha_2, \ldots, \alpha_n$. Unfortunately, in general, there is no canonical choice for such an ordering. There are, however, standard conventions for types A, B, C, and D. As noted in Section 0.3, for type A, one may identify the vector space $V$ with the codimension one subspace of $\mathbb{R}_0^{n+1} \subset \mathbb{R}^{n+1}$ on which the standard basis coordinates sum to zero, and naturally choose $\Pi_A = \{\alpha_i = e_i - e_{i+1} : 1 \le i \le n\}$. With respect to this ordering, the Cartan matrix, is

$$C_A(i,j) = \begin{cases} 2 & i = j \\ -1 & i = j \pm 1 \\ 0 & \text{otherwise} \end{cases}.$$

For types B, C, and D, the vector space $V$ is identified with $\mathbb{R}^n$ and the simple roots are coordinatized as:

$$\Pi_B = \{\alpha_i = e_i - e_{i+1} : 1 \le i \le n-1\} \cup \{\alpha_n = e_n\} \tag{1}$$

$$\Pi_C = \{\alpha_i = e_i - e_{i+1} : 1 \le i \le n-1\} \cup \{\alpha_n = 2e_n\} \tag{2}$$

$$\Pi_D = \{\alpha_i = e_i - e_{i+1} : 1 \le i \le n-1\} \cup \{\alpha_n = e_{n-1} + e_n\} \tag{3}$$

giving Cartan matrices:

$$C_B(i,j) = \left(\begin{array}{c|c} C_{A_{n-1}} & \begin{matrix} 0 \\ \vdots \\ 0 \\ -1 \end{matrix} \\ \hline 0 \dots 0 \quad -2 & 2 \end{array}\right) \quad C_C(i,j) = \left(\begin{array}{c|c} C_{A_{n-1}} & \begin{matrix} 0 \\ \vdots \\ 0 \\ -2 \end{matrix} \\ \hline 0 \dots 0 \quad -1 & 2 \end{array}\right) \quad C_D(i,j) = \left(\begin{array}{c|c} C_{A_{n-1}} & \begin{matrix} \vdots \\ 0 \\ -1 \\ 0 \end{matrix} \\ \hline \dots 0 - 1 \quad 0 & 2 \end{array}\right)$$

Taking determinants, we find the index of connection is $n+1$ for type $A_n$, is 2 for both $B_n$ and $C_n$, and is 4 for type $D_n$.

## 0.5 Affine Reflection Groups

We will now transition our attention to a class of infinite groups generated by *affine reflections* (across hyperplanes not necessarily containing the origin) in Euclidean space. We will see that these groups are intimately connected to finite Weyl groups and the crystallographic structure introduced in Section 0.4. We follow Chapter 4 of Humphreys [19].

Define an **affine hyperplane** in real vector space $V$ by

$$H_{\alpha,k} := \{\lambda \in V : (\lambda, \alpha) = k\}.$$
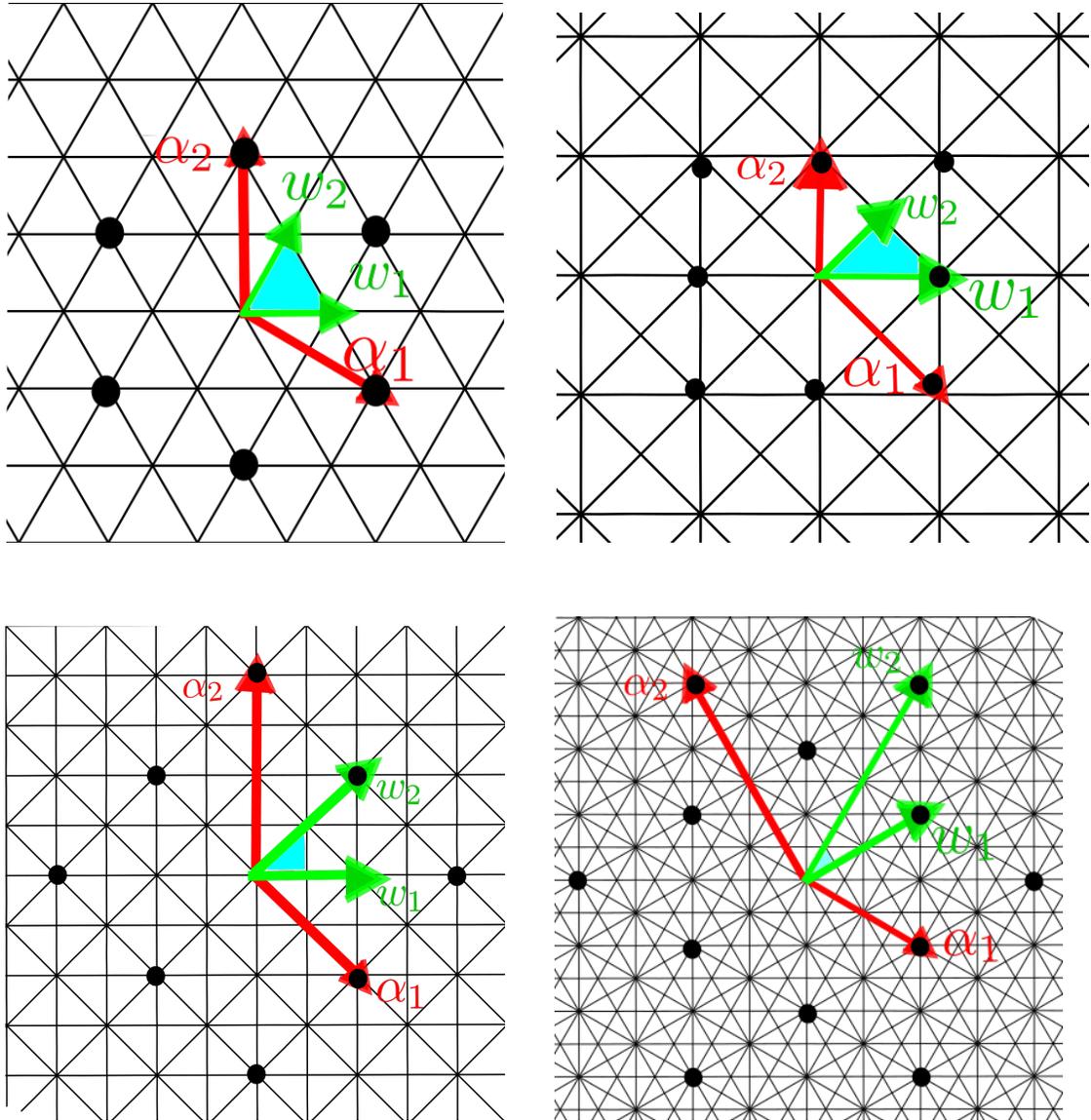
Figure 5: The superimposed root lattices and weight lattices for $A_2$, $B_2$, $C_2$ and $G_2$ with index of connection 3, 2, 2, 1 respectively. The darker points are roots, and the fundamental alcoves are shaded blue.

Notice that $H_{\alpha,k} = H_{-\alpha,-k}$ and that $H_{\alpha,0} = \alpha^{\perp}$. The hyperplane $H_{\alpha,k}$ may be obtained by translating $H_{\alpha}$ by $\frac{k}{(\alpha,\alpha)}\alpha = \frac{k}{2}\alpha^{\vee}$ (where $\alpha^{\vee}$ is the coroot corresponding to $\alpha$).

Translations in a vector space $V$ are normalized by the general linear group, $GL(V)$. Algebraically, this amounts to nothing more than the fact that for any $A \in GL(V)$ and $x, \lambda \in V$ one has $A(A^{-1}x + \lambda) = x + A\lambda$. We may then define the **affine group** $\mathrm{Aff}(V)$ as the semidirect product of $GL(V)$ and the group of translations by elements of $V$.

Define the **affine reflection** across $H_{\alpha,k}$ by

$$s_{\alpha,k}(\lambda) := \lambda - ((\lambda, \alpha) - k)\alpha^{\vee}.$$

Geometrically, $s_{\alpha,k}$ fixes $H_{\alpha,k}$ pointwise and sends the zero vector to $k\alpha^{\perp}$. $H_{\alpha,k}$ agrees with the original definition of (linear) reflection when $k = 0$. One can always write an affine reflection as a linear reflection followed by a translation: $s_{\alpha,k}(\lambda) = s_{\alpha}(\lambda) + k\alpha^{\vee}$ Given a crystallographic root system $\Phi$, define an arrangement of hyperplanes

$$\mathcal{H} := \{H_{\alpha,k} : \alpha \in \Phi, k \in \mathbb{Z}\}.$$

This arrangement $\mathcal{H}$ is acted upon naturally by both the Weyl group, $W = W(\Phi)$, and the coroot lattice, $\Lambda_{\Phi^{\vee}}$. For all $w \in W$, one may verify:

$$wH_{\alpha,k} = H_{w\alpha,k} \quad \text{and} \quad ws_{\alpha,k}w^{-1} = s_{w\alpha,k}.$$

For any coroot $\lambda$ and $x \in V$:

$$H_{\alpha,k} + \lambda = H_{\alpha,k+(\lambda,\alpha)} \quad \text{and} \quad \lambda + s_{\alpha,k}(x - \lambda) = s_{\alpha,k+(\lambda,\alpha)}(x).$$

Given a crystallographic root system $\Phi$, define the **affine Weyl group** $W_a = W_a(\Phi)$ to be the subgroup of $\text{aff}(V)$ generated by all affine reflection $s_{\alpha,k}$ where $\alpha \in \Phi$, $k \in \mathbb{Z}$.

**Proposition 0.8.** $W_a$ *is the semidirect product of* $W$ *and the translation group corresponding to the coroot lattice* $\Lambda_{\Phi^\vee}$.

*Proof.* The $W$ action on $V$ takes coroots to coroots, and thus normalizes $\Lambda_{\Phi^\vee}$. Since, moreover, $\Lambda_{\Phi^\vee}$ and $W$ intersect trivially, their semidirect product $L \rtimes W$ is well defined and we need only check equality of $L \times W$ and $W_a$ as sets. Because $s_{\alpha,k}(x) = s_\alpha(x) + k\alpha^\vee$, the generators of $W_a$ are all in $L \rtimes W$ and thus $W_a \subset L \times W$. On the other hand, translation by a coroot can be expressed as a composition of affine reflections: $x + k\alpha^\vee = s_{\alpha,k}(s_{\alpha,0}(x))$ so $L \subset W_a$. Since $W \subset W_a$ as well, we have $L \times W \subset W_a$. $\qquad\square$

To study finite Weyl groups geometrically, we observed a free transitive action of the group on the regions, or chambers of the complement of the corresponding Coxeter arrangement. A similar strategy works for affine Weyl groups. The complement of the affine Coxeter arrangement, $V^0 := V \smallsetminus (\bigcup_{H \in \mathcal{H}} H)$ is a collection of connected components called **alcoves** (these are the triangular regions in Figure 5). Each alcove is defined by a (finite) set of inequalities of the form $k_\alpha < (\lambda, \alpha) < k_\alpha + 1$, for $\alpha \in \Phi^+$, $k_\alpha \in \mathbb{Z}$ and, as such, is an open set. Define the **fundamental alcove** to the be particular alcove

$$A := \{\lambda \in V : 0 < (\lambda, \alpha) < 1 \quad \forall \alpha \in \Phi^+\}.$$

It is a general fact that $\alpha_{max} - \alpha$ is a sum of simple roots for all $\alpha \in \Phi^+$. This allows one to remove superfluous inequalities and obtain a simpler description of the fundamental alcove:

**Proposition 0.9.**

$$A = \{\lambda \in V : (a_i, \lambda) > 0 \quad \forall \alpha_i \in \Pi \text{ and } (\alpha_{max}, \lambda) < 1\}.$$

**Proposition 0.10.** *Because the simple roots are a simple system one may express* $\alpha_{max} = \sum_{\alpha_i \in \Pi} n_i \alpha_i$ *for some* $n_i$ *unique integers. The fundamental alcove is a simplex and is the interior of the convex hull of the zero vector and vertices* $v_i$, *each a scalars of fundamental weight* $\omega_i$ *with coefficient* $c_i = \dfrac{2}{n_i(\alpha_i, \alpha_i)}$. *That is:*

$$\bar{A} = \text{Conv}\left(\left\{\frac{2}{n_i(\alpha_i, \alpha_i)}\omega_i : 1 \leq i \leq n\right\} \cup \{\mathbf{0}\}\right).$$

*Proof.* Recall that, by definition, the fundamental weights satisfy $\dfrac{2(\omega_i, \alpha_j)}{(\alpha_i, \alpha_i)} = 0$ for $i \neq j$, and $(\alpha_i, \omega_i) = \dfrac{(\alpha_i, \alpha_i)}{2}$.

$$(\alpha_{max}, c_i\omega_i) = 1$$
$$c_i\left(\sum n_j\alpha_j, \omega_i\right) = 1$$
$$c_i n_i(\alpha_i, \omega_i) = 1$$
$$c_i n_i(\alpha_i, \omega_i) = 1$$
$$c_i n_i \frac{(\alpha_i, \alpha_i)}{2} = 1$$
$$c_i = \frac{2}{n_i(\alpha_i, \alpha_i)}.$$

$\square$

The numbers $n_i$ are sometimes called the **marks** [41].

For type A, every root $\alpha$ has $(\alpha, \alpha) = 2$ and $\alpha_{max} = \sum_{\alpha_i \in \Pi} \alpha_i$ so $c_i = 1$ for all $i$, and the fundamental alcove is simply the interior of the convex hull of the zero vector and the fundamental weights.

For type B (with respect to a suitable choice for the generic hyperplane defining positive and negative roots) one has $\alpha_{max} = e_1 + e_2 = \alpha_1 + 2(\alpha_2 + \alpha_3 + \cdots + \alpha_n)$ and the

non-zero vertices of the fundamental alcove are:

$$\left( \frac{2}{1 \times 2}\omega_1, \frac{2}{2 \times 2}\omega_2, \frac{2}{2 \times 2}\omega_2, \dots, \frac{2}{2 \times 2}\omega_{n-1}, \frac{2}{2 \times 1}\omega_n \right) = \left( \omega_1, \frac{\omega_2}{2}, \frac{\omega_3}{2}, \dots, \frac{\omega_{n-1}}{2}, \omega_n \right).$$

In particular, for $B_2$, $\alpha_{max} = 1\alpha_1 + 2\alpha_2$ and $(\alpha_1, \alpha_1) = 2$ and $(\alpha_2, \alpha_2) = 1$ and the non-zero vertices of the alcove (as shown in Figure 5) are exactly the fundamental weights:

$$v_1 = \frac{2}{1 \times 2}\omega_1 = \omega_1$$

$$v_2 = \frac{2}{2 \times 1}\omega_2 = \omega_2.$$

For type C,(again with respect to a suitable choice for the generic hyperplane defining positive and negative roots) $\alpha_{max} = 2e_1 = 2(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}) + \alpha_n$ and the non-zero vertices of the fundamental alcove are:

$$\left( \frac{2}{2 \times 2}\omega_1, \frac{2}{2 \times 2}\omega_2, \frac{2}{2 \times 2}\omega_2, \dots, \frac{2}{2 \times 2}\omega_{n-1}, \frac{2}{1 \times 4}\omega_n \right) = \left( \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}, \dots, \frac{\omega_{n-1}}{2}, \frac{\omega_n}{2} \right).$$

In particular, for $C_2$, $\alpha_{max} = 2\alpha_1 + 1\alpha_2$ and $(\alpha_1, \alpha_1) = 2$ and $(\alpha_2, \alpha_2) = 4$. Thus,

$$v_1 = \frac{2}{2 \times 2}\omega_1 = \frac{\omega_1}{2}$$

$$v_2 = \frac{2}{1 \times 4}\omega_2 = \frac{\omega_2}{2}.$$

For $G_2$, $\alpha_{max} = 3\alpha_1 + 2\alpha_2$. It is common to coordinatize such that $\alpha_1 = (1, -1, 0)$ $\alpha_2 = (-1, 2, -1)$ and so $(\alpha_1, \alpha_1) = 2$ and $(\alpha_2, \alpha_2) = 6$. Thus,

$$v_1 = \frac{2}{3 \times 2}\omega_1 = \frac{1}{3}\omega_1$$

$$v_2 = \frac{2}{2 \times 6}\omega_2 = \frac{1}{6}\omega_2.$$

## 0.6 Partial Orders on Coxeter Groups

We have seen that finite Coxeter systems not only have an algebraic description in terms of the group presentation, but also a geometric description as a finite group generated by reflections with an associated root system. There is a third combinatorial description in terms of "reduced words": Let $(W, S)$ be a Coxeter system with finite generating set $S$. Every element of $W$ may be expressible (non-uniquely) as a word in the letters from $S$. Consider the word length function $\ell : W \to \mathbb{Z}$ with respect to $S$, which sends each element $w \in W$ to the length of a shortest word describing it. If $\ell_S(w) = r$ and $w = s_{i_1} s_{i_2} \cdots s_{i_r}$, we call $s_{i_1} s_{i_2} \cdots s_{i_r}$ a **reduced word** for $w$. Reduced words are generally not unique, but they satisfy two important properties:

*THE EXCHANGE PROPERTY:* Let $w = s_1 s_2 \cdots s_r$ be a reduced word and consider $s \in S$. If $\ell_S(sw) < \ell_S(w)$ then $sw = s_1 \cdots \hat{s}_i \cdots s_r$ for some $1 \le i \le r$.

*THE DELETION PROPERTY:* If $w = s_1 s_2 \cdots s_r$ and $\ell_s(w) < r$, then we have $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$ for some $1 \le i < j \le r$.

**Theorem 0.11.** *(Chapter 1 of Björner and Brenti [9]) If $W$ is a group with a generating set $S$ of involutions, the following are equivalent:*

- $(W, S)$ *is a Coxeter system.*

- $(W, S)$ *satisfies the exchange property.*

- $(W, S)$ *satisfies the deletion property.*

There are several natural partial orders on the set of elements in a Coxeter system. We call "$\le$" an **order relation** or a **partial order** on a set $P$ if it is reflexive ($x \le x$ for all $x \in P$), transitive ($x \le y$ and $y \le z$ implies $x \le z$ for all $x, y, z \in P$), and antisymmetric ($x \le y$ and $y \le x$ implies $x = y$). A set along with a partial order is called a **poset**. A poset $P$ is further said to be **graded** if there exists a rank

function $rk : P \to \mathbb{Z}$ with the property that for all $x \leq y$ in $P$, every maximal chain $x = z_0 < z_1 < \cdots < z_k = y$ is the same length $k = rk(y) - rk(x)$.

**Definition 1.** The **right (respectively left) weak order** on Coxeter system $(W, S)$ is defined for all $\nu, \mu \in W$ by

$$\nu \leq \mu \iff \ell_S(\nu) = \ell_S(\mu) + \ell_S(\nu^{-1}\mu)$$

$$\left(\nu \leq \mu \iff \ell_S(\nu) = \ell_S(\mu) + \ell_S(\mu\nu^{-1})\right)$$

It is elementary to show that these are order relations. The right and left weak orders are generally not the same partial order, but inversion induces a poset isomorphism between the two. It is therefore not uncommon to talk abstractly about "the weak order" on $(W, S)$. The weak order is graded with rank function $\ell_S$.

The right (left) weak order may be thought of in terms of *prefixes* (*suffixes*) of reduced words: $\nu \leq \mu$ if and only if there exists a reduced word of $\nu$ which is a prefix (suffix) of a reduced word for $\mu$. Note that here inversion equates to reading a word backwards and switches prefix and suffix. We say that $s \in S$ is a **left descent** of an element $w$ if $\ell(sw) < \ell(w)$. Similarly, if $\ell(ws) < \ell(w)$ we say that $s$ is a **right descent** of $w$.

It follows from the exchange property (see Björner and Brenti chapter 3 [9]) that the weak order is, in general, a **meet-semilattice**: there exist a greatest lower bound for any pair of elements $\nu, \mu \in W$ called their **meet** and denoted $\nu \wedge \mu$. If $W$ is finite, then the weak order has a maximum element, often called "the longest element" and denoted $\omega_0$. It then follows from general lattice theory (see proposition 3.3.1 of EC1 [37]) that the weak order for a finite Coxeter system is a **lattice**: a poset with both a meet an a **join** (a least upper bound for every pair of elements $\nu, \mu \in W$ denoted $\nu \vee \mu$.) Finite lattices will always have a unique smallest element, denoted $\hat{0}$ and a unique largest element, denoted $\hat{1}$. It is perhaps unfortunate that "lattice" refers to
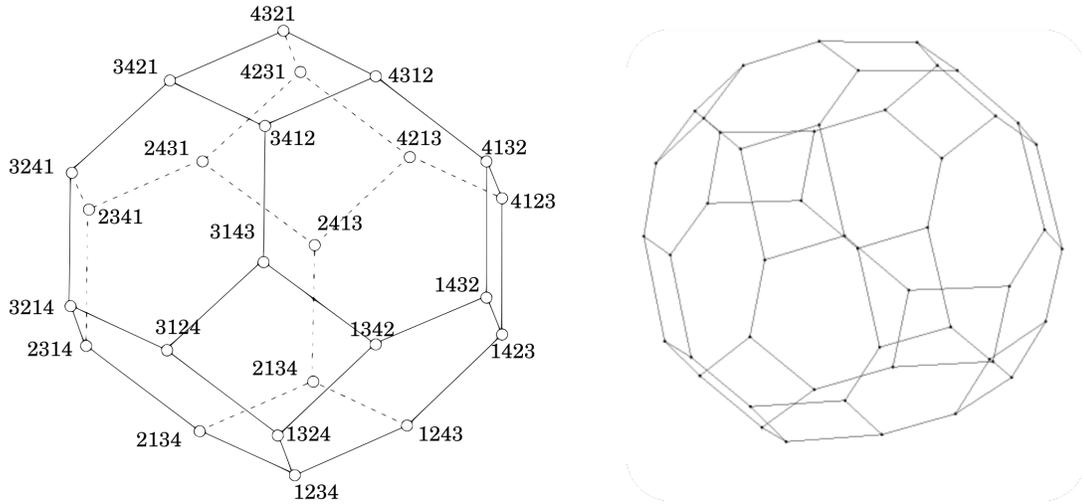
Figure 6: The graph of the Permutohedron of type $A_3$ is also the Hasse diagram of the weak order on the Weyl group of $A_3$ (i.e. the symmetric group $\mathfrak{S}_4$). The Permutohedron of type $B_3$ is shown on the right.

both a type of poset, and a discrete additive subgroup of Euclidean space, but let us address any ambiguity as it may arise.

We say an element $y$ **covers** an element $x$ in poset $P$ if $x < y$ and there does not exist $x \lneqq z \lneqq y$. We say that $x$ is covered by $y$ and write $x \lessdot y$. (Note that left (respectively right) descents equate to covers in the left (respectively right) weak order.) For locally finite posets (ones with finite intervals), one often considers the **Hasse diagram**: a directed graph with vertex set the elements of $P$ and an edge from $x$ to $y$ if and only if $x \lessdot y$. Hasse diagrams are typically drawn so that all edges are directed up. Forgetting orientation, the Hasse diagram of a the weak order for a Coxeter system of rank $n$ will be an $n$-regular graph; the Cayley graph of $W$ with respect to generating set $S$.

The weak order for a finite Coxeter system $(W, S)$ also has a nice geometric interpretation in terms of the **Coxeter arrangement**. Recall that the connected components of $V \smallsetminus (\cup_{H \in \mathcal{A}} H)$, called **chambers**, are acted upon freely and transitively by $W$. Each hyperplane in $\mathcal{A}$ has a positive side corresponding to the direction of
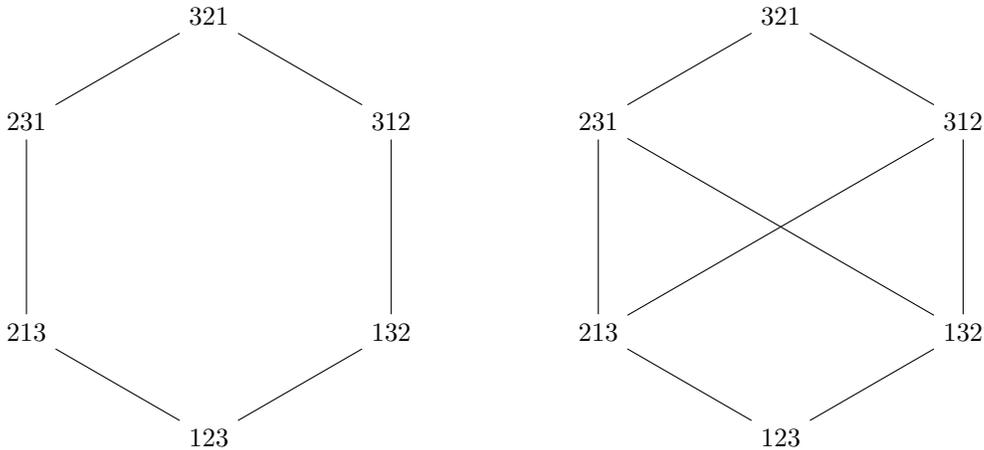
Figure 7: The hasse diagrams of the weak order and the (strong) Bruhat order for the Weyl group of $A_2$.

its positive root, and the intersection of all of the positive half-spaces we called the **fundamental chamber**.

Picking a generic point $p$ in the fundamental chamber, its $W$-orbit will have one point in each chamber and give a natural bijection between chambers and elements of $W$. The convex hull of these points is called the **permutohedron** of type $W$, and its 1-skeleton is isomorphic to the Hasse diagram of the weak order (see Figure 6). We refer the reader to Fomin and Reading [16] and Björner and Brenti [9] for more on the weak order, permutohedra, and related combinatorics.

A second partial order on Coxeter systems $(W, S)$ is the **strong order** or **Bruhat order**. We say $\nu \leqslant \mu$ if there is a reduced word (again, in letters from $S$) for $\nu$ which appears as *any* subword of $\mu$ (not necessarily a prefix or suffix). Since prefixes and suffixes are both subwords, Bruhat order refines both the right and left weak orders, and removes their inherent "sidedness". (See Figure 7.) Bruhat order is (generally) not a lattice or meet-semilattice, but we will see in Section 0.7 that (at least in types A and B) it can be completed to one in a somewhat natural way. Bruhat order arises as the inclusion order of closures of Bruhat cells from the corresponding semisimple Lie group.

# 0.7 Dedekind-MacNeille Completion of Bruhat Order and Alternating Sign Matrices

Given a lattice $L$, one may consider the greatest lower bound or least upper bound of any finite set of elements, extending the binary meet and join operators $\wedge, \vee : L \times L \to L$ to the maps $\wedge, \vee : \underbrace{L \times \cdots \times L}_{n \text{ times}} \to L$. It is straightforward to verify that $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ and $x_1 \vee x_2 \vee \cdots \vee x_n$ are well-defined regardless or reordering or parenthesization.

In general, however, the meet and join operators need not extend to arbitrary (infinite) subsets of $L$. Take, for example, the rational numbers $\mathbb{Q}$ with their usual total order. The element $3 \vee 3.1 \vee 3.14 \vee 3.141 \vee \ldots$ is not well defined because the $\pi \notin \mathbb{Q}$. If the meet and join operators do extend to all subsets of a lattice $L$, we say that $L$ is a **complete lattice**.

**Definition 2.** (MacNeille 1937 [24]) The **Dedekind-MacNeille completion** of a poset $P$ is the smallest complete lattice $L$ containing $P$ as a subposet.

The motivating example is that the Dedekind-MacNeille completion of $\mathbb{Q}$ is $\mathbb{R}$. On the other extreme, the Dedekind-MacNeille completion of any locally-finite lattice $L$ is merely $L$ itself. In general, the completion may be constructed explicitly using a generalization of the Dedekind cuts used to construct the real numbers from the rationals. We refer the reader to [8] for more details on general constructions. We will need one particular property later:

**Proposition 0.12.** *(Siegfried & Schröder, Proposition 5.3.7, p. 121 [35]) A poset $P$ is **join-dense** and **meet-dense** in its Dedekind–MacNeille completion; that is, every element of the completion is a join of some set of elements of $P$, and is also the meet of some set of elements in $P$. The Dedekind–MacNeille completion is characterized among lattice completions of $P$ by this property.*

The only posets we have considered which are not a finite lattices, are Bruhat orders. One may wonder how many element are in their Dedekind-MacNeille completions, and if the elements have natural combinatorial descriptions. For types A and B, Lascoux and Schützenburger showed that the Bruhat orders "exhibits clivage [1]" and that a poset exhibits clivage if and only if its Dedekind-MacNeille completion is a "distributive lattice".

A **distributive lattice** is a lattice where for all $x, y, z \in L$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

One can show that this property is equivalent to

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Stanley calls distributive lattices "the most important class of lattices from the combinatorial point of view," and gives a great overview in EC1 Sections 3.4 and 3.5 [38]. The fundamental theorem of finite distributive lattices (sometimes called Birkhoff's representation theorem) is particularly appealing:

**Theorem 0.13.** *Let $L$ be a finite lattice. Then $L$ is distributive if and only if there is a unique (up to isomorphism) poset $P$ for which $L \cong J(P)$.*

We say that, in a lattice $L$, an element $\hat{0} \neq x \in L$ is **join-irreducible** if $x$ cannot be expressed as the join of two other elements. In a finite lattice, it turns out that an element is join-irreducible if and only if it covers exactly one element. Further, if $L$ is a distributive lattice, then the poset $P$ from theorem 0.7 is merely the subposet of the join-irreducibles.

---

[1] A finite poset is said to exhibit "clivage" [23] or be "dissective" [34] if every join-irreducible element generates a principal order filter whose complement is a principal order ideal.

In the left (respectively right) weak order, the join-irreducibles are exactly the elements with a unique left (respectively right) descent. Such elements are called left (respectively right) **grassmannian** elements. Recall that Bruhat order refines both the left and right weak orders so for an element to be join-irreducible in the MacNeille completion of Bruhat order, it is necessary that it have *at most* one left descent and *at most* one right descent. On the other hand, it must have *at least* one left descent and *at least* one right descent to not be the identity ($\hat{0}$ in the poset). We call elements with exactly one left descent and one right descent **bigrassmannian** elements.

In the type A setting (the symmetric group $\mathfrak{S}_n$ with generating set the adjacent transpositions, $s_i = (i, i+1)$ for $1 \le i \le n-1$) the number of descents of the permutation $\pi$ can be recovered from its oneline notation. If the $i$th entry in the oneline notation is greater than the $i+1$st entry, then $i$ is a left descent of $\pi$. If the letter $i+1$ appears to the left of the letter $i$, then $i$ is a right descent of $\pi$.

Given a triple $(a, b, c)$ with $0 \le a < b < c \le n$ one may construct a bigrassmannian permutation in oneline notation by writing

$$1\,2\,\ldots\,a-1\,c\,c+1\,\ldots\,n-1\,n\,a\,a+1\,\ldots\,b-1\,b$$

This map is a bijection and therefore demonstrates that there are $\binom{n+1}{3}$ bigrassmannian permutations in $\mathfrak{S}_n$.

Lascoux and Schützenburger showed that, for type A, the bigrassmannian permutations are exactly the join-irreducibles in the Dedekind-MacNeille completion of Bruhat order. They also gave a beautiful combinatorial description of all of the elements in terms of **alternating sign matrices** and **monotone triangles**:

**Definition 3.** *Alternating Sign Matrices* or ASMs, are square matrices with entries 0, 1, or −1 whose rows and columns sum to 1 and whose non-zero entries in each row and column alternate in sign.

ASMs were first considered by Mills, Robbins, and Rumsey [26] as a natural generalization of permutation matrices. As motivation, they showed that using Dodgeson condensation to compute determinants amounts to summing over ASMs instead of permutations. A conjectural product formula was given in their seminal paper, but enumerating ASMs remained an open problem for many years. Their formula was first proved by Zeilberger in 1995 [45], and later Kuperberg found a different proof [21] using the Yang Baxter equation and methods from statistical physics.

**Theorem 0.14.** *(Zeilberger 1996 [45]) There number of $n \times n$ ASMs is*

$$ASM(n) = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!} = \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!}.$$

At the time of the Mills, Robbins, Rumsey paper, this product formula was already known to enumerate two other classes of combinatorial objects: descending plane partitions (DPPs) and totally symmetric self complementary plane partitions (TSSCPPs). In an attempt to establish a bijection with these other objects, Mills, Robbins, and Rumsey transformed their alternating sign matrices into certain triangular arrays [11]:

**Definition 4.** A **monotone triangle** (or strict Gelfand pattern or a gog triangle) is a number triangle with $n$ rows, the $k$th row containing exactly $k$ entries between 1 and $n$. There is strict increase across rows and weak increase diagonally or down columns.

The Mills, Robbins, Rumsey bijection from ASMs to monotone triangles is as follows: Let the $k$th row of the triangle equal the positions of 1's in the sum of the first $k$ rows of an alternating sign matrix. (see Figure 8). In particular, the identity

$$
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\leftrightarrow
\begin{array}{cccccc}
3 & & & & & \\
2 & 4 & & & & \\
2 & 4 & 5 & & & \\
2 & 3 & 4 & 6 & & \\
1 & 2 & 3 & 4 & 6 & \\
1 & 2 & 3 & 4 & 5 & 6
\end{array}
\leftrightarrow
\begin{array}{ccccc}
& 2 & & & \\
1 & 2 & & & \\
1 & 2 & 2 & & \\
1 & 1 & 1 & 2 & \\
0 & 0 & 0 & 0 & 1
\end{array}
$$
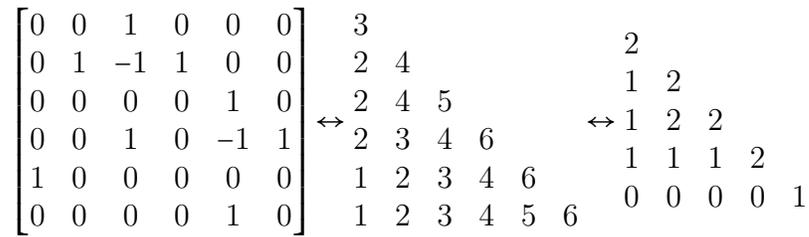
Figure 8: An ASM with its corresponding monotone triangle and reduced monotone triangle

matrix will always correspond to the monotone triangle

$$
\begin{array}{cccc}
1 & & & \\
1 & 2 & & \\
\vdots & \vdots & \ddots & \\
1 & 2 & \cdots & n
\end{array}.
$$
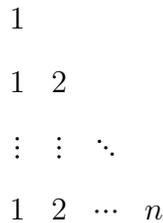
Because this is the $\hat{0}$ in the lattice of monotone triangles and the partial order is componentwise comparison, we may consider **reduced monotone triangles** by subtracting this triangle from all of the others (again, see Figure 8).

Unfortunately, to this day, there are no explicit bijections between ASMs and TSSCPPs or DPPs. There are, however some partial results [39] and determinantal formulas proving that many statistics on these objects are equinumerous [6]. Despite all this, the definition of monotone triangles was not in vain.

**Theorem 0.15.** *(Lascoux and Schützenberger, 1996 [23]) The Dedekind-MacNeille completion of Bruhat order on permutations is isomorphic to the lattice of monotone triangles ordered componentwise.*

Lascoux and Schützenberger further showed that the join-irreducible elements are *exactly* the bigrassmannian permutations [23] We mentioned earlier, using a bijection with triples $0 \leq a < b < c \leq n$, that the number of bigrassmannian permutation in $\mathfrak{S}_n$

is the **tetrahedral number**, $\binom{n}{3}$ which is counted by the coefficients of

$$\frac{1}{(1-z)^4} = 1 + 4z + 10z^2 + 20z^3 + 35z^4 + \dots.$$

One can also show that bigrassmannian permutations correspond to the reduced monotone triangles determined by fixing a single entry and making everything else as small as possible (so as to still be a monotone triangle). There are $n-1$ ways to fix the entry in the top row, $n-2$ was to fix either of the entries in the second row, $n-3$ ways to fix any of the three entries in the third row, etc. for a total which verifies the tetrahedral count:

$$1(n-1) + 2(n-2) + 3(n-3) + \dots + (n-1)1 = \frac{(n-1)n(n+1)}{6}.$$

One may also associate each $n \times n$ ASM $M$ with its $(n-1) \times (n-1)$ North-East **corner-sum matrix**:

$$NE(M)_{i,j} := \sum_{\substack{a \le i \\ b > j}} M_{a,b} \quad \text{for } 1 \le i, j \le n-1.$$

One can show that the component-wise partial order on these corner sum matrices is the same as the order on monotone triangles. For more on alternating sign matrices, monotone triangles, corner-sum matrices, and their history we refer to [12].

Lascoux and Schützenberger also studied the Dedekind-MacNeille completion of type B Bruhat order, showing that there are $\frac{n(2n^2+1)}{3}$ or **octahedral** many join-irreducibles in its Dedekind-MacNeille completion. All of them are (necessarilly) bigrassmannian, but there are $\binom{n}{4}$ bigrassmannians which are not join-irreducibles. We give an explicit description of the join-irreducibles and discuss various subtleties in Chapter 2.

Geck and Kim [17] showed that type D Bruhat order does not exhibit clivage and that consequently, its Dedekind-MacNeille completion is not a distributive lattice. We leave this case, along with the exceptional types, untouched.

## 0.8   The Waldspurger and Meinrenken Theorems

The main contribution of this thesis is to make combinatorially explicit for types A B and C the contents of the following theorems of Waldspurger and Meinrenken from the early 2000s:

**Theorem 0.16.** *(Waldspurger [44])*

*Let $W$ be a finite group generated by reflections in a Euclidean vector space $V$. Let $\mathcal{C}_\Omega \subset V$ be an open dominant chamber, and $\mathcal{C} \subset V$ the closed cone over the positive roots of $\Phi(W)$. Associate with each group element $w$ the relatively open cone $C_w :=$ $(id - w)\mathcal{C}_\Omega$. Then the cones $C_w$ are all disjoint and their union covers $\mathcal{C}$. That is,*

$$\mathcal{C} = \bigsqcup_{w \in W} C_w.$$

Along with Waldspurger's original proof, there are two elegant alternative proofs in Meinrenken [25] and Bibikov and Zhgoon [7]. All are quite topological in nature, and as we will not be reusing any of the machinery from these proofs, we omit them. Meinrenken's paper also contains a natural analogue of Waldspurger's result for affine Weyl groups:

**Theorem 0.17.** *(Meinrenken [25])*

*Let the affine Weyl group for a crystallographic Coxeter system be denoted $W_a$ and recall that $W_a = \Lambda_{\Phi^\vee} \rtimes W$ where the coroot lattice $\Lambda_{\Phi^\vee}$ acts by translations. Let $A$ denote the open fundamental alcove, with $0 \in \bar{A}$. Define relatively open and possibly degenerate simplices $A_w = (id - w)A$ for $w \in W_a$. Then the $A_w$ are all disjoint, and*
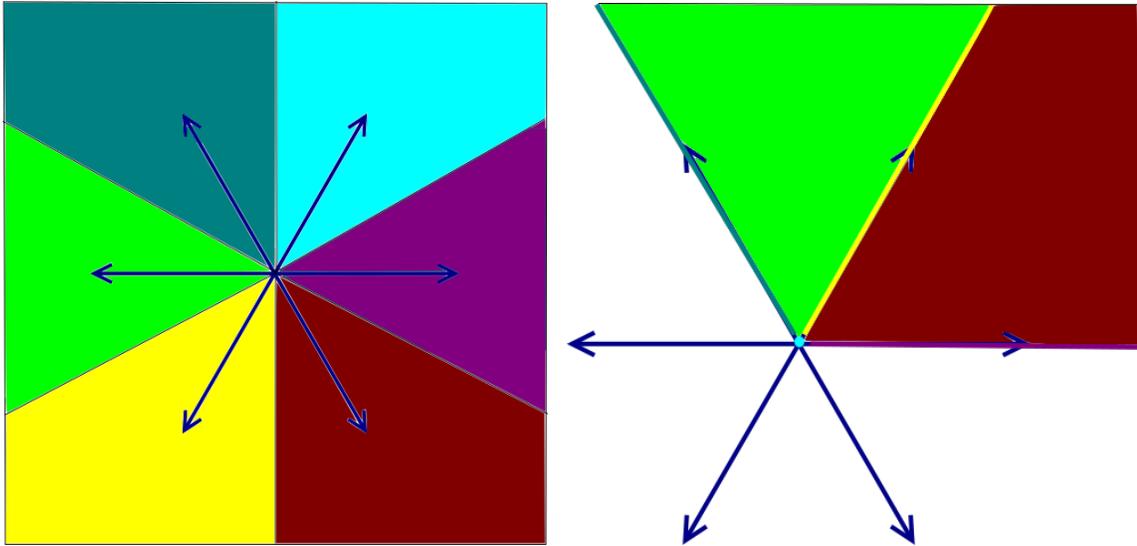
Figure 9: The chambers of the Coxeter arrangement for $A_2$ are on the left, and the corresponding Waldspurger decomposition is on the right. In the Waldspurger decomposition, three of the chambers collapse to rays, and the fundamental chamber collapses to a point at the origin.

*their union is all of $V$. That is,*

$$V = \bigsqcup_{w \in W_a} A_w.$$

We will define the **Meinrenken tile** $\mathfrak{M} := \bigsqcup_{w \in W} A_w$, restricting to the finite Weyl as a subgroup the affine Weyl group. This restriction is convenient since the coroot lattice action merely translates the Meinrenken tile:

$$V = \bigsqcup_{v \in \Lambda_{\Phi^\vee}} \mathfrak{M} + v.$$

Even in low dimensions the Meinrenken tile need not be convex (see Figure 10).
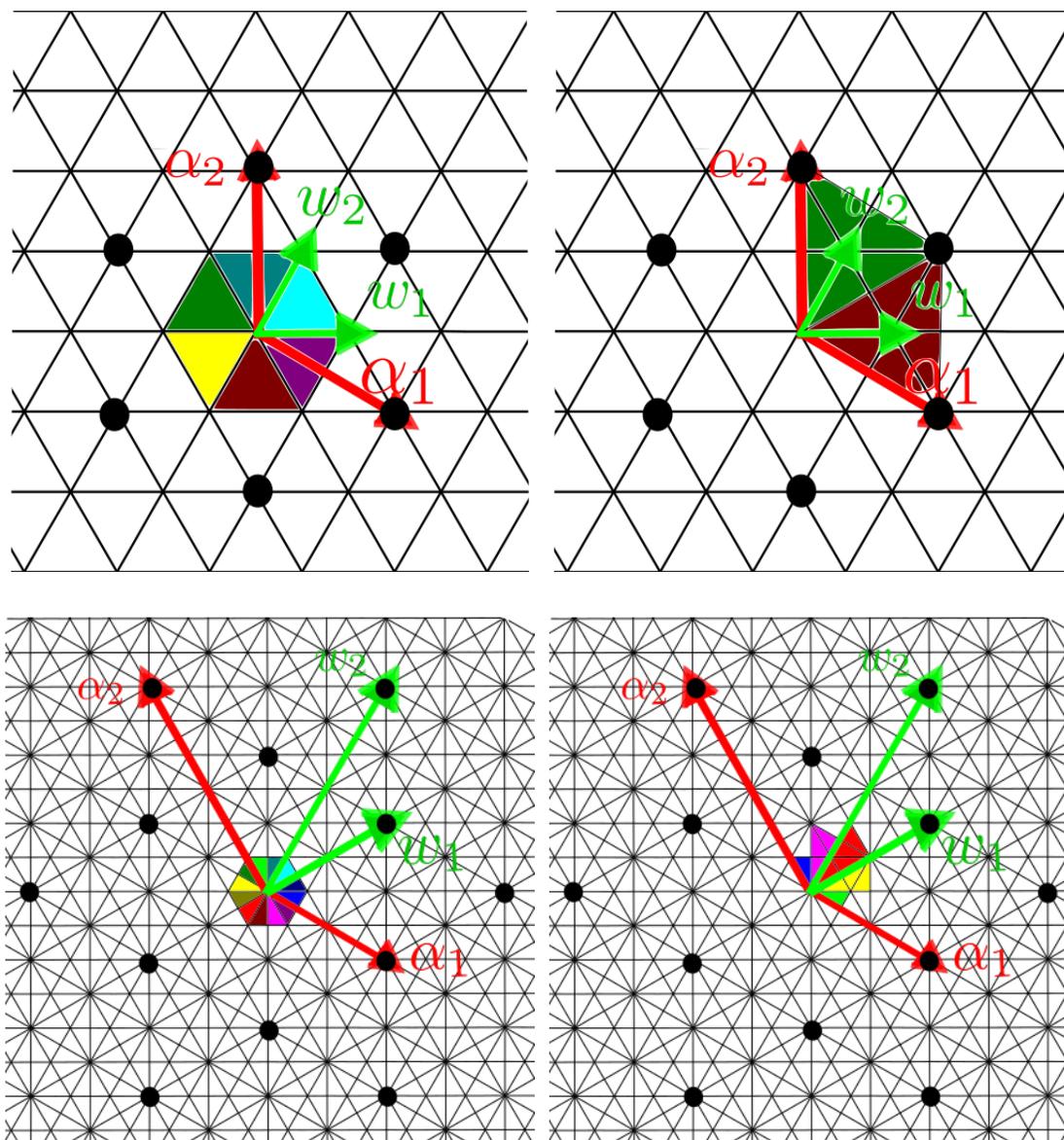
Figure 10: The subfigures on the left show a classical fundamental domain for the action of the coroot lattice in $\tilde{A}_2$ (on top) and $\tilde{G}_2$ (on bottom). On the right are the corresponding Meinrenken tiles. The Meinrenken tile for $\tilde{A}_2$ consists of a vertex at the origin, three open line segments, and two triangles. The Meinrenken tile for $\tilde{G}_2$ consists of a vertex at the origin, six open line segments, and five triangles. (See chapter 2 for types B and C pictures)

The remainder of this thesis is organized as follows:

We begin Chapter 1 with a combinatorial algorithm, assigning to each permutation $\pi \in \mathfrak{S}_n$, its **Waldspurger transform**, an $(n-1) \times (n-1)$ matrix denoted $\mathbf{WT}(\pi)$. We show that columns of $\mathbf{WT}(\pi)$ are vectors in root coordinates which describe the vertices of the cone $C_w = C_\pi$ from Waldspurger's theorem and the simplex $A_w = A_\pi$ from Meinrenken's theorem.

In Section 1.2 we classify the row and column vectors of Waldspurger matrices, showing that they satisfy certain unimodality conductions. We call such vectors "UM vectors" and we give explicit bijections between UM vectors and unimodal Motzkin paths, abelian ideals in the Lie algebra, $\mathfrak{sl}_n$, tableau with bounded hook lengths, and coroots in a certain polytope studied by Panyushev, Peterson, and Kostant [32].

In Section 1.3 we prove that componentwise comparison of Waldspurger matrices is isomorphic to Bruhat order on permutations. Summing all of the entries of a Waldspurger matrix gives the rank of the corresponding permutation in the lattice of monotone triangles. Inspired by this, we extend the Waldspurger transform to alternating sign matrices and exhibit a lattice isomorphism between ASM Waldspurger matrices and monotone triangles. We observe that an ASM Waldspurger matrix is minimal with respect to a single fixed entry, if and only if it is $\mathbf{WT}(\pi)$ for $\pi$ a bigrassmannian permutation, just as in the case of monotone triangles.

In Section 1.4 we show that the rank function for this lattice admits a geometric interpretation and that its Hasse diagram embeds naturally inside of the Meinrenken tile. Section 1.5 concludes the chapter with an investigation of the diagonal vectors of Waldspurger matrices. We show that they are in natural bijection with Motzkin paths, and that they partition the ASM lattice into disjoint intervals, each containing a unique involution whose cycle decomposition gives a non-crossing pairing.

In Chapter 2, we turn to types B and C. We define a generalized Waldspurger transform $\mathbf{WT}_\Phi$ for any crystalographic root system $\Phi$. If $\Phi$ is of rank $n$ and $w \in$

$W(\Phi)$ then we say $\mathbf{WT}_\Phi(w)$ is an $n \times n$ **type $\Phi$ Waldspurger matrix**. We define **Waldspurger order** to be the componentwise order on Waldspurger matrices. While in type A, Waldspurger order is the same as Bruhat order, this fails for type $B$ when $n > 3$. In order to describe $\mathbf{WT}_B$ and $\mathbf{WT}_C$ combinatorially, we introduce $\mathfrak{CS}_{2n} \subset \mathfrak{S}_{2n}$, the subgroup of "centrally symmetric" permutations, i.e. permutations whose permutation matrices are invariant under 180° rotation.

In Section 2.1 we recall the classical fact that $\mathfrak{CS}_{2n}$ is isomorphic to the type $B_n$ Weyl group and show that moreover, componentwise order on $\mathbf{WT}(\mathfrak{CS}_{2n})$ is isomorphic to type $B_n$ Bruhat order. The geometric realization of the $B_n$ Weyl group as the group $\pm\mathfrak{S}_n$ of $n \times n$ **signed permutations** is obtainable from the $\mathfrak{CS}_{2n}$ representation via a "folding" isomorphism, and we extend this folding map to $\mathbf{WT}(\mathfrak{CS}_{2n})$.

Hoping to recover the Dedekind-MacNeille completion of type B Bruhat order, Section 2.2 looks at the Waldspurger transform of centrally symmetric alternating sign matrices, $\mathbf{WT}(HTASM_{2n})$. We prove that this set is a distributive lattice with the same octahedral number of join-irreducibles as the Dedekind-MacNeille completion of type B Bruhat order (hereafter called $BASM_n$). Despite this, we show that the two posets are not isomorphic. Proposition 0.7 implies that $BASM_n$ may be viewed as a subposet of $\mathbf{WT}(HTASM_{2n})$. In this setting, we compare their underlying posets of join-irreducibles (which we call $P_1$ and $P_2$ respectively) noting that they share all but tetrahedral many elements. This culminates in Theorem 2.2 where we characterize the elements of $P_2$, the "type B base" of Lascoux and Schützenberger.

Section 2.3 returns to the transforms $\mathbf{WT}_B$ and $\mathbf{WT}_C$. We show that folding elements of $\mathbf{WT}(\mathfrak{CS}_{2n})$ vertically gives type $C_n$ Waldspurger matrices, and that folding horizontally gives type $B_n$ Waldspurger matrices. Section 2.4 looks at the small examples of $B_2$ and $C_2$ and Section 2.5 characterizes the row and column vectors of type $B$ and $C$ Waldspurger matrices. Section 2.6 explains why folding causes

Waldspurger order to disagree with Bruhat order for types B and C, and Section 2.7 closes the chapter with some open problems.

There are several symmetries of the type A Meinrenken tile which are observable in Figure 10 and Figure 1.3. Chapter 3 formally characterizes these symmetries, showing in Section 3.1 that they hold in all dimensions. The Meinrenken tile is topologically "half open" in the sense that the coroot lattice action induces a bijection between $\partial\overline{\mathfrak{M}} \cap \mathfrak{M}$ and $\overline{\mathfrak{M}} \cap (V \smallsetminus \mathfrak{M})$.

Section 3.2 is largely enumerative and compares three different notions of dimension associated (via Meinrenken's theorem) with a permutation:

- the linear dimension $:= \dim A_\pi$

- the affine dimension $:= \dim(A_\pi \cup \partial\overline{M}) = \dim(\Delta_\pi)$

- the combinatorial dimension $CD(\pi) := \#\{\text{distinct non-zero columns of } \mathbf{WT}(\pi)\}$.

Linear dimension is well understood and is determined entirely by the permutation's cycle structure. Affine dimension is similar to linear dimension, but there is a doubling effect coming from the "half-openness" of the Meinrenken tile. We show that permutations of maximum combinatorial dimension are exactly the SIF permutations, that is, the $\pi \in \mathfrak{S}_n$ which stabilize no interval other than $[n] = \{1, 2, \ldots, n\}$. We pose the more general enumeration of permutations via their combinatorial dimension as an open problem, Question 9.

Inspired by a theorem of Bibikov and Zhgoon, Section 3.3 defines a dual graph on the set of maximum dimensional cones $C_w$ from Waldspurger's theorem. For type A, we have $C_\pi$ of maximum dimension iff $\pi$ is an $n$-cycle, so we may identify vertices with $n$-cycles. Edges in this graph come in two distinct flavors; $s_i s_{i+1}$ for $i \in [n-1]$ (products of two noncommuting adjacent transpositions) and $s_i s_j$ for $1 \le i < i+1 < j \le n$ (product of two commuting adjacent transpositions). Considering only the first flavor of edges gives a regular graph, while considering only the second flavor gives an irregular graph

whose degree sequence we study. The irregular graph is highly disconnected and we conjecture that its number of connected components is given by the $n$th Pell number (see page 117). We conclude the thesis with analogous conjectures for types $B$ and $D$.

# List of Abbreviations and Symbols

$A$ (open) fundamental alcove for affine Coxeter arrangment. 29

$A_w$ relatively open simplex corresponding to Weyl group element $w$ in Meinrenken decomposition: $(id - w)A$. 30

$B$ bilinear form on $V$ defined on simple root basis $B(\alpha_s, \alpha_t) = -\cos\left(\dfrac{\pi}{m(s,t)}\right)$. 4

$\mathcal{C}$ (closed) cone over the positive roots.. 8

$\mathcal{C}_\Omega$ dominant chamber, i.e. the (open) cone over the fundamental weights.. 5

$C_\Phi$ Cartan matrix. 10

$C_w$ relatively open cone corresponding to Weyl group element $w$ in Waldspurger decomposition: $(id - w)\mathcal{C}_\Omega$. 29

$\Delta_\pi$ a simplex on the boundary of $\mathfrak{M}$, the Meinrenken tile $\left\{ \sum\limits_{i=1}^{n-1} a_i c_i \middle| \quad a_i \in \mathbb{R}_{\geq 0} \text{ and } \sum a_i = 1 \right\}$. 40, 97

$\Lambda_{\Phi^\vee}$ coroot lattice $\mathbb{Z}\alpha_1^\vee + \mathbb{Z}\alpha_2^\vee + \cdots + \mathbb{Z}\alpha_n^\vee$. 16

$\Lambda_\Omega$ weight lattice $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \cdots + \mathbb{Z}\omega_n$. 11

$\mathfrak{M}$ Meinrenken tile: $\bigsqcup\limits_{w \in W} (id - w)A$. 30, 40

$n_i$ marks i.e. the coefficients of the simple roots in the longest root $\alpha_{max} = \sum_{\alpha_i} n_i \alpha_i$.
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$\Omega$ set of fundamental weights $w_i$ defined in terms of simple roots so that $\frac{2(\alpha_i, \omega_j)}{(\alpha_i, \alpha_i)} = \delta_{i,j}$ for all $i, j \in [n]$. 12

$\Phi^\vee$ dual root system to $\Phi$. 10

$\Pi$ positive simple roots. 8

$\rho_s$ reflection across the hyperplane orthogonal to the geometric realization of element $s$ of Coxeter system $(W, S)$. 4

$t_\alpha$ reflection across the hyperplane $\alpha^\perp$. 7

$W_a$ affine Weyl group, (finite Weyl group semidirect product with the coroot lattice) $\Lambda_{\Phi^\vee} \rtimes W$. 29

# Chapter 1

# Type A

Recall that for type $A_{n-1}$ the Weyl group is the symmetric group on $n$ elements, $\mathfrak{S}_n$. The group acts naturally on $\mathbb{R}^n$ by permuting coordinates, but fixes the one dimensional subspace spanned by the vector of all ones. Since permutation matrices are orthogonal matrices, this action stabilizes the subspace on which the standard basis vectors sum to zero $\mathbb{R}_0^n \subset \mathbb{R}^n$. Each chambers, or connected components of $\mathbb{R}_0^n \setminus \bigcup_{1 \le i < j \le n} (e_i - e_j)^\perp$, is a simplicial cone. In Section 0.4 we showed that one such chamber is $\mathcal{C}_\Omega = \left\{ \sum_{i=1}^{n-1} a_i \omega_i \mid a_i \in \mathbb{R}_{>0} \right\} \subset \mathbb{R}_0^n$ where the $w_i$ were the **fundamental weights** defined by the equations $(w_i, e_j - e_{j+1})\delta_{i,j}$. We called $\mathcal{C}_\Omega$ the weight cone, or fundamental chamber. Because the $\mathfrak{S}_n$ action on the chambers is free and transitive, each chamber is $\pi(\mathcal{C}_\Omega)$ for a unique $\pi \in \mathfrak{S}_n$ and chambers are naturally identified with permutations.

The setup for Waldspurger's theorem also associates each permutation $\pi$ with a cone $C_\pi = (id - \pi)\mathcal{C}_\Omega$. Unlike the cone $\pi(\mathcal{C}_\Omega)$, however, the cone $C_\pi$ may be degenerate (see Figure 9.) Waldspurger's theorem states that the $C_\pi$ are all disjoint, and their union is equal to $\mathcal{C}$, the closed cone over the positive roots:

$$\mathcal{C} = \bigsqcup_{\pi \in \mathfrak{S}_n} C_\pi.$$

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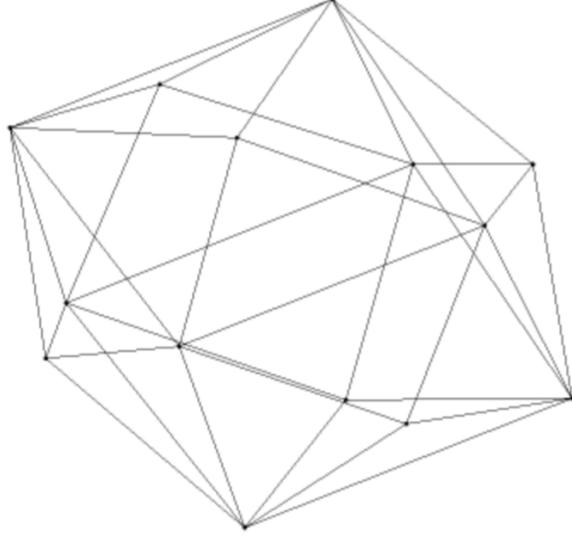
Figure 1.1: The polytope polar to the type $A_3$ permutohedron has 24 facets, each corresponding to a permutation of 4. It will tile $\mathbb{R}^3$ when translation by (co)root vectors.

Inside the open simplicial cone $\mathcal{C}_\Omega$, is an open simplex (i.e. the fundamental alcove):

$$A = \left\{ \sum_{i=1}^{n-1} a_i \omega_i \quad \mid \quad a_i \in \mathbb{R}_{\geq 0} \quad \sum a_i < 1 \right\}.$$

Hence, $\pi(A) \subset \pi(\mathcal{C}_\Omega)$ for each $\pi \in \mathfrak{S}_n$ and the simplices $\pi(A)$ are all disjoint.

The closure of the union of the $\pi(A)$ is a polytope which is **polar** or **dual** to the type A permutohedron (see Figure 1.1). Polytopes and their duals will not play a significant role in this thesis, so we refer the reader to Ziegler's book [46] for exact definitions and methods for constructing polars. What is important for our purposes is that the dual permutohedron is a fundamental domain for the translating action of the coroot lattice (which in type A is the same as the root lattice.) Among other things, this implies that $A$ is a fundamental domain for the action of the **affine symmetric group** [1], the finite symmetric group semidirect product with the coroot lattice $\tilde{\mathfrak{S}}_n \cong \mathfrak{S}_n \rtimes \mathbb{Z}^{n-1}$.

---

[1]There are several combinatorial ways of working with the affine symmetric group [5] which may be interesting to consider in the context of Meinrenken's theorem. In this thesis we always work with the finite symmetric group, and only consider translation via coroots when necessary.

The setup for Meinrenken's theorem associates each affine permutation $\pi$ with the simplex $A_\pi := (id - \pi)A$. Unlike the simplices $\pi(A)$, the simplices $A_\pi$ may be degenerate (see Figure 10.) Meinrenken's theorem states that the $A_\pi$ are all disjoint and that their union is the entire vector space $\mathbb{R}_0^n$. That is,

$$\bigsqcup_{\pi \in \mathfrak{S}_n} A_\pi = \mathbb{R}_0^n.$$

The study of the classical cones $\pi(\mathcal{C}_\Omega)$ and simplices $\pi(A)$ has intimate connections with the combinatorics of permutations. One may wonder about the combinatorial significance of the geometry of the cones $C_\pi$ and simplices $A_\pi$. In the next section we give both a combinatorial description by considering what we call the **Waldspurger transform $\mathbf{WT}(\pi)$** of the permutation $\pi$. $\mathbf{WT}(\pi)$ is an $(n-1) \times (n-1)$ matrix constructed from the $n \times n$ permutation matrix via a **transformation diagram** such as the one in Figure 1.2.

## 1.1  Waldspurger Transform of Permutations

**Definition 5.** Let $\pi \in \mathfrak{S}_n$ be expressed as an $n \times n$ permutation matrix. (For aesthetics, our examples put the entries of $\pi$ on a grid, leave off the zeros, and use stars instead of ones.) Define the $(n-1) \times (n-1)$ **Waldspurger matrix $\mathbf{WT}(\pi)$** by filling in the spaces between the entries of the permutation matrix $\pi$ as follows:

If an entry is on or above the main diagonal, count the number of stars above and to the right, and put that count in the space. If the entry is on or below the main diagonal, count the number of stars below and to the left and put that count in the
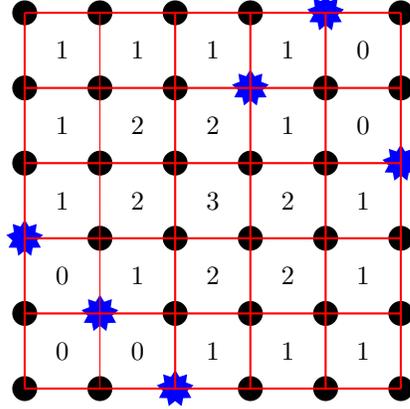
Figure 1.2: The transformation diagram for the permutation $456213 \in \mathfrak{S}_6$. The columns vectors give root coordinates for six points over which $C_{456213}$ is a cone. $A_{456213}$ is the convex hull of these vectors and zero.

space. Stated more tersely,

$$
\mathbf{WT}(\pi) := \begin{cases} \displaystyle\sum_{\substack{a \leq i \\ b > j}} \pi_{a,b} & i \leq j \\[2em] \displaystyle\sum_{\substack{a > i \\ b \leq j}} \pi_{a,b} & i \geq j \end{cases} .
$$

Note that entries on the diagonal are still well-defined. As an example, the transformation diagram for the permutation $456213 \in \mathfrak{S}_6$ is given in Figure 1.2.

**Theorem 1.1.** *Let $c_1, c_2, \ldots, c_{n-1}$ be the columns of the matrix $\mathbf{WT}(\pi)$. Expressing $\omega_i$, the ith fundamental weight, and the permutation matrix $\pi$ in root coordinates, for all $i \in [n-1]$, one has the following equality:*

$$
c_i = (id - \pi)\omega_i.
$$

Geometrically, this tells us that the columns of $\mathbf{WT}(\pi)$ are vectors in simple root coordinates which describe the Waldspurger and Meinrenken theorems via the

following correspondence:

$$C_\pi = \left\{ \sum_{i=1}^{n-1} a_i c_i \quad | \quad a_i \in \mathbb{R}_{\geq 0} \right\} \tag{1.1}$$

$$A_\pi = \left\{ \sum_{i=1}^{n-1} a_i c_i \quad | \quad a_i \in \mathbb{R}_{\geq 0} \text{ and } \sum a_i \leq 1 \right\} \tag{1.2}$$

Recall that **Meinrenken tile** is defined to be the set $\mathfrak{M} := \bigsqcup_{\pi \in \mathfrak{S}_n} A_\pi$ and that it tiles all of space under the translation by elements of the (co)root lattice. It will, at times, be convenient for us to study the topological boundary of $\mathfrak{M}$. This boundary, it turns out, is also built out of simplices described in terms of Waldspurger matrices:

$$\Delta_\pi := \left\{ \sum_{i=1}^{n-1} a_i c_i \quad | \quad a_i \in \mathbb{R}_{\geq 0} \text{ and } \sum a_i = 1 \right\}. \tag{1.3}$$

The proof of Theorem 1.1 follows immediately from the following lemma by multiplying both sides of the equation by $C^{-1}$, the inverse of the Cartan matrix and looking at the columns.

**Lemma 1.** *Let $P$ be the $(n-1) \times (n-1)$ matrix for the permutation $\pi \in \mathfrak{S}_n$ expressed in root coordinates. Let $C$ be the $(n-1) \times (n-1)$ Cartan matrix, $I$ the $(n-1) \times (n-1)$ identity matrix, and $\mathbf{WT}(\pi)$ be the $(n-1) \times (n-1)$ Waldspurger matrix for $\pi$. Then*

$$(\mathbf{I} - \mathbf{P}) = \mathbf{WT}(\pi)\mathbf{C}.$$
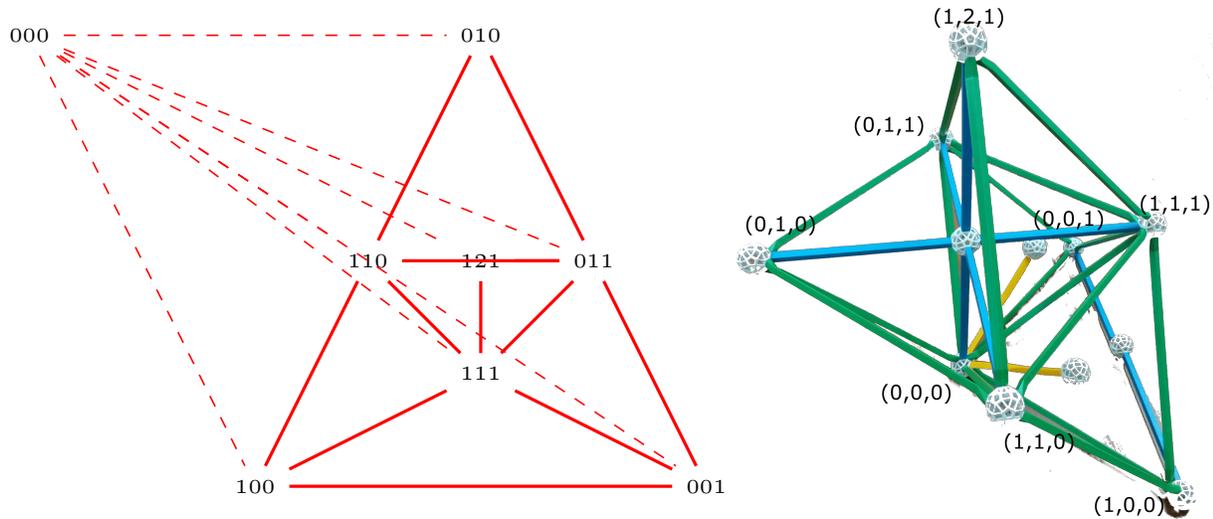
Figure 1.3: Waldspurger's theorem decomposes the $A_3$ root cone (on the left) into a point at the origin, 6 one-dimensional cones (dotted rays), 11 two-dimensional cones, and 6 three-dimensional cones. Points are labeled in root coordinates. The $A_3$ Meinrenken tile is shown on the right constructed with Zometools. (The two yellow edges and one blue edge coming out from the origin are the fundamental weights.)

*Proof.* For notational brevity, let $D := \mathbf{WT}(\pi)$. We use the fact that $C = A^T A$ where $A$ is the $n \times (n-1)$ matrix

$$
A = \begin{pmatrix}
1 & 0 & 0 & \dots & 0 \\
-1 & 1 & 0 & \dots & 0 \\
0 & -1 & 1 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 1 \\
0 & 0 & 0 & \dots & -1
\end{pmatrix}
$$

to rewrite the conclusion :

$$
P = I - DA^T A
$$

We may then multiply both sides by $A$ on the left to obtain

$$AP = A - ADA^T A.$$

Because $A$ is the change of basis matrix between standard euclidean coordinates and simple root coordinates on $\mathbb{R}_0^n$, we have $AP = \pi A$. Making this substitution and canceling the $A$'s on the right we obtain:

$$\pi = I - ADA^T$$

This is what we will verify.

Multiplying $A$ and $D$, we see that $(AD)_{i,j} = D_{i,j} - D_{i-1,j}$ with the understanding $D_{0,k} := 0$ for all $k$. One more multiplication gives us that

$$(ADA^T)_{i,j} = D_{i,j} - D_{i-1,j} - D_{i,j-1} + D_{i-1,j-1}$$

once again, with the understanding that if either $i = 0$ or $j = 0$ then $D_{i,j} := 0$

**Case 1.** If $i = j$ then

$$
\begin{aligned}
(ADA^T)_{i,j} &= D_{i,j} - D_{i-1,j} - D_{i,j-1} + D_{i-1,j-1} \\
&= \sum_{\substack{a\leq i \\ b>j}} \pi_{a,b} - \sum_{\substack{a\leq i-1 \\ b>j}} \pi_{a,b} - \sum_{\substack{a>i \\ b\leq j-1}} \pi_{a,b} + \sum_{\substack{a>i-1 \\ b\leq j-1}} \pi_{a,b} \\
&= \sum_{k\neq j} \pi_{i,k} \\
&= \begin{cases} 0 & \pi_{i,j} = 1 \\ 1 & \pi_{i,j} = 0 \end{cases}
\end{aligned}
$$

To understand the second-to-last inequality, observe that we are summing over the following terms of permutation matrices:

$$
\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\
\cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\
\cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
-
\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\
\cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\
\cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
-
\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\
\cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\
\cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
+
\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\
\cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\
\cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

Thus, $(I - ADA^T)_{i,j} = \pi_{i,j}$ for this case.

**Case 2.** If $i < j$ then

$$
(ADA^T)_{i,j} = D_{i,j} - D_{i-1,j} - D_{i,j-1} + D_{i-1,j-1}
$$

$$
= \sum_{\substack{a \leq i \\ b > j}} \pi_{a,b} - \sum_{\substack{a \leq i-1 \\ b > j}} \pi_{a,b} - \sum_{\substack{a \leq i \\ b > j-1}} \pi_{a,b} + \sum_{\substack{a \leq i-1 \\ b > j-1}} \pi_{a,b}
$$

$$
= -\pi_{i,j}
$$

This last equality is, again, easier to understand visually:

$$
\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\
\cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\
\cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
-
\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\
\cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\
\cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
-
\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\
\cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\
\cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
+
\begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\
\cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\
\cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

Thus, $(I - ADA^T)_{i,j} = \pi_{i,j}$ for this case as well.

**Case 3.** If $i > j$ then

$$(ADA^T)_{i,j} = D_{i,j} - D_{i-1,j} - D_{i,j-1} + D_{i-1,j-1}$$

$$= \sum_{\substack{a>i\\b\leq j}} \pi_{a,b} - \sum_{\substack{a>i-1\\b\leq j}} \pi_{a,b} - \sum_{\substack{a>i\\b\leq j-1}} \pi_{a,b} + \sum_{\substack{a>i-1\\b\leq j-1}} \pi_{a,b}$$

$$= -\pi_{i,j}$$

As before, the final equality is apparent with a visual:

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\ \cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\ \cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} - \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\ \cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\ \cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} - \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\ \cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\ \cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \cdots \\ \cdots & \pi_{i,j-1} & \pi_{i,j} & \pi_{i,j+1} & \cdots \\ \cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & \pi_{i+1,j+1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus, $(I - ADA^T)_{i,j} = \pi_{i,j}$ in this final case.

$\square$

## 1.2   UM Vectors

Suppose $v$ is the $k$-th column of the Waldspurger matrix associated to the permutation $\pi$. It is evident from the transformation diagram that $v_1 = 0$ or $v_1 = 1$ since the one in the first row of $\pi$ can either be to the left or to the right of $v_1$. By similar reasoning, for $i \leq k$ we have $v_i = v_{i-1}$ or $v_i = v_{i-1} + 1$ and for $i > k$ we have $v_i = v_{i-1}$ or $v_i = v_{i-1} - 1$ with $v_n = 0$ or $v_n = 1$. In other words, $v$ will start with a zero or a one, weakly increase (by steps of 0 or 1) until the $k$th entry, and then weakly decrease (by steps of 0 or 1), to the last entry.

**Definition 6.** A **Motzkin path** is a lattice path in the integer plane $\mathbb{Z} \times \mathbb{Z}$ consisting of steps $(1,1),(1,-1),(1,0)$ which starts and ends on the x-axis, but never passes
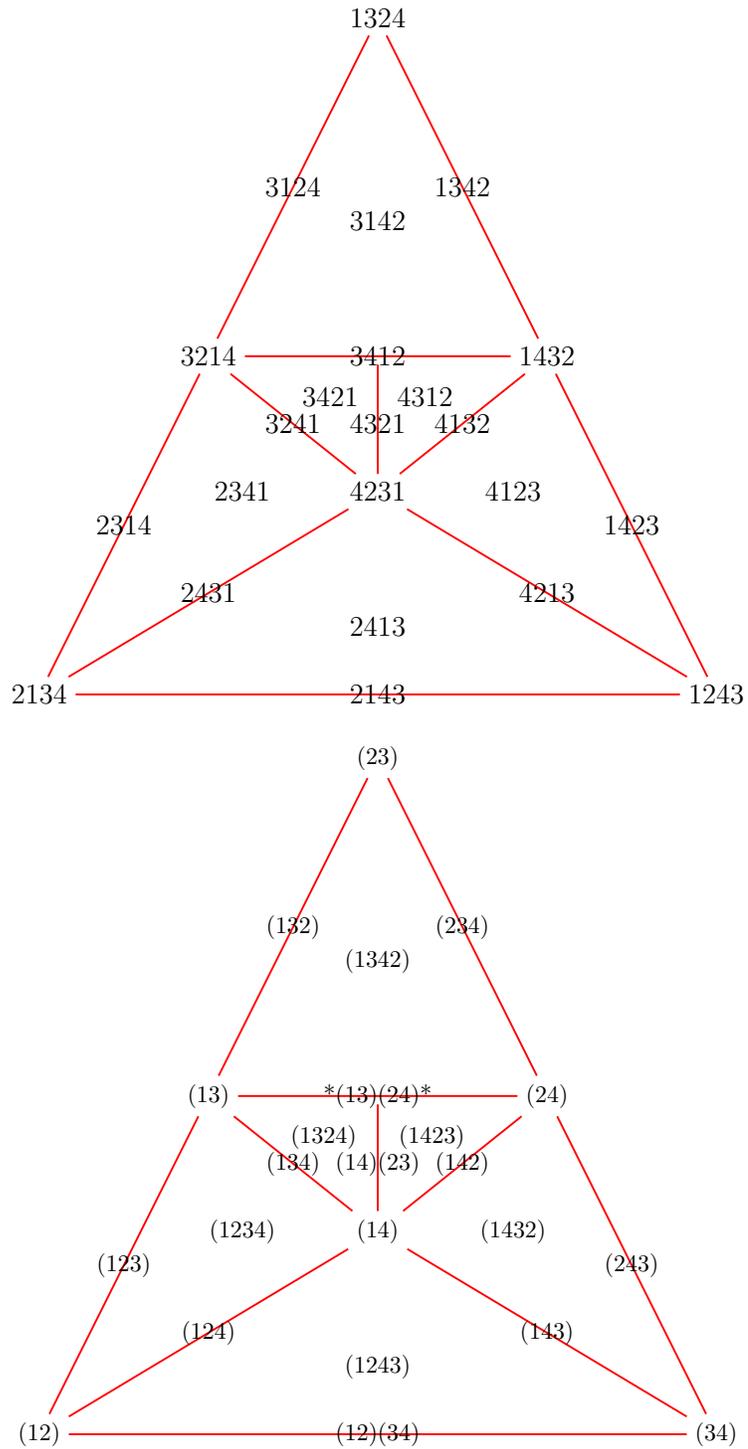
Figure 1.4: A slice of the Root cone $A_3 = \mathfrak{S}_4$

below it. A Motzkin path is **unimodal** if all occurrences of the step $(1,1)$ are before the occurrences of $(1,-1)$. For brevity, we will henceforth refer to unimodal Motzkin paths as *UMP*'s.

**Lemma 2.** (counting UMPs)

*There are $2^{n-1}$ UMPs between $(0,0)$ and $(0,n)$.*

*Proof. (induction)*

**Base case:** There is only one UMP of length one, and only two UMPs of length two.

**Induction hypothesis:** Suppose there are $2^{k-1}$ UMPs of length $k$ for all $k \leq n-1$. Consider an arbitrary UMP of length $n$.

We will partition UMPs of length $n$ into four (nondisjoint) classes:

(A) Those starting with an up-step.

(B) Those ending with an up-step.

(C) Those starting and ending with an up-step.

(D) Those starting and ending with a flat step.

By the principle of inclusion-exclusion, $\#\text{UMPs} = \#A + \#B - \#C + \#D$.

**Class A:**

The first step of the UMP is $(1,0)$. Cutting off this step, we have an arbitrary UMP of length $n-1$ and so by induction, that there are $2^{n-2}$ such UMPs.
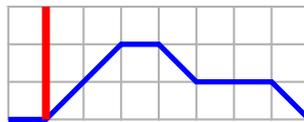


**Class B:**

The last step of the UMP is $(1,0)$. Cutting off this last step, we have an arbitrary

UMP of length $(n-1)$ and so by induction, that there are $2^{n-2}$ such UMPs.



**Class C:**

If the first and last steps of a UMP are both $(1,0)$ then the UMP was counted by both of the previous cases. Cutting off both the first and last steps we have an arbitrary UMP of length $n-2$. There are, by induction, $2^{n-3}$ such UMPs.



**Class D:**

The first and last steps of the UMP are $(1,1)$ and $(1,-1)$, respectively. Cutting these steps once again, we see by induction, that there are $2^{n-3}$ such UMPs.
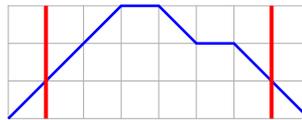


So we see that there are $2^{n-2} + 2^{n-2} - 2^{n-3} + 2^{n-3} = 2^{n-1}$ UMPs of length $n$.  □

**Definition 7.** A *UM vector* is any vector that appears as a column in $\mathbf{WT}(\pi)$ for some permutation $\pi$.

**Theorem 1.2.** *There is a bijective correspondence between UM vectors of length $n-1$ and UMPs with $n$ steps. Consequently, there are $2^n$ UM vectors of length $n$.*

*Proof.* We will define a map from UM vectors to UMPs which is easily inverted. What is less clear, a priori, is that our map is surjective; equivalently, does every vector with integer entries starting and ending with a zero or one having no ascents after the first decent appear as the column vector of $\mathbf{WT}(\pi)$ for some permutation $\pi$? This we will show by induction.

A UM vector must start with a zero or a one, weakly increase by one until its entry on the diagonal, and then weakly decrease by one until its final entry, a zero or one. Any row vector of a Waldspurger matrix must also be a UM vector with its maximum also on the diagonal. Padding a UM vector with zeros on each end gives the $x$ coordinates for a UMP of length $n$. For example,

$$(1,2,3,3,2,1,0) \leftrightarrow (0,1,2,3,3,2,1,0,0) \leftrightarrow$$ 

To show surjectivity, consider an arbitrary UMP, and use the bijection above to turn it into a vector $v$ with integer entries starting and ending with a zero or one having no ascents after the first decent. Suppose $v$ starts with a zero, that is $v = (0, v_2, v_3, ..., v_{n-1})$. By induction, there exists a permutation $\pi$ of $n-1$ whose Waldspurger transform has column vector $(v_2, v_3, ..., v_{n-1})$. Let $\hat{\pi}$ be the permutation of $n$ which sends 1 to 1, and $k$ to $\pi(k-1)$ for $k \in \{2, ..., n\}$. Then $\mathbf{WT}(\hat{\pi})$ has $v$ as a column vector.

A similar trick works for vectors ending with a zero.

If the vector starts and ends with a one, $v = (1, v_2, ..., v_{n-2}, 1)$, then by induction, there exists a permutation $\pi$ of $n-2$ whose Waldspurger transform has column vector $(v_2 - 1, v_3 - 1, ..., v_{n-2} - 1)$. Let $\hat{\pi}$ be the permutation of $n$ which sends 1 to $n$, and $n$ to 1 and $k$ to $\pi(k-1)$ for $k \in \{2, ..., n-1\}$. Then $\mathbf{WT}(\hat{\pi})$ has $v$ as a column vector. $\quad\square$

It turns out UM vectors were implicitly considered by Kostant, Panyushev, and Peterson in the context of Lie theory as abelian ideals in the nilradical of the Lie algebra $\mathfrak{sl}_n$. Rather than introduce more machinery, we give the following combinatorial definition:

Figure 1.5: Abelian ideals in the nilradical of the Lie algebra $\mathfrak{sl}_n$ correspond to certain order filters in the poset of positive roots and to Ferrer's shapes with hooklengths less than or equal to $n$. In this figure, each of the positive roots are given as vectors in simple root coordinates.

**Definition 8.** An **abelian ideal** is an order filter $I$ in the poset of positive roots possessing the property that if $r_1 \in I$ and $r_2 \in I$ then $r_1 + r_2 \notin I$.

While abelian ideals are define for all types, in type A they are naturally in bijection with Ferrer's shapes with bounded hooklength: Given $\lambda$ a partition of the integer $k$, that is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)}$ and $\sum_{i=1}^{\ell(\lambda)} \lambda_i = k$, one may consider its **Young diagram**, an arrangement of $k$ cells in $\ell(\lambda)$ left aligned rows, the $i$th row having $\lambda_i$ cells. Give the cell in the $i$th row and $j$th column coordinates $(i, j)$. The **hook** $H_\lambda(i, j)$ is the set of cells $(a, b)$ such that $a = i$ and $b \geq j$ or $a \geq i$ and $b = j$. The **hook-length** $h_\lambda(i, j)$ is the number of cells in the hook $H_\lambda(i, j)$. Every order filter in the type A positive roots poset may be identified with a Ferrer's shape given the correspondence in Figure 1.5. The ideal will be abelian iff the shape has all hooklengths less than or equal to $n$.

**Proposition 1.3.** *UM vectors are in bijection with Ferrer's shapes with hook lengths bounded above by n and also with abelian ideals in the nilradical of the Lie algebra $\mathfrak{sl}_n$.*

*Proof Sketch.* Take any UM vector and write it as a sum of positive roots by recursively subtracting the highest root whose nonzero entries (in simple root coordinates) correspond to positive nondecreasing entries in the UM vector. For example, the

vector $(0, 1, 2, 1) = (0, 1, 1, 0) + (0, 0, 1, 1)$. These vectors generate an order filter in the poset of positive roots which corresponds to a Ferrer's shape with bounded hook length, as seen in Figure 1.5. This map is clearly injective. Kostant and Panyushev showed that there are $2^{n-1}$ such abelian ideals, and so we conclude that it is bijective.

$\square$

In 2004 Rudi Suter [41] showed that these Ferrer's shapes with bounded hook-lengths exibit a dihedral symmetry when considered as a subposet of Young's lattice.

**Question 1.** Does this group action have a meaningful interpretation with respect to the combinatorics of Waldspurger matrices or the geometry of the Waldspurger and Meinrenken decompositions?

Kostant, Panyushev, and Peterson showed [32] (again for all types) that abelian ideals could be faithfully expressed as the sum of their generators, and that such vectors were exactly the coroots inside a certain polytope. For type A (where roots and coroots are equivalent) we show that, in root coordinates, these vectors are precisely our UM vectors.

**Theorem 1.4.** *UM vectors are exactly the roots c (in root coordinates) such that $-1 \le (c, r) \le 2$ for every positive root $r$. They are roots inside the polytope defined by affine hyperplanes at heights $-1$ and $2$ orthogonal to every positive root.*

*Proof.* We will show that our $2^{n-1}$ UM vectors satisfy the inequalities coming from the defining hyperplanes. Explicitly, suppose that $\bar{x}$ is a UM vector and $\bar{y}$ is a positive root (both expressed in root coordinates). Then the dot product is expressed as $(x, y) = x^t \cdot y = \bar{x}^t A^t A \bar{y} = \bar{x}^t C \bar{y}$ where $A$ is the matrix defined in Theorem 1.1 and $C$ is the Cartan matrix. Suppose that $\bar{y} = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots 0)^t$ where the first one

is in position $i$ and the last one is in position $j$. Then

$$\bar{x}^t C \bar{y} = 2 \left( \sum_{k=i}^{j} x_k \right) - x_{i-1} - x_{j+1} - 2 \left( \sum_{k=i+1}^{j-1} x_k \right)$$

$$= -x_{i-1} + x_i + x_j - x_{j+1}$$

Because $(x_1, \ldots, x_{n-1})$ is a UM vector, $x_i$ and $x_{i-1}$ can differ by at most one, and likewise $x_j$ and $x_{j+1}$ can differ by at most one. This yields that

$$-2 \le x_i - x_{i-1} + x_j - x_{j+1} \le 2.$$

However, the $-2$ is unattainable by the unimodality of UM vectors. Suppose that $x_{i-1} > x_i$, that is $x_{i-1} = x_i + 1$. Then $x_j \ge x_{j+1}$, that is, $x_j = x_{j+1}$ or $x_j + 1 = x_{j+1}$. Either way, $x_i - x_{i-1} + x_j - x_{j+1} = x_i - (x_i + 1) + x_j - x_{j+1} > -2$. Thus

$$-1 \le x_i - x_{i-1} + x_j - x_{j+1} \le 2$$

showing that our UM vectors are all inside the polytope.

We appeal to the enumeration given by Kostant, Panyushev, and Peterson to show that these are all of the root vectors in the polytope. $\square$

**Question 2.** Every Waldspurger matrix has UM vectors for its row and column vectors, with each row and column having it maximum entry on the diagonal. Are all matrices with this property $\mathbf{WT}(\pi)$ for some $\pi \in \mathfrak{S}_n$? If not, can such matrices be characterized?

The answer to this first question is no. One may verify that the $2 \times 2$ identity matrix has UM vectors for rows and columns, but is not $\mathbf{WT}(\pi)$ for any $\pi \in \mathfrak{S}_3$. The second question has a surprising and beautiful answer (Theorem 1.6).

# 1.3 Entropy, Alternating Sign Matrices, and a Generalized Waldspurger Transform

In section 0.7 we referred to a result of Lascoux and Schützenberger stating that the componentwise order on monotone triangles (or equivalently reduced monotone triangles) was isomorphic to Dedekind-MacNeille completion of Bruhat order on permutations. Knowing this, it is straightforward to show that summing all of the entries in a reduced monotone triangle gives its rank in the lattice. It is natural to ask if this statistic restricts in a meaningful way to a statistic on permutations. Lascoux and Schützenberger showed that it in fact does, and that the rank of a permutation in the Dedekind-MacNeille completion of Bruhat order is half its **entropy**.

**Definition 9.** The **entropy** (alternatively called variance in the literature [27]) of a permutation $\pi$ is

$$E(\pi) := \sum_{i=1}^{n} (\pi(i) - i)^2.$$

Entropy is a statistic which also showed up in our study of Waldspurger matrices.

**Theorem 1.5.** *For $\pi \in \mathfrak{S}_n$, let its **Waldspurger height** $h(\pi)$ be the sum of the entries of $\mathbf{WT}(\pi)$ then*

$$h(\pi) := \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{WT}(\pi)_{i,j} = \frac{1}{2} E(\pi).$$

*Proof.* Consider what each "star" in the transformation diagram contributes to the entries in the Waldspurger matrix. A star contributes one to every entry enclosed in the right triangle between itself and the main diagonal, and one half to every entry on the main diagonal whose box is cut by the triangle.
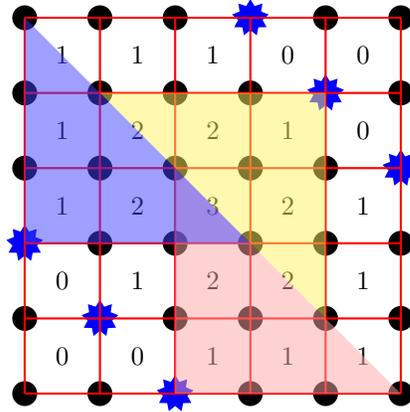
That is, the star in the $i$th column contributes $(\pi(i) - i)^2/2$ to the entries of the Waldspurger matrix. We conclude that

$$\sum_{i=1}^{n}(\pi(i) - i)^2/2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{WT}(\pi)_{i,j}.$$

$\square$

This theorem suggests at least some kind of relationship between Waldspurger matrices and monotone triangles or ASMs. Inspired by this, we extend the domain of the Waldspurger transform from the set of permutation matrices to a broader class of square matrices. Recall from section 1.1 that an entry on the diagonal of the Waldspurger matrix counted both the number of stars "above and to the right" in the transformation diagram, and the number of stars "below and to the left". Desiring to preserve this **diagonal property**, we define the generalized Waldspurger transform only for sum-symmetric matrices. An $n \times n$ matrix $M$ is **sum-symmetric**[2] if its $i$th row sum equals its $i$th column sum for all $1 \le i \le n$e write $M \in SS_n$.

---

[2]these matrices are sometimes called "line-sum-symmetric" in literature [15]. Other sources use the name sum-symmetric to describe a different class of matrices altogether [1].

**Definition 10.** From an $n \times n$ sum-symmetric matrix $M$, define the $(n-1) \times (n-1)$ matrix, $\mathbf{WT}(M)$ where

$$\mathbf{WT}(M)_{i,j} = \begin{cases} \displaystyle\sum_{\substack{a \leq i \\ b > j}} M_{a,b} & i \leq j \\[2em] \displaystyle\sum_{\substack{a > i \\ b \leq j}} M_{a,b} & i \geq j \end{cases}.$$

Sum-symmetric matrices are a natural choice for the domain of $\mathbf{WT}$ because they are exactly the matrices for which the diagonal property is preserved, and one can show that the map $\mathbf{WT} : SS_n \twoheadrightarrow \mathrm{Mat}_{n-1}$ is linear and surjective with kernel the diagonal matrices. Restricting the domain to the set of ASMs, however, reveals the answer to Question 2. If $M \in ASM$ we say that $\mathbf{WT}(M)$ is an **ASM Waldspurger matrix**.

**Theorem 1.6.** *The restriction of the Waldspurger transform to alternating sign matrices has as its image all $M \in \mathrm{Mat}_{n-1}$ such that columns and rows of $M$ are UM vectors with maximums on the diagonal. Component-wise comparison of these matrices is exactly the same order as is defined on the ASM lattice via monotone triangles.*

*Proof.* Because an ASM can be uniquely recovered from its non-diagonal entries, and because the kernel of $\mathbf{WT}$ consists of only diagonal matrices, we know that $\mathbf{WT}$ gives a bijection between ASMs and Waldspurger ASMs. We also have the bijection between ASMs and monotone triangles described in section 0.7. We will show that the composition of these two bijections

$$\text{Monotone Triangles} \to \text{ASMs} \to \text{Waldspurger ASMs}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \leftrightarrow \begin{matrix} 3 \\ 2 \ 4 \\ 2 \ 4 \ 5 \\ 2 \ 3 \ 4 \ 6 \\ 1 \ 2 \ 3 \ 4 \ 6 \\ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \end{matrix} \leftrightarrow \begin{matrix} 2 \\ 1 \ 2 \\ 1 \ 2 \ 2 \\ 1 \ 1 \ 1 \ 2 \\ 0 \ 0 \ 0 \ 0 \ 1 \end{matrix}$$



Figure 1.6: An ASM and corresponding monotone triangle, reduced monotone triangle, and generalized Waldspurger matrix. (The blue 9-sided stars represent 1's, and the green six-sided stars represent -1's in the transformation diagram.)

has an explicit description in terms of "painting instructions" and that it is order preserving. The map is easy to describe, but it will take a little work to verify that it is well-defined, surjective and order preserving.

The map, from monotone triangles to Waldspurger matrices is as follows: Subtract off the monotone triangle corresponding to the identity permutation, and then consider the entries of this reduced monotone triangle as "painting instructions." The $(i,j)$th entry of the reduced triangle tells us how much paint to load our brush with for a left-to-right stroke beginning at the $(i,j)$th entry of the corresponding Waldspurger matrix. As a working example, consider Figure 1.6. The two at the top of the reduced triangle is "painted" onto the $(1,1)$ and $(1,2)$ entries of the associated Waldspurger matrix. The one in the next row is painted onto the $(2,1)$ entry, and the two after it is painted onto the $(2,2)$ and $(2,3)$ entries.

$$\begin{bmatrix}001\\010\\100\end{bmatrix}$$

$$\begin{bmatrix}001\\100\\010\end{bmatrix} \qquad \begin{bmatrix}010\\001\\100\end{bmatrix}$$

$$\begin{bmatrix}0\ 1\ 0\\1{-}1\ 1\\0\ 1\ 0\end{bmatrix}$$

$$\begin{bmatrix}010\\100\\001\end{bmatrix} \qquad \begin{bmatrix}100\\001\\010\end{bmatrix}$$

$$\begin{bmatrix}100\\010\\001\end{bmatrix}$$

$$\begin{bmatrix}11\\11\end{bmatrix}$$

$$\begin{bmatrix}11\\01\end{bmatrix} \qquad \begin{bmatrix}10\\11\end{bmatrix}$$

$$\begin{bmatrix}10\\01\end{bmatrix}$$

$$\begin{bmatrix}10\\00\end{bmatrix} \qquad \begin{bmatrix}00\\01\end{bmatrix}$$

$$\begin{bmatrix}00\\00\end{bmatrix}$$

```
            3
           23
          123
     3              2
    13             23
   123            123
            2
           13
          123
     2              1
    12             13
   123            123
            1
           12
          123
```

```
            2
           11
     2              1
    01             11
            1
           01
     1              0
    00             01
            0
           00
```
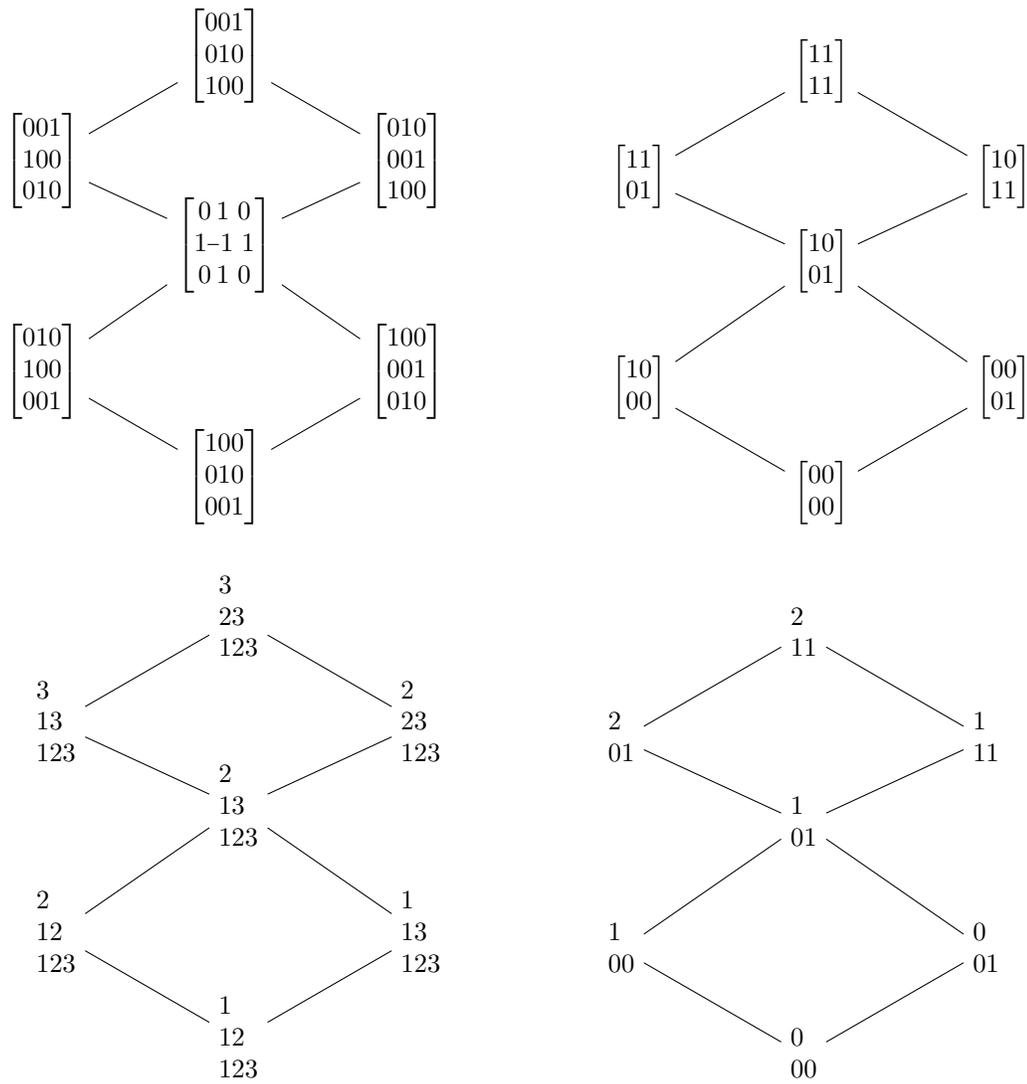
Figure 1.7: The Dedekind-MacNeille completion of Bruhat order $A_2$ viewed as ASMs, Generalized Waldspurger Matrices, monotone triangles, and reduced monotone triangles

Figure 1.8: There are four ways to peel the UM vector 1233332221. It may peel into three, four, five, or six parts, depending on which 3 is on the diagonal of the Waldspurger matrix it is appearing in.

To check that our painting map is well defined, we must check that it gives a matrix with unimodal rows and columns with maximums on the diagonal. The left-to-right painting process ensures that the entries in each row of the Waldspurger matrix will increase weakly by one up to the diagonal. The fact that rows of the reduced triangle are weakly increasing guarantees that the row of the Waldspurger matrix will be weakly decreasing by ones after the diagonal. The conditions on the columns are a bit more disguised, but the fact that reduced monotone triangles increase weakly up columns guarantees that the columns of the Waldspurger matrix will increase weakly up to the diagonal. Finally, the fact that reduced monotone triangles decrease by at most one in the ↘ direction, guarantees that the columns of the Waldspurger matrix will decrease weakly above the diagonal. This follows from induction on the size of the monotone triangle. Suppose that the lower-left corner or the monotone trianges maps onto a generalized Waldspurger matrix of dimension one less. Then painting a new diagonal will preserve the unimodality in rows and columns, and keep the maximums on the diagonal.

This painting map has an inverse "peeling" operation. UM vectors by themselves are not in bijection with rows of reduced monotone triangles, but, if one knows that the UM vector is to appear in row $k$, our painting map will have an inverse "peeling" operation into $k$ entries as seen in Figure 1.8.

To peel a UM vector into $k$ parts, create a diagram as in Figure 1.8 and specify $k$ starting points, one at the top of each of the $k$ columns. First draw a path from the $k$th starting point to the end, staying as far up and to the right as possible. Then do

$$
\begin{array}{ccccc}
1 & 1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 & 1 \\
1 & 2 & 3 & 2 & 1 \\
1 & 2 & 2 & 2 & 1 \\
1 & 1 & 1 & 1 & 1
\end{array}
$$

Figure 1.9: the number of ways of fixing each entry in a $5 \times 5$ Waldspurger matrix (also the top element in the Waldspurger version of the ASM lattice.)

the same with the $(k-1)$st point. Note that the unimodality condition on the UM vector guarantees that this path will be weakly shorter than the first one. Continue in this way until all of the vertices are exhausted. Record the length of the paths to get the corresponding row in the associated reduced monotone triangle. □

*Alterative Proof*[3]. In Section 0.7 we considered an $(n-1) \times (n-1)$ North-East corner-sum matrix $NE(M)$ for each $n \times n$ ASM $M$ and asserted that the component-wise partial order on corner-sum matrices was the same as the one on monotone triangles. It follows easily from the definition of $\mathbf{WT}$ and $NE$ that

$$
\mathbf{WT}(M)_{i,j} = \begin{cases}
NE(M)_{i,j} & \text{if } i \leq j \\
NE(M)_{i,j} + j - i & \text{if } i \geq j
\end{cases}
$$

and therefore,

$$
\mathbf{WT}(M_1) \leq \mathbf{WT}(M_2) \Leftrightarrow NE(M_1) \leq NE(M_2).
$$

□

The relationship between the tetrahedral poset and ASMs has been studied elsewhere [40], but Waldspurger matrices provide a new prospective. Bigrassmannian permutations have Waldspurger matrices determined by fixing a single entry and then "falling down" as quickly as possible. That is, $\pi \in \mathfrak{S}_n$ is bigrasmannian iff there exists a row, column, value triple $(i, j, k)$ for which $\mathbf{WT}(\pi)_{i,j} = k$ and all other entries of $\mathbf{WT}(\pi)$ are as small as possible (so as to still have UM vectors for columns and
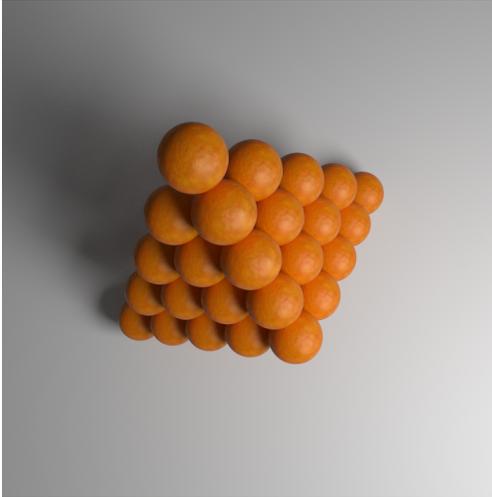
Figure 1.10: A tetrahedron of oranges sitting on its edge and viewed from on top. Stable configurations of oranges correspond to Waldspurger matrices, with the full tetrahedron corresponding to the Waldspurger matrix from Figure 1.9.

rows with maximums on the diagonals). More poetically, $\mathbf{WT}(\pi)$ for bigrassmannian $\pi$ describes an arrangement of oranges[4] in a tetrahedral orange basket (held up so that one edge is parallel with the ground) where only one orange may be removed without causing a tumble.

Let us verify the enumeration: In the set of $n \times n$ Waldspurger matrices, the number of ways of fixing a single entry to be a one is $n^2$, to be a two is $(n-2)^2$, to be a three is $(n-3)^2$, etc. (See figure 1.9.) This sum of alternating squares is equal to $\binom{n}{3}$, the tetrahedral number, which we know from Section 0.7, counts bigrassmannian permutations. (See [28] for more on the tetrahedral numbers.)

The Waldspurger decomposition thus identifies each bigrassmannian permutation $\pi \in \mathfrak{S}_n$ with a triple $(i, j, k)$ where $1 \le i, j, \le n-1$ and $1 \le k \le \min\{i, j, n-i, n-j\}$. This is reminiscent of something from Section 0.7 where we exhibited a simple bijection between bigrasmannian permutations in $\mathfrak{S}_n$ and triples $(a, b, c)$ with $0 \le a < b < c \le n$. With a bit of work, one can show that the correspondence between these two types of triples is as follows:

---

[4]Thank you Ryan Holmquist for the help making Figure 1.10 in Blender!

| given (i,j,k) if $i \geq j$ | given (i,j,k) if $i \leq j$ | given (a,b,c) |
|:---:|:---:|:---:|
| $a = j - k$ | $a = i - k$ | $i = b$ |
| $b = i$ | $b = j$ | $j = c - b + a$ |
| $c = i + k$ | $c = j + k$ | $k = \min\{b - a, c - a\}$ |

**Question 3.** It seems curious that this $(a, b, c) \leftrightarrow (i, j, k)$ bijection does not depend on $n$, and thus it plays nice with the inclusion $\mathfrak{S}_2 \subset \mathfrak{S}_3 \subset \cdots \subset \mathfrak{S}_n \subset \ldots$. Does this fact have any deeper consequences?

## 1.4  Centers of Mass and Geometric Realizations of Hasse Diagrams

Our definition for Waldspurger matrices was geometrically motivated, but we have seen that they are also very combinatorially related to the ASM lattice. It is then natural to ask how this partial order and the geometry are connected. One classical invariant of posets with a distinctly geometric flavor is the notion of order dimension. The order dimension of a poset $P$ is the smallest $n$ for which $P \cong Q \subset \mathbb{R}^n$ where the elements of $Q$ are ordered componentwise. In [34], Reading computed the order dimension of Bruhat orders for types A and B, the former being $\dim(A_n) = \lfloor \frac{(n+1)^2}{4} \rfloor$. This tells us, in particular, that there is no way of embedding the lattice of $3 \times 3$ Waldspurger matrices in dimension less than 4 in a way that preserves componentwise comparison. On the other hand, for each of these $3 \times 3$ matrices, we have an associated simplex $\Delta_M \subset \mathbb{R}^3$ and may consider the natural map which takes $\Delta_M$ to its center of mass.

If one replaces each simplex $\Delta_\pi$ (where $\pi \in \mathfrak{S}_n$) with its center of mass, one gets back a translate of the vertex set of the classical permutohedron. If one instead considers the centers of mass for each $\Delta(M)$ where $M$ is an alternating sign matrix,

one obtains every lattice point on the interior of the permutohedron as well; some appearing with multiplicities. (see Figure 1.12). For example, the two generalized Waldspurger matrices below have the same center of mass.

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

.

**Proposition 1.7.** *The Waldspurger height, $\sum \mathbf{WT}(M)_{i,j}$ is not only the rank of an ASM $M$ in the lattice, it is also the height of the center of mass of $\Delta_M$ inside of the Meinrenken tile in the direction of $\rho$, the sum of the positive roots.*

*Proof.* We want to show that projection of the center of mass of $\Delta_M$ onto $\rho$ is (up to scalar multiple) equal to the sum of the entries in $\mathbf{WT}(M)$. By the definition of $\Delta_M$, its center of mass is a scalar times the vector of column sums of $\mathbf{WT}(M)$. We will be done if we can show that projection of a vector $v$ onto $\rho$ is (up to scalar multiple) $\rho$ times the sum of the entries of $v$.

Projection of a vector $v$ onto $\rho$ in root coordinates, is $\frac{v^T C \rho}{\rho^T C \rho} \rho$. The denominator is just a scalar, and the numerator is

$$
v^T C \rho = v^T \begin{bmatrix} 2 & -1 & 0 & \dots & \dots \\ -1 & 2 & -1 & 0 & \dots \\ 0 & -1 & 2 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} n \\ 2(n-1) \\ 3(n-2) \\ \vdots \\ (n-2)3 \\ (n-1)2 \\ n \end{bmatrix} = v^T \theta
$$
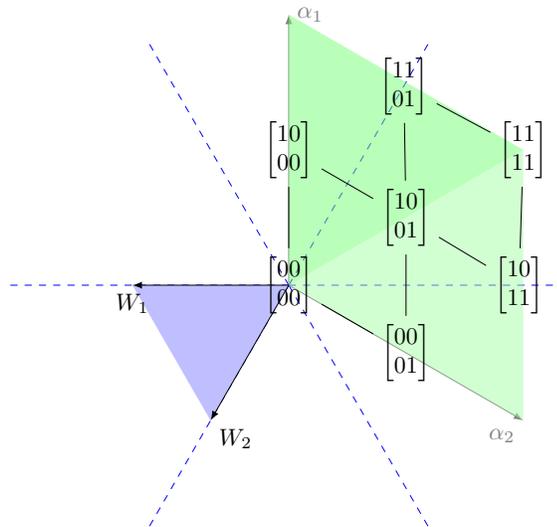
Figure 1.11: Place $\mathbf{WT}(M)$ at the baricenter of $\Delta_M$ for each $M \in ASM$ to get a geometric realization of the Hasse diagram inside the Meinrenken tile

where $\theta$ is the vector of all ones. We conclude that, up to scalars, this projection is the sum of the entries of $v$. $\qquad\square$

**Question 4.** It is a classical result in Ehrhart theory [33] that the number of lattice points inside of the permutohedron is the number of forests on the vertex set $[n] = \{1, 2, \ldots n\}$. We have exhibited a surjective map from ASMs to these same points. Is there an interpretation of these multiplicities in terms of forest structures?

## 1.5    M Vectors and Interval Decompositions of the ASM Lattice

In Section 1.2 we defined UM vectors abstractly as the vectors which appeared as rows or columns of Waldspurger matrices. We then showed that they corresponded naturally to unimodal Motzkin paths and could be alternatively defined as vectors starting and ending with a zero or one, increasing by zero or one up to a given point, and decreasing by zero or one thereafter. The vectors which appear as *diagonals*
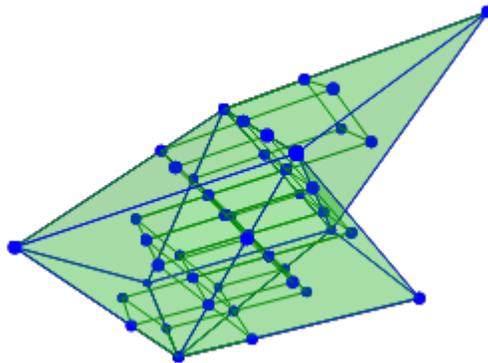
Figure 1.12: There are 38 lattice points in the permutohedron and 42 Alternating sign matrices in this dimension. Four of the interior points have multiplicity two.

of Waldspurger matrices possess a similar classification in terms of (not necessarily unimodal) Motzkin paths. The same vector/path correspondence (see the proof of Theorem 1.2) allows us to define **M vectors**– vectors of length $n$ corresponding to Motzkin paths of length $n+2$. That is, an M vector is vector of non-negative integers starting and ending with a zero or one, with adjacent entries differing by zero or one.

**Theorem 1.8.** *The diagonal of an $n \times n$ Waldspurger matrix is an M vector of length $n$. Moreover, each M vector appears as the diagonal of* $\mathbf{WT}(\pi)$ *for some permutation $\pi$. Grouping Waldspurger matrices by their diagonal M vectors partitions the Waldspurger ASM lattice into disjoint intervals. Each interval contains a unique involution whose cycle decomposition gives a non-crossing partial pairing.*

*Proof.* It is known that the number of involutions in $\mathfrak{S}_n$ with cycle decomposition a non-crossing partial pairing is the $n$th Motzkin number [29]. Because of this, it will suffice for us to only verify three things:

1. Adjacent diagonal entries of Waldspurger matrices must stay the same, increase by one, or decrease by one.

2. The set of ASMs with diag($\mathbf{WT}(M)$) = $v$ for $v$ an M vector forms an interval in the ASM lattice.

3. Given an M vector $v$, there exists an involution $\pi$ whose cycle decomposition gives a non-crossing pairing with the diagonal of $\mathbf{WT}(\pi)$ equal to $v$.

The first item follows from considering local movements within the transformation diagram.

- If $\mathbf{WT}(M)_{i,i} = k$ then both $\mathbf{WT}(M)_{i,i+1}$ and $\mathbf{WT}(M)_{i+1,i}$ are in $\{k, k-1\}$.

  1. If $\mathbf{WT}(M)_{i,i+1} = \mathbf{WT}(M)_{i+1,i} = k$, then $\mathbf{WT}(M)_{i+1,i+1} \in \{k, k+1\}$.

  2. If one or both of $\mathbf{WT}(M)_{i,i+1}$ and $\mathbf{WT}(M)_{i+1,i}$ is equal to $k-1$, then $\mathbf{WT}(M)_{i+1,i+1} \in \{k-1, k\}$.

Now for the second item: Because the ASM lattice is distributive, and the join-irreducibles correspond to Waldspurger matrices specified by a single entry, there is a unique minimal Waldspurger matrix with a given prescribed diagonal– the join of Waldspurger matrices specified by each diagonal entry. Since the partial order on Waldspurger matrices is componentwise, the join of all elements with a prescribed diagonal will also have that diagonal, meaning there is a unique maximal Waldspurger matrix with each prescribed diagonal.

Finally, the third item: Given an M vector, $m = (m_1, m_2, \ldots, m_{n-1})$, construct an involution $\pi \in \mathfrak{S}_n$ iteratively from the identity permutation as follows:

Let $m_i$ be the first entry of $M$ which equals one, and let $m_{j+1}$ be the first entry after $m_i$ which equals zero. By the previous proposition, $m_j = 1$ and, by construction, all entries between $m_i$ and $m_j$ are greater than or equal to one. Multiply the involution by the transposition $(i, j)$, subtract one from $m_i, m_{i+1}, \ldots, m_{j-1}, m_j$.

Repeat until the the $M$ vector is reduced to all zeros. Every involution introduced during this process does not cross any of the previous ones, and the process is reversible. (see Figure 1.13.)

$\square$

$$1232321 \xrightarrow{(1,8)} 0121010 \xrightarrow{(2,5)} 0010010 \xrightarrow{(3,4)} 0000010 \xrightarrow{(6,7)} 0000000$$
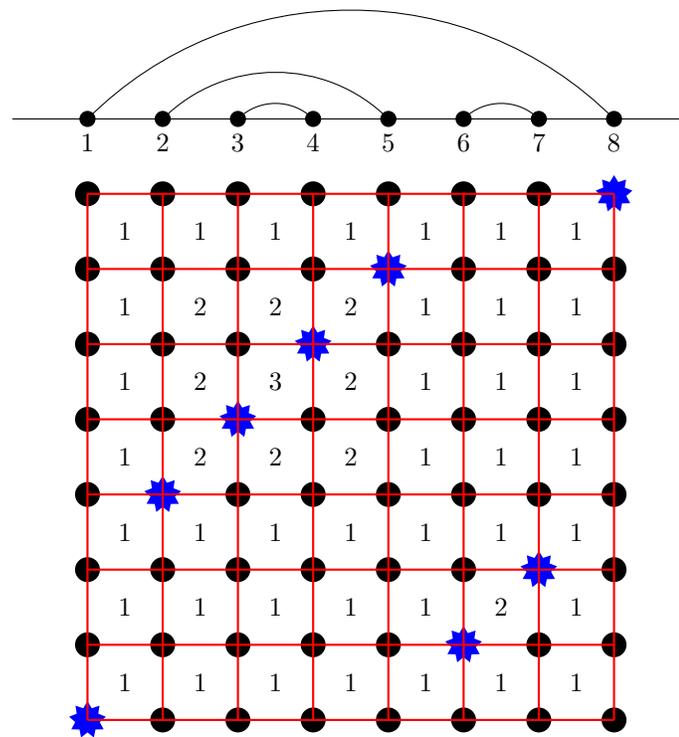


Figure 1.13: The permutation 85432761 is the unique permutation $\pi \in \mathfrak{S}_8$ with cycle decomposition a non-crossing pairing, and with $\mathrm{diag}(\mathbf{WT}(\pi)) = 1232121$.

```
                            00300
                         ╱    │    ╲
                       ╱      │      ╲
                     ╱        │        ╲
                  02000     00200     00020
                ╱   │    ╳    │    ╳    │   ╲
              ╱     │  ╱   ╲  │  ╱   ╲  │     ╲
            ╱       │╱      ╲│╱      ╲│       ╲
         10000    01000    00100    00010    00001
```
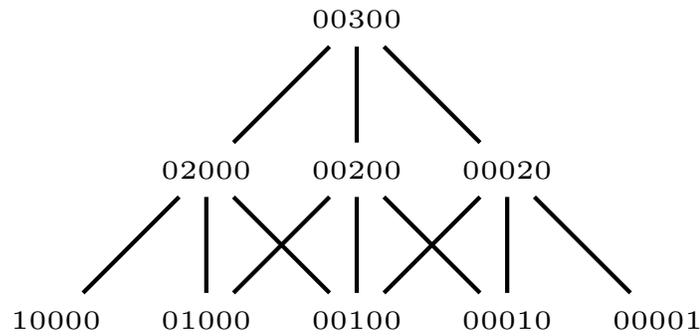
Figure 1.14: The lattice of M vectors of length five ordered componentwise is isomorphic to the lattice of order ideals in this poset

Like ASM Waldspurger matrices, M vectors with the componentwise comparison order form a distributive lattice. Whereas ASM Waldspurger matrices corresponded to order ideals in the tetrahedral poset (see Figure 1.10) M vectors correspond to order ideals in the poset in Figure 1.14 This poset is, in fact, a slice of the tetrahedral poset.

**Question 5.** Let $N(\alpha)$ denote the number of Waldspurger matrices with diagonal vector $\alpha$. One may write

$$n! = \sum_{\alpha \in \text{M Vectors}} N(\alpha)$$

Can one express $N(\alpha)$ only in terms of $\alpha$?

**Question 6.** (likely less tractable) Let $M(\alpha)$ denote the number of ASM Waldspurger matrices with diagonal vector $\alpha$. One may express the number of ASMs as

$$ASM(n) = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!} = \sum_{\alpha \in \text{M Vectors}} M(\alpha)$$

Can one express $M(\alpha)$ only in terms of $\alpha$?

# Chapter 2

# Types B and C

For general crystallographic root systems, $\Phi$, define the Waldspurger Transform of a Weyl group element $g$ to be the matrix

$$\mathbf{WT}_\Phi(g) := (Id - R_g)C_\Phi^{-1}$$

where $R_g$ is the matrix of $g$ in the coordinates of the simple roots of $\Phi$, and $C_\Phi^{-1}$ is the inverse of the Cartan matrix[1]. We call $\mathbf{WT}_\Phi(g)$ a **type $\Phi$ Waldspurger matrix** and define the **Waldspurger Order** $(\Phi, \succeq)$ as the componentwise order on type $\Phi$ Waldspurger matrices. If no root system is specified, we will assume type A, so that $\mathbf{WT} = \mathbf{WT}_A$ is the Waldspurger transform already discussed. Recall that type A

---

[1]For type A, the columns of $C_\Phi^{-1}$ are the (non-zero) vertices of the fundamental alcove in root coordinates. As noted in Section 0.5, one must, in general, scale the $i$th column of $C_\Phi^{-1}$ by the $i$th "mark" to get the $i$th non-zero vertex of the fundamental alcove. While it may be more natural geometrically to include this scaling in the definition of $\mathbf{WT}_\Phi$, it would cause type $B$ and type $C$ Waldspurger matrices to have non-integer entries. Prioritizing the combinatorics above the geometry, we have chosen (at least for the duration of this thesis) to leave this scaling as an afterthought.

Waldspurger order is isomorphic to type A Bruhat order and that

$$\mathbf{WT}(\pi)_{i,j} = \begin{cases} \sum_{\substack{a \leq i \\ b > j}} \pi_{a,b} & i \leq j \\ \\ \sum_{\substack{a > i \\ b \leq j}} \pi_{a,b} & i \geq j \end{cases}.$$

It is natural to ask which phenomena of those we observed in type A will hold more generally. In particular, how does the combinatorics of the Waldspurger decomposition relate to the Dedekind-MacNeille completion of Bruhat order for type B?

Lascoux and Schützenberger showed that the Dedekind-MacNeille completion of Bruhat order for type B is a distributive lattice, and gave a description of the join-irreducible elements as a subset of the bigrassmannian elements [23]. They showed that, while the number of bigrassmannian elements is $\binom{n+4}{4} + \binom{n+3}{3} - \binom{n+2}{2}$, which is counted by the coefficients of

$$\frac{1}{(1-z)^5} + \frac{1}{(1-z)^4} = 1 + 6z + 19z^2 + 45z^3 + 90z^4 + 161z^5 + \dots \tag{2.1}$$

the number of join-irreducibles or elements of the "base" is the **octahedral number** $\frac{n(2n^2+1)}{3}$ which is counted by the coefficients of

$$\frac{(1+z)^2}{(1-z)^4} = 1 + 6z + 19z^2 + 44z^3 + 85z^4 + 146z^5 + \dots. \tag{2.2}$$

Geck and Kim [17] gave a more in-depth treatment of exactly when bigrassmannian elements fail to be part of the base. (Figure 4 in the appendix shows transformation diagrams for the $15 = 161 - 146$ bigrassmannian elements which are not in the base of $B_6$.) Reading gave a combinatorial description of the base in terms of signed

monotone triangles [34]. Recently, Anderson gave another combinatorial description of the base in terms of type B Rothe diagrams and essential sets [3].

Despite all this, the story is still a bit unsatisfying; there are two posets in competition for the title of "type B alternating sign matrices":

1. The partial order on monotone triangles coming from centrally symmmetric $2n \times 2n$ ASMs (hereafter call HTASM for half-turn-alternating-sign-matrices).

2. The Dedekind-MacNeille completion of Bruhat order for the Weyl group of $B_n$ (hereafter called BASM).

Reading noted in [34] that these two are not isomorphic for $n > 3$. While the number of elements in the former is given by a nice product formula (Theorem 2.2) there is not even a conjecture for the size of BASM (though we present some data later). Moreover, up until now, the elements of BASM have had no explicit combinatorial description. In this chapter we show that $BASM_n$ can be seen as a sublattice of $\mathbf{WT}(HTASM_{2n})$. Both posets happen to be distributive lattices with octrahedral many join-irreducible elements. We show that all but tetrahedral many of these join-irreducibles are the same, and we describe the ones which differ.

We then turn to the geometry of the Waldspurger decomposition for types B and C, showing that $\mathbf{WT}_B$ and $\mathbf{WT}_C$ can be computed via "folding" centrally symmetric Waldspurger matrices. This folding obfuscates some of the poset theory, but introduces some interesting questions. In particular, Bruhat order and Waldspurger order are not isomorphic for type $B_n$ when $n > 3$. We discuss the complications which arise, and end the chapter with some open questions.

## 2.1   Centrally Symmetric Permutation Matrices

Recall from Section 0.4 that for type B, the underlying vector space was $V = \mathbb{R}^n$, and our roots $\Phi$ consisted of all integer vectors in $V$ of length 1 or $\sqrt{2}$, for a total of $2n^2$

roots. We chose the simple roots: $\alpha_i = e_i - e_{i+1}$, for $1 \le i \le n-1$ and $\alpha_n = e_n$ a shorter root. The type C roots are the same $\alpha_i = e_i - e_{i+1}$ but $\alpha_n = 2 \cdot e_n$ so there is a unique long simple root instead of a unique short simple root.

These root systems share a Weyl group of size $2^n n!$ given by the presentation

$$\langle s_1, s_2, \ldots, s_n \mid s_i^2 = s_n^2 = (s_i s_{i+1})^3 = (s_{n-1} s_n)^4 = 1 \quad \forall 1 \le i < n-1 \rangle.$$

The geometric realization represents this group with the set of $n \times n$ **signed permutation matrices**, $\pm \mathfrak{S}_n$. That is, $\pm \mathfrak{S}_n$ is the set of all $n \times n$ matrices with entries in $\{0, 1, -1\}$ having exactly one nonzero entry in each row and column. We will need this representation in Section 2.3 when we investigate $\mathbf{WT}_B$ and $\mathbf{WT}_C$, but in the meantime, there is another representation which will prove more combinatorially convenient.

Call a square $n \times n$ matrix **centrally symmetric** if it is preserved under $180°$ rotation; that is if $M_{i,j} = M_{n-i,n-j}$ for all $1 \le i, j \le n-1$. Let $\mathfrak{C}\mathfrak{S}_n \subset \mathfrak{S}_n$ denote the set of permutations whose $n \times n$ permutation matrices are centrally symmetric.

**Proposition 2.1.** *The group $\mathfrak{C}\mathfrak{S}_{2n} \subset \mathfrak{S}_{2n}$ is isomorphic to the group $\pm \mathfrak{S}_n$ of signed permutations via a "folding move".*

*Proof.* If $\pi$ is a $2n \times 2n$ centrally symmetric permutation matrix, we may "fold" it to obtain $\pi^\star$, a signed permutation on $n$, by letting

$$\pi_{i,j}^\star = \pi_{i,j} - \pi_{2n-i+1,j}$$

The map is invertible because $\pi$ was a permutation matrix, meaning that

$$\pi_{i,j}^{\star} = \begin{cases} 1 & \text{if } \pi_{i,j} = 1 \\ -1 & \text{if } \pi_{2n-i+1,j} = 1 \\ 0 & \text{otherwise} \end{cases}$$

i.e. there will never be any collisions in the folding and the map is a bijection.

To see that multiplication is preserved, observe that generators map to generators:

| abstract generators | generators of $\mathfrak{CS}_{2n}$ | generators of $\pm\mathfrak{S}_n$ |
|---|---|---|
| $\forall 1 \leq i < n, \quad s_i$ | $(i, i+1)(2n-i+1, 2n-i)$ | $\begin{pmatrix} 1 & 0 & \cdots & & & & & \\ 0 & 1 & & & & & & \\ \vdots & & \ddots & & & & & \\ & & & 0 & 1 & & & \\ & & & 1 & 0 & & & \\ & & & & & \ddots & \vdots & \\ & & & & & 1 & 0 & \\ & & & & \cdots & 0 & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ i \\ i+1 \\ \\ \\ \end{matrix}$ |
| $s_n$ | $(n, n+1)$ | $\begin{pmatrix} 1 & 0 & \cdots & & \\ 0 & 1 & & & \\ \vdots & & \ddots & & \vdots \\ & & & 1 & 0 \\ & & \cdots & 0 & -1 \end{pmatrix}$ |

$\square$

In Section 2.3 we will define a similar vertical folding map on centrally symmetric type A Waldspurger matrices:

$$\mathcal{F} : \mathbf{WT}_{A_{2n-1}}(\mathfrak{CS}_{2\mathfrak{n}}) \longrightarrow \mathrm{Mat}_n$$

Where

$$\mathcal{F}(M)_{i,j} := \begin{cases} M_{i,j} + M_{2n-i+1,j} & \text{for all } 1 \leq i, j < n \\ M_{i,j} & \text{for all } i = n, j \leq n \end{cases}.$$

Folding will allow us to describe the maps $\mathbf{WT}_B$ and $\mathbf{WT}_C$ in terms of our original type A Waldspurger transform.

Before doing this, we wish to study the lattice BASM. Using the correspondence of generators from the proof of Proposition 2.1, it is straightforward to verify the following:

**Proposition 2.2.** *An element $\pi \in \mathfrak{C}\mathfrak{S}_{2n}$ represents a bigrassmannian element of type $B_n$ if and only if when considered as a permutation, $\pi$ has exactly one or two left descents, and exactly one or two right descents.*

In [9, Chapter 8], Björner and Brenti show that componentwise order on $NE(\mathfrak{C}\mathfrak{S}_{2n})$ is isomorphic to type B Bruhat order. Combining this result with Theorem 1.6 we obtain the following:

**Proposition 2.3.** *The Bruhat order on the Weyl group of $B_n$ is isomorphic to the componentwise comparision order on $\mathbf{WT}(\mathfrak{C}\mathfrak{S}_{2n})$.*

## 2.2 Centrally Symmetric Alternating Sign Matrices

In the 1980's many beautiful conjectural product formulas arose from symmetry classes of ASMs and plane partitions, similar in nature to one in Theorem 0.7. In particular, Kuperberg was able to prove this early conjecture of Robbins:

**Theorem 2.4.** *(Kuperberg 2002 [22]) The number of $n \times n$ ASMs invariant under rotation by $180°$, also called half-turn ASMS or HTASMs is*

$$\frac{HTASM(2n)}{ASM(n)} = (-3)^{\binom{n}{2}} \prod_{i,j} \frac{3(j-i)+2}{j-i+n}.$$

Because elements of Dedekind-MacNeille completion of Bruhat order for type A corresponded to ASMs, and type B Bruhat order is the restriction of type A Bruhat

order to centrally symmetric permutations, it is natural to consider extending the folding map $\mathfrak{CS}_{2n} \to \pm\mathfrak{S}_n$ to the larger domain of $HTASM_{2n}$. Unfortunately, the folding map fails to be injective on ASMs. For example, the following two $6 \times 6$ ASMs are folded to the same $3 \times 3$ matrix:

$$
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 & 0 \\
1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 \\
0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & -1 & 1 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\mapsto
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & -1 \\
1 & -1 & 1
\end{bmatrix}.
$$

Though folding will be convenient for studying the geometry of the type B Waldspurger decomposition, these collisions in folding HTASMs will obfuscate their partial order.

We will thus start by focusing on the componentwise order on $\mathbf{WT}(HTASM_{2n})$, asking how this relates to the BASM lattice. While $\mathbf{WT}(HTASM_{2n})$ is (by definition) a subposet of $\mathbf{WT}(ASM_{2n})$, it follows from Proposition 0.7 that $BASM_n$ is as well, and we will rely heavily on these embeddings.

**Theorem 2.5.** *The set $\mathbf{WT}(HTASM_{2n})$ partially ordered componentwise, is a distributive lattice.*

*Proof.* First notice that $\mathbf{WT}(HTASM_{2n})$ is a lattice. The componentwise min or max of two centrally symmetric matrices is centrally symmetric. Moreover, componentwise min and max preserve the properties required to be a type A Waldspurger matrix. Hence, for any $M_1, M_2 \in HTASM_{2n}$,

$$
\mathbf{WT}(M_1) \vee \mathbf{WT}(M_2) = [\max\{\mathbf{WT}(M_1)_{i,j}, \mathbf{WT}(M_2)_{i,j}\}]_{i,j=1}^{2n-1} \in \mathbf{WT}(HTASM_{2n})
$$

$$\mathbf{WT}(M_1) \wedge \mathbf{WT}(M_2) = [\min\{\mathbf{WT}(M_1)_{i,j}, \mathbf{WT}(M_2)_{i,j}\}]_{i,j=1}^{2n-1} \in \mathbf{WT}(HTASM_{2n}).$$

Distributivity follows from distributivity of the type A case as well. For any HTASM, $M$, we may express $\mathbf{WT}(M)$ as the join of $\mathbf{WT}(A_1), \mathbf{WT}(A_2), \ldots, \mathbf{WT}(A_k)$ where the $A$'s are type A bigrassmannian elements and $k$ is minimal. If any of the $A$'s are centrally symmetric, then they are join-irreducibles in HTASM. For $\mathbf{WT}(M)$ to be centrally symmetric, all of the non-centrally symmetric $A$'s must come in pairs. The componentwise maxes of these pairs are the remaining join-irreducibles in HTASM.

$\square$

Lascoux and Schützenberger showed that $BASM_n$ was a distributive lattice with octahedral many join-irreducibles. We will show that the same holds for $\mathbf{WT}(HTASM_{2n})$. We will then describe both sets of join-irreducibles in terms of elements in $\mathbf{WT}(ASM_{2n})$.

In order to do this, recall what we showed in the type A case: In the lattice $\mathbf{WT}(ASM_{2n})$, each join-irreducible is $\mathbf{WT}(\pi)$ for some bigrassmannian permutation. $\mathbf{WT}(\pi)$ is uniquely determined by a single matrix entry $(i, j)$ with value $1 \le k \le \min\{i, j, 2n - i, 2n - j\}$. All other entries are made as small as possible so that rows and columns are UM vectors. That is,

- Each row and column starts and ends with a 0 or 1.

- Adjacent entries in each row and column differ by 0 or 1.

- Rows and columns are unimodal, i.e. entries can't increase after they decrease.

- Each row and column has its maximum (possibly not unique) on the matrix's diagonal.

Join-irreducibles in $\mathbf{WT}(ASM_{2n})$ can thus be identified with **triples**, $(i, j, k)$ where $i$ is the row, $j$ the column, and $k$ the specified entry with $1 \le k \le \min\{i, j, 2n-i, 2n-j\}$. From here on we will always assume given some $i$ and $j$, that $k$ is between 1 and $\min\{i, j, 2n - i, 2n - j\}$, and will call such values of $k$ **valid**.

For example the triple $(3, 2, 2)$ in $\mathbf{WT}(ASM_6)$ corresponds to the bigrassmannian permutation given in one-line notation by 451236.

We use the join operator on $\mathbf{WT}(ASM_{2n})$ to define a map which takes a collection of triples to the join of their corresponding join-irreducibles

$$J_\Delta : \{\text{triples defining bigrassmannian permutations in } \mathfrak{S}_{2n}\} \longrightarrow \mathbf{WT}(ASM_{2n})$$

and ask when $J_\Delta$ of a particular set of tiples will be centrally symmetric.

**Proposition 2.6.** *For $i, j, \in [2n-1]$, and valid $k$,*

$$J_\Delta((i, j, k), (2n-i, 2n-j, k)) \in \mathbf{WT}(HTASM_{2n})$$

*Moreover, restricting to the set of triples $(i, j, k)$ where $j < n$ or $(i \leq n$ and $j = n)$ gives a bijection onto the set of join-irreducibles of $\mathbf{WT}(HTASM_{2n})$.*

*Proof.* By definition of $J_\Delta$,

$$J_\Delta((i, j, k)(2n-i, 2n-j, k)) = J_\Delta((i, j, k)) \vee J_\Delta((2n-i, 2n-j, k)).$$

We also know $J_\Delta((i, j, k)) = \mathbf{WT}(\pi)$ for some bigrassmannian permutation $\pi$ as likewise $J_\Delta((2n-i, 2n-j, k)) = \mathbf{WT}(\pi')$. Moreover, $\mathbf{WT}(\pi)$ and $\mathbf{WT}(\pi')$ are related by 180° rotation, and because join in $\mathbf{WT}(ASM_{2n})$ is componentwise, their join is centrally symmetric. Regardless of the $i$ and $j$, it turns out for valid $k$, the matrix $J_\Delta((i, j, k)(2n-i, 2n-j, k))$ is always a join-irreducible in $\mathbf{WT}(HTASM_{2n})$. Indeed, decreasing the $(i, j)$ and $(2n-i, 2n-j)$ entries by one gives the only HTASM Waldspurger matrix covered by $J_\Delta((i, j, k)(2n-i, 2n-j, k))$ in $\mathbf{WT}(HTASM_{2n})$. This decrementation only changes two entries of the Waldspurger matrix (or only one if $(i, j) = (n, n)$ in which case only subtract one from this entry– not two.) It is thus

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 12 & 12 & 12 \\ 1 & 12 & 123 & 123 \\ 1 & 12 & 123 & 1234 \\ 1 & 12 & 123 & \\ 1 & 12 & 12 & \\ 1 & 1 & 1 & \end{pmatrix}$$

Figure 2.1: Each entry in the matrix above corresponds to a valid $(i, j, k)$ triple. Join-irreducible in $\mathbf{WT}(HTASM_8)$ are in bijection with these triples. There are 44 digits in this figure and 44 is the fourth octahedral number.

a uniquely minimal alteration preserving the properties on page 76 (necessary to be the Waldspurger transform of some ASM) and preserving central symmetry.

The restrictions on $i$ and $j$ given in the statement of the proposition serve to ensure that each join-irreducible in $\mathbf{WT}(HTASM_{2n})$ is obtained only once. See figure 2.1 for a pictorial explanation. There are

$$(n-1)(2n-1) + (n-2)(2n-3) + \cdots + (1)(3)$$

valid triples $(i, j, k)$ with $j < n$ and

$$n + (n-1) + (n-2) + \cdots + 1$$

valid triples $(i, j, k)$ with $i \leq n$ and $j = n$ for a total of

$$\sum_{k=1}^{n-1} k(2k+1) + \binom{n+1}{2} = n(2n^2+1)/3$$

or octahedral many distinct join-irreducibles. $\qquad \square$

**Corollary.** *There exist two subposets $P_1$ and $P_2$ of $\mathbf{WT}(HTASM_{2n})$ with $|P_1| = |P_2| = n(2n^2+1)/3$ and*
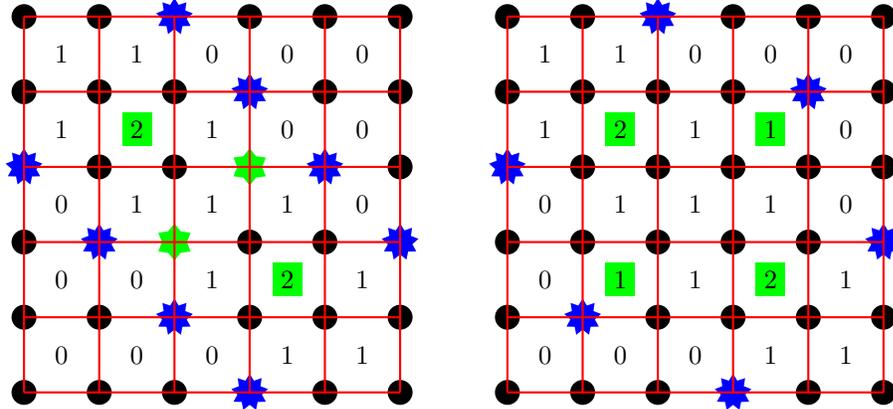
$$\mathbf{WT}(HTASM_{2n}) = J(P_1)$$

Figure 2.2: The transformation diagram for $J_\Delta((2,2,2),(4,4,2))$ in $\mathbf{WT}(HTASM_6)$ on the left and for $J_\Delta((2,2,2),(4,4,2),(6-2,6-2,1),(6-4,6-4,1))$ on the right. These correspond to elements of $P_1$ and $P_2$ respectively. The left shows the smallest centrally symmetric ASM Waldspurger matrix with $(2,2)$ entry a 2. The right shows the smallest centrally symmetric (permutation) Waldspurger matrix with the $(2,2)$ entry a 2. It follows from Proposition 0.7 all of the elements in $P_2$ must come from permutations.

$$BASM_n = J(P_2).$$

We see from the proof of Proposition 2.2 that the poset $P_1$ is a very natural octahedral poset. Similar to the tetrahedral poset in figure 1.10, it even possesses a geometric description: One may think of its elements in $P_1$ as oranges in an octahedral orange basket sitting on its edge. $\mathbf{WT}(HTASM_{2n})$ is the lattice of all stable configurations of oranges in this basket. A configuration is naturally identified with the set of oranges that may be removed without causing a tumble– i.e. matrices $M \in \mathbf{WT}(HTASM_{2n})$ correspond to antichains or order ideals in $P_1$.

Though $J(P_1) = \mathbf{WT}(HTASM_{2n})$ is a complete lattice *containing* type B Bruhat order as a subposet, it is not minimal in the sense of Dedekind-MacNeille completion. By Proposition 0.7 every element of $BASM_n \subset \mathbf{WT}(HTASM_{2n})$ must be expressible as the join of some set of elements in $\mathbf{WT}(\mathfrak{CS}_{2n})$. The element $J_\Delta((2,2,2),(4,4,2)) \in \mathbf{WT}(HTASM_6)$ (with transformation diagram given on the left of figure 2.2) cannot be expressed as such a join, however, because there is no $\pi \in \mathfrak{CS}_{2n}$ with $\mathbf{WT}(\pi)_{2,2} =$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 12 & 12 & \mathbf{12} & 12 \\ 1 & 12 & 12\mathbf{3} & \mathbf{123} & 123 \\ 1 & \mathbf{12} & \mathbf{123} & \mathbf{1234} & 1234 \\ 1 & 12 & 123 & 1234 & \mathit{12345} \\ 1 & 12 & 123 & 1234 \\ 1 & 12 & 123 & 123 \\ 1 & 12 & 12 & 12 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 2.3: In $\mathbf{WT}(HTASM_{2n})$, every join-irreducible is the $J_\Delta((i,j,k),(2n-i,2n-j,k))$ for some $(i,j,k)$ in the fundamental octahedron (see figure 2.1). Here, the italicized entries in green correspond to the join-irreducibles in $BASM_8 \subset \mathbf{WT}(HTASM_{2n})$ which are $J_\Delta$ of a singleton. Entries in black correspond to $J_\Delta$ of a doubleton, and the bold entries in red correspond to $J_\Delta$ of a four-tuple.

2 and $\mathbf{WT}(\pi) \leq J_\Delta((2,2,2),(4,4,2))$. Indeed, $P_1$ has $\binom{n}{3}$ elements coming from $HTASM_{2n} \smallsetminus \mathfrak{CS}_{2n}$ all of which share this difficulty.

Before characterizing the join-irreducibles in $BASM_n$, let us define the **fundamental octahedron** to be the set of triples

$$\{(i,j,k) \quad | \quad j < n \text{ or } (i \leq n \text{ and } j = n)\}.$$

Note that every point in the fundamental octahedron corresponded to a unique join-irreducible in $\mathbf{WT}(HTASM_{2n})$ (see figure 2.2). The same is true for $BASM_n \subset \mathbf{WT}(HTASM_{2n})$.

**Theorem 2.7.** *Every join-irreducible in $BASM_n$ (i.e. every element in $P_2$) is $J_\Delta$ of exactly one, two, or four join-irreducibles of $\mathbf{WT}(ASM_{2n})$. Let $M$ be a centrally symmetric $(2n-1) \times (2n-1)$ Waldspurger matrix. Then $M$ is in $P_2$ if and only if one of the following three hold:*

*1. $M = J_\Delta((n,n,k))$ for $1 \leq k \leq n$.*

*2. $M = J_\Delta((i,j,k)$ and $(2n-i,2n-j,k))$*

   *for $i > n$, $j < n$ and $k \leq \min\{i,j\}$*

*or for $i < n$, $j = n$ and $k \le \min\{i, j\}$*

*or for $i = n$, $j < n$ and $k \le \min\{i, j\}$.*

*or for $i, j < n$ and $k \le \min\{i, j, 2n - i - 1, 2n - j - 1\}$.*

3. $M = J_\Delta((i, j, k), (2n - i, 2n - j, k), (2n - i, j, k - \min\{n - i, n - j\}), (i, 2n - j, k - \min\{n - i, n - j\}))$

   *for $i, j < n$ and $2 \le \min\{n - i, n - j\} < k \le \min\{i, j\}$*

*Proof.* These cases partition the fundamental octahedron into three parts as shown in figure 2.3. Before getting into details, first check that the enumeration is correct: By Lascoux and Scützenberger's result, we know that there are $\frac{n(2n^2+1)}{3}$ elements in $P_2$. We have $n$ elements of the first type, $2\binom{n}{2} + \binom{n+1}{3} + \sum_{k=1}^{n-1} k^2$ elements of the second type, and $\binom{n}{3}$ elements of third type and one can check that indeed,

$$n + 2\binom{n}{2} + \binom{n+1}{3} + \sum_{k=1}^{n-1} k^2 + \binom{n}{3} = \frac{n(2n^2 + 1)}{3}.$$

From Proposition 0.7 it follows that every element in $P_2$ must be $\mathbf{WT}(\pi)$ for some permutation $\pi \in \mathfrak{CS}_{2n}$. Further, we know that each of $\mathbf{WT}(\pi) \in \mathbf{WT}(\mathfrak{CS}_{2n})$ is $J_\Delta(S)$ for some set $S$ of $(i, j, k)$ triples. To determine $S$, we need only specify triples $(i, j, k)$ in the fundamental octahedron, as central symmetry implies that for such triples

$$(i, j, k) \in S \iff (2n - i, 2n - j, k) \in S.$$

We will show that given a triple $(i, j, k)$ in the fundamental octahedron, there is a unique minimal set $S$ such that $J_\Delta(S) = \mathbf{WT}(\pi)$ for some $\pi \in \mathfrak{CS}_{2n}$ and that $\mathbf{WT}(\pi)$ is the smallest (permutation) Waldspurger matrix whose $(i, j)$ entry has value $k$.

**Case 1:**

If $i = j = n$ then $J_\Delta(i, j, k)$ is also an element in $P_1$ and is $\mathbf{WT}(\pi)$ for $\pi$ a centrally symmetric bigrassmannian permutation. It covers exactly one element in $BASM_n$

because it is uniquely minimal amoung $(2n-1) \times (2n-1)$ Waldspurger matrices whose $(n,n)$ entry is $k$.

**Case 2:**

The set of $(i,j,k)$ triples in the fundamental octahedron which fall into the second case come in four flavors. There is no hope of reducing the size of $S$ without breaking central symmetry, but we must check that $\mathbf{WT}^{-1}(J_\Delta((i,j,k),(2n-i,2n-j,k)))$ is indeed a permutation matrix, and not just an ASM. We will argue this with subcases, using the transformation diagram.

*Subcase 1:*

For triples $(i,j,k)$ in the fundamental octahedron with $i,j < n$ and $k < \min\{i,j,2n-i-1,2n-j-1\}$ it is easy to see that

$$\mathbf{WT}^{-1}(J_\Delta((i,j,k),(2n-i,2n-j,k))) = \left( \begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right)$$

which is a permutation matrix since $A$ and $B$ are permutation matrices for bigrassmannian permutations. Indeed,

$$A = \mathbf{WT}_n^{-1}(J_\Delta((i,j,k))$$

$$B = A \text{ rotated by } 180°.$$

*Subcase 2:*

For triples in the fundamental octahedron with $i \geq n$, notice that $J_\Delta((i,j,k)(2n-i,2n-j,k))$ will contain a $2(i-n)+1 \times 2(n-j)+1$ rectangular submatrix centered at $(n,n)$ with all values equal to $k$, as shown in figure 2.4 Any entry on the diagonal of the transformation diagram which is inside this box gets a star. Outside of this box, the placement of stars on the left half of the diagram coinsides with the stars in the transformation diagram of $\mathbf{WT}_{2n}^{-1}(J_\Delta((i,j,k))$ and the placement of the stars on the

right half of the diagram coinsides with the stars in $\mathbf{WT}_{2n}^{-1}(J_\Delta((2n-i, 2n-j, k))$. That is, $\mathbf{WT}^{-1}(J_\Delta((i, j, k), (2n-i, 2n-j, k)))$ is the permutation which maps $1, 2, \ldots j$ to the same place $\mathbf{WT}_{2n}^{-1}(J_\Delta((i, j, k)))$ would, fixes $j+1, \ldots, 2n-j$, and maps $2n-j+1, \ldots, 2n$ to the same place $\mathbf{WT}_{2n}^{-1}(J_\Delta((2n-i, 2n-j, k)))$ would.
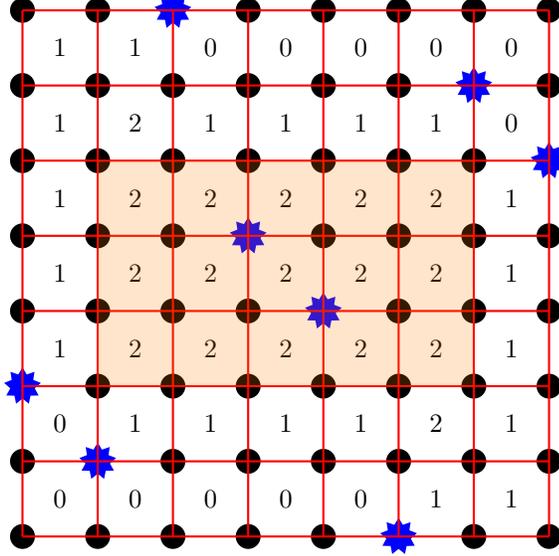


Figure 2.4: The transformation diagram for the join-irreducible $J_\Delta((5, 2, 2), (3, 6, 2))$ in $BASM_4$

### Subcase 3:

For triples in the fundamental octahedron with $j = n$ and $i \leq n$, an argument analogous to the one in Subcase 2 works by considering the transpose of the transformation diagram.

### Case 3:

We already saw in figure 2.2 that there are elements in $P_1$ which are $\mathbf{WT}(M)$ for $M \in HTASM_{2n} \smallsetminus \mathfrak{CS}_{2n}$ and that by Proposition 0.7, such elements cannot be in $P_2$. We will see exactly what these $\binom{n}{3}$ elements should be replaced with.

Let us attempt to build a transformation diagram in a minimal way by fixing the $(i, j)$ entry with value $k$ where $j \leq i < n$ and $2 \leq \min\{n - i, n - j\} \leq k \leq \min\{i, j\}$. We will consider the case when $i \leq j$ and $n-j \leq n-i$, and a similar argument will work for $i > j$. To preserve central symmetry, fixing the $(i, j)$ entry with value $k$ necessitates

fixing the $(2n - i, 2n - j)$ entry with value $k$. After this, consider the vertical strip between the specified entries. There are $2n - 2j$ vertical lines in this strip on which we need to place stars. Making the matrix as small as possible necessitates placing the $k$ stars above and to the right of entry $(i, j)$ *as close to* $(i, j)$ as possible, that is, on the left most $k$ of the $2n - 2j$ vertical lines. By symmetry, we are forced to place the $k$ stars below and to the left of $(2n - i, 2n - j)$ *as close to* $(2n - i, 2n - j)$ as possible, that is on the right most $k$ of the $2n - 2j$ vertical lines. We now have placed $2k$ stars on $2n - 2j$ vertical lines but by assumption $n - j < k$. Conclude by the pigeonhole principle that we have placed two stars in the same column and that our quest for minimality has lead us to creating the transformation diagram of some $M \in HTASM_{2n} \smallsetminus \mathfrak{CG}_{2n}$.

Backing up, we see that creating the transformation diagram of some $\pi \in \mathfrak{CG}_{2n}$ (after having fixed $(i, j, k)$ and $(2n - i, 2n - j, k)$) necessitates placing at most $n - j$ of the $k$ stars above and to the right of $(i, j)$ in the vertical strip and at most $n - j$ of the $k$ stars below and to the left of $(2n - i, 2n - j)$ in the vertical strip. That is, we are prescribing two additional triples: $(2n - i, j, k - (n - j))$ and $(i, 2n - j, k - (n - j))$.

We have now shown that it is *necessary* for a centrally symmetric (permutation) Waldspurger matrix with $(i, j)$ entry $k$ (where $j \le i < n$ and $2 \le \min\{n - i, n - j\} \le k \le \min\{i, j\}$) to be above $J_\Delta((i, j, k), (2n - i, 2n - j, k), (2n - i, j, k - \min\{n - i, n - j\}), (i, 2n - j, k - \min\{n - i, n - j\}))$. It turns out, moreover, that $J_\Delta((i, j, k), (2n - i, 2n - j, k), (2n - i, j, k - \min\{n - i, n - j\}), (i, 2n - j, k - \min\{n - i, n - j\}))$ will always be the Waldspurger transform of some permutation, and showing this will conclude our proof.

To this end, we give the permutation explicitly: Let $a = k - \min\{i, j\}$. Let $b = n - j + a$ and let $c = k + a$. Write down the identity permutation of $\mathfrak{G}_{2n}$ in oneline notation, and place cuts after positions $a, b, c, n, 2n - c, 2n - b, 2n - a$. This partitions the numbers $1, 2, \ldots, 2n$ into 8 parts, (the first and last part may be empty). We will not touch the

first and last parts, but label the other 6 from left to right with the letters C-O-R-S-E-T (or N-O-R-M-A-S[2]). We then reorder the blocks to R-E-C-T-O-S (or R-A-N-S-O-M). This yields the oneline notation for the permutation whose transformation diagram we have been considering, $J_\Delta((i,j,k),(2n-i,2n-j,k),(2n-i,j,k-\min\{n-i,n-j\}),(i,2n-j,k-\min\{n-i,n-j\}))$.

For example, if $(i,j,k)=(3,3,3)$ and $n=4$, then $(a,b,c)=(0,1,3)$ and

$$|\underset{C}{\underline{1}}|\underset{O}{\underline{23}}|\underset{R}{\underline{4}}|\underset{S}{\underline{5}}|\underset{E}{\underline{67}}|\underset{T}{\underline{8}}| \quad \Longleftrightarrow \quad |\underset{R}{\underline{4}}|\underset{E}{\underline{67}}|\underset{C}{\underline{1}}|\underset{T}{\underline{8}}|\underset{O}{\underline{23}}|\underset{S}{\underline{5}}|$$

Consider the transformation diagram and verify that indeed,
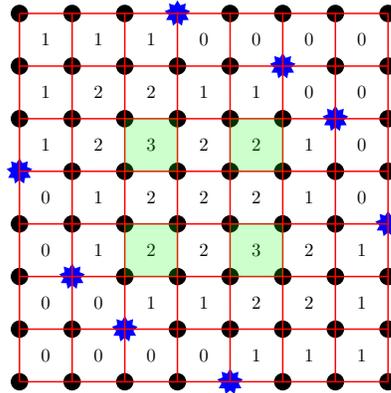


$$\mathbf{WT}(46718235) = J_\Delta((3,3,3),(5,5,3),(5,3,2),(3,5,2)).$$

□

We now have an explicit description of the elements of $P_2 \subset BASM_n \subset \mathbf{WT}(HTASM_{2n})$ as Waldspurger matrices. We used this to compute the size of $BASM_n$ for $n = 2, 3, 4, 5, \ldots$ to be

$$2, 10, 132, 4824, \ldots$$

---

[2]We chose to label these parts so that the rearrangement would be an anagram. This is totally unnecessary. In the process, we discovered the following useless fact: these two pairs of words are the only anagrams on six letters in the English language related by the involution 351624.

Note that this is indeed less than the size of $HTASM_{2n}$ for $n = 2, 3, 4, 5 \ldots$ which is

$$2, 10, 140, 5544, 622908, 198846076, \ldots.$$

Given some $M \in \mathbf{WT}(HTASM_{2n})$ it is relatively straight forward to express it as $J_\Delta(S)$ for $S$ a minimal set of $(i, j, k)$ triples, and to check if the triples in the fundamental octahedron are accompanied by those suggested by Theorem 2.2. This, at the very least, gives hope that a more intrinsic description of the elements of $BASM_n$ may be possible.

**Question 7.** Is there a faster way to compute $\#J(P_2)$ (the size of $BASM_n$) and does it admit a closed formula?

## 2.3 $\mathbf{WT}_B$ and $\mathbf{WT}_C$ via Folding

We now consider the definitions of the maps $\mathbf{WT}_B$ and $\mathbf{WT}_C$ given at the beginning of this chapter, and work towards combinatorial descriptions. This necessitates familiarizing ourselves with type B and C root coordinates and we follow the conventions established in Section 0.4.

Let $P$ be the change of basis matrix that gives the simple roots of $B_n$ in terms of the standard basis vectors, and define $Q$ analogously for $C_n$. That is,

$$P_{i,j} = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i = j + 1 \\ 0 & \text{otherwise} \end{cases} \qquad Q_{i,j} = \begin{cases} 1 & \text{if } i = j, j < n \\ -1 & \text{if } i = j + 1 \\ 2 & \text{if } i = j = n \\ 0 & \text{otherwise} \end{cases}.$$

One can then verify that

$$P_{i,j}^{-1} = \begin{cases} 1 & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases} \qquad Q_{i,j}^{-1} = \begin{cases} 1 & \text{if } i = j, j < n \\ 1/2 & \text{if } i = j = n \\ 0 & \text{otherwise} \end{cases}.$$

With respect to this ordering on the simple roots, one can further verify that the inverses of the Cartan matrices for the root systems $B_n$ and $C_n$ are, respectively:

$$(C_{B_n}^{-1})_{i,j} = \begin{cases} \min(i,j) & \text{if } j < n \\ i/2 & \text{if } j = n \end{cases} \qquad (C_{C_n}^{-1})_{i,j} = \begin{cases} \min(i,j) & \text{if } i < n \\ j/2 & \text{if } i = n \end{cases}.$$

Next, if we let $S = Q(C_{C_n}^{-1})$, and let $R = P(C_{B_n}^{-1})$, one may verify that

$$S = \begin{cases} 1 & \text{if } j \geq i \\ 0 & \text{otherwise} \end{cases} \qquad R = \begin{cases} 1 & \text{if } j \geq i, j \neq n \\ 1/2 & \text{if } j = n \\ 0 & \text{otherwise} \end{cases}.$$

**Theorem 2.8.** $\mathcal{F}$ *is a bijection between centrally symmetric Waldspurger matrices of type $A_{2n-1}$, and Waldspurger matrices of type $C_n$ and the following diagram commutes:*

$$\begin{array}{ccc} \pm\mathfrak{S}_n & \xrightarrow{\mathbf{WT}_{C_n}} & \mathbf{WT}_{C_n}(\pm\mathfrak{S}_n) \\ \uparrow{\scriptstyle\star} & & {\scriptstyle\mathcal{F}}\uparrow \\ \mathfrak{CS}_{2n} & \xrightarrow{\mathbf{WT}} & \mathbf{WT}(\mathfrak{CS}_{2n}) \end{array}$$

*Proof.* We will show that $\mathcal{F}(\mathbf{WT}(\pi))_{i,j}$ and $\mathbf{WT}_{C_n}(\pi^\star)_{i,j}$ are summing over the same parts of the permutation matrix $\pi$. On the one hand,

$$\mathcal{F}(\mathbf{WT}(\pi))_{i,j} = \begin{cases} \mathbf{WT}(\pi)_{i,j} + \mathbf{WT}(\pi)_{2n-i+1,j} & \text{for all } 1 \le i, j < n \\[2ex] \mathbf{WT}(\pi)_{i,j} & \text{for all } i = n, j \le n \end{cases}$$

$$= \begin{cases} \displaystyle\sum_{\substack{a \le i \\ b > j}}^{2n} \pi_{a,b} + \sum_{\substack{a > 2n-i+1 \\ b \le j}}^{2n} \pi_{a,b} & i \le j < n \\[3ex] \displaystyle\sum_{\substack{a > i \\ b \le j}}^{2n} \pi_{a,b} + \sum_{\substack{a > 2n-i+1 \\ b \le j}}^{2n} \pi_{a,b} & j \le i < n \\[3ex] \displaystyle\sum_{\substack{a > i \\ b \le j}}^{2n} \pi_{a,b} & i = n \end{cases} .$$

On the other hand,

$$\mathbf{WT}_{C_n}(\pi^\star)_{i,j} = \left(Id - (Q^{-1}\pi^\star Q)C_{C_n}^{-1}\right)_{i,j}$$

$$= \left(C_{C_n}^{-1} - (Q^{-1}\pi^\star S)\right)_{i,j}$$

$$= \left(C_{C_n}^{-1}\right)_{i,j} - \left((Q^{-1}\pi^\star S)\right)_{i,j}$$

$$= \begin{cases} \min(i,j) & \text{if } i < n \\ j/2 & \text{if } i = n \end{cases} - \begin{cases} \displaystyle\sum_{a \le i, b \le j} \pi^\star_{a,b} & \text{if } i < n \\ \frac{1}{2}\displaystyle\sum_{a \le i, b \le j} \pi^\star_{a,b} & \text{if } i = n \end{cases}$$

$$= \begin{cases} \min(i,j) - \displaystyle\sum_{a \le i, b \le j}^{2n} \pi_{a,b} - \pi_{2n-a+1,b} & \text{if } i < n \\ \frac{j}{2} - \frac{1}{2}\displaystyle\sum_{a \le i, b \le j}^{2n} \pi_{a,b} - \pi_{2n-a+1,b} & \text{if } i = n \end{cases}$$

$$= \begin{cases} i - \displaystyle\sum_{a \le i, b \le j}^{2n} \pi_{a,b} - \pi_{2n-a+1,b} & \text{if } i \le j < n \\ j - \displaystyle\sum_{a \le i, b \le j}^{2n} \pi_{a,b} - \pi_{2n-a+1,b} & \text{if } j \le i < n \\ \frac{j}{2} - \frac{1}{2}\displaystyle\sum_{a \le i, b \le j}^{2n} \pi_{a,b} - \pi_{2n-a+1,b} & \text{if } i = n \end{cases}$$

$$= \begin{cases} \displaystyle\sum_{a \le i}^{2n} \pi_{a,b} - \sum_{a \le i, b \le j}^{2n} \pi_{a,b} + \sum_{\substack{a > 2n-i+1 \\ b \le j}}^{2n} \pi_{a,b} & \text{if } i \le j < n \\ \displaystyle\sum_{b \le j}^{2n} \pi_{a,b} - \sum_{a \le i, b \le j}^{2n} \pi_{a,b} + \sum_{\substack{a > 2n-i+1 \\ b \le j}}^{2n} \pi_{a,b} & \text{if } j \le i < n \\ \frac{j}{2} - \frac{1}{2}\displaystyle\sum_{a \le i, b \le j}^{2n} \pi_{a,b} - \pi_{2n-a+1,b} & \text{if } i = n \end{cases}$$

$$= \begin{cases} \displaystyle\sum_{\substack{a \le i \\ b > j}}^{2n} \pi_{a,b} + \sum_{\substack{a > 2n-i+1 \\ b \le j}}^{2n} \pi_{a,b} & i \le j < n \\ \displaystyle\sum_{\substack{a > i \\ b \le j}}^{2n} \pi_{a,b} + \sum_{\substack{a > 2n-i+1 \\ b \le j}}^{2n} \pi_{a,b} & j \le i < n \\ \displaystyle\sum_{\substack{a > i \\ b \le j}}^{2n} \pi_{a,b} & i = n \end{cases}.$$

The last equality is explained pictorially on the next page. $\qquad\qquad\square$

The case $i \leq j < n$ says

$$\left(\begin{array}{ccccc} \ddots & \cdots & & & \\ \cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \begin{array}{c}\text{sum these}\\ \text{entries}\end{array} & \\ \cdots & \pi_{i,j-1} & \pi_{i,j} & & \\ \cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & & \\ & & & & \ddots \end{array}\right) + \left(\begin{array}{ccccc} \ddots & \cdots & & & \\ \cdots & \begin{array}{c}\text{sum these}\\ \text{entries}\end{array} & \pi_{2n-i+1,j-1} & \pi_{2n-i+1,j} & \\ \cdots & & \pi_{2n-i,j-1} & \pi_{2n-i,j} & \\ \cdots & & \pi_{2n-i-1,j-1} & \pi_{2n-i-1,j} & \\ & & & & \ddots \end{array}\right)$$

$$= \left(\begin{array}{ccccc} \ddots & \cdots & & & \\ \cdots & \begin{array}{c}\text{sum these}\\ \text{entries}\end{array} & \pi_{i-1,j} & \pi_{i-1,j+1} & \\ \cdots & & \pi_{i,j} & \pi_{i,j+1} & \\ \cdots & & \pi_{i+1,j} & \pi_{i+1,j+1} & \\ & & & & \ddots \end{array}\right) - \left(\begin{array}{ccccc} \ddots & \cdots & & & \\ \cdots & \begin{array}{c}\text{sum these}\\ \text{entries}\end{array} & \pi_{i-1,j} & \pi_{i-1,j+1} & \\ \cdots & & \pi_{i,j} & \pi_{i,j+1} & \\ \cdots & & \pi_{i+1,j} & \pi_{i+1,j+1} & \\ & & & & \ddots \end{array}\right) + \left(\begin{array}{ccccc} \ddots & \cdots & & & \\ \cdots & \pi_{2n-i+1,j-1} & \pi_{2n-i+1,j} & \begin{array}{c}\text{sum these}\\ \text{entries}\end{array} & \\ \cdots & \pi_{2n-i,j-1} & \pi_{2n-i,j} & & \\ \cdots & \pi_{2n-i-1,j-1} & \pi_{2n-i-1,j} & & \\ & & & & \ddots \end{array}\right)$$

The $j \leq i < n$ case says

$$\left(\begin{array}{ccccc} \ddots & \cdots & & & \\ \cdots & \begin{array}{c}\text{sum}\\ \text{these}\\ \text{entries}\end{array} & \pi_{i-1,j} & \pi_{i-1,j+1} & \\ \cdots & & \pi_{i,j} & \pi_{i,j+1} & \\ \cdots & & \pi_{i+1,j} & \pi_{i+1,j+1} & \\ & & & & \ddots \end{array}\right) - \left(\begin{array}{ccccc} \ddots & \cdots & & & \\ \cdots & \begin{array}{c}\text{sum these}\\ \text{entries}\end{array} & \pi_{i-1,j} & \pi_{i-1,j+1} & \\ \cdots & & \pi_{i,j} & \pi_{i,j+1} & \\ \cdots & & \pi_{i+1,j} & \pi_{i+1,j+1} & \\ & & & & \ddots \end{array}\right)$$

$$= \left(\begin{array}{ccccc} \ddots & \cdots & & & \\ \cdots & \pi_{2n-i-1,j-1} & \pi_{2n-i-1,j} & \begin{array}{c}\text{sum these}\\ \text{entries}\end{array} & \\ \cdots & \pi_{2n-i,j-1} & \pi_{2n-i,j} & & \\ \cdots & \pi_{2n-i+1,j-1} & \pi_{2n-i+1,j+1} & & \\ & & & & \ddots \end{array}\right) + \left(\begin{array}{ccccc} \ddots & \cdots & & & \\ \cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \begin{array}{c}\text{sum these}\\ \text{entries}\end{array} & \\ \cdots & \pi_{i,j-1} & \pi_{i,j} & & \\ \cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & & \\ & & & & \ddots \end{array}\right)$$

Finally, the case $i = n$ says

$$\left(\begin{array}{ccccc} \ddots & \cdots & & & \\ \cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \pi_{i-1,j+1} & \\ \cdots & \begin{array}{c}\text{sum these}\\ \text{entries}\end{array} & \pi_{i,j} & \pi_{i,j+1} & \\ \cdots & & \pi_{i+1,j} & \pi_{i+1,j+1} & \\ & & & & \ddots \end{array}\right) = \frac{1}{2}\left(\begin{array}{ccccc} \ddots & \cdots & & & \\ \cdots & \pi_{i-1,j-1} & \pi_{i-1,j+1} & & \\ \cdots & \begin{array}{c}\text{sum}\\ \text{these}\\ \text{entries}\end{array} & \pi_{i,j} & \pi_{i,j+1} & \\ \cdots & & \pi_{i+1,j} & \pi_{i+1,j+1} & \\ & & & & \ddots \end{array}\right)$$

$$- \frac{1}{2}\left(\begin{array}{ccccc} \ddots & \cdots & & & \\ \cdots & \begin{array}{c}\text{sum these}\\ \text{entries}\end{array} & \pi_{i-1,j} & \pi_{i-1,j+1} & \\ \cdots & & \pi_{i,j} & \pi_{i,j+1} & \\ \cdots & & \pi_{i+1,j} & \pi_{i+1,j+1} & \\ & & & & \ddots \end{array}\right) + \frac{1}{2}\left(\begin{array}{ccccc} \ddots & \cdots & & & \\ \cdots & \pi_{i-1,j-1} & \pi_{i-1,j} & \begin{array}{c}\text{sum these}\\ \text{entries}\end{array} & \\ \cdots & \pi_{i,j-1} & \pi_{i,j} & & \\ \cdots & \pi_{i+1,j-1} & \pi_{i+1,j} & & \\ & & & & \ddots \end{array}\right)$$

Because transposition and the map $\star : \mathfrak{C}\mathfrak{S}_{2n} \to \pm\mathfrak{S}_n$ commute, we will from now on abuse notation and identify centrally symmetric permutations with their images in $\pm\mathfrak{S}_n$.

**Proposition 2.9.** $\mathbf{WT}_{C_n}(\pi^\top) = (\mathbf{WT}_{B_n}(\pi))^\top$ *for any* $\pi \in \pm\mathfrak{S}_n$.

*Proof.*

$$
\begin{aligned}
\mathbf{WT}_{C_n}(\pi^\top) &= Id - (Q^{-1}\pi^\top Q)C_{C_n}^{-1} \\
&= C_{C_n}^{-1} - (Q^{-1}\pi^\top S) \\
&= (C_{B_n}^{-1})^\top - (R^\top \pi^\top (P^{-1})^\top) \\
&= Id - (P^{-1}\pi R)^\top (C_{B_n}^{-1})^\top \\
&= (Id - (P^{-1}\pi R)C_{B_n}^{-1})^\top \\
&= (\mathbf{WT}_{B_n}(\pi))^\top.
\end{aligned}
$$

$\square$

Informally, this proposition tells us that while the geometry of the Waldspurger decompositions and Meinrenken tiles for types B and C may be different in higher dimensions, their combinatorics will remain essentially the same.

## 2.4   $B_2$ and $C_2$ in Detail

There are exactly eight centrally symmetric $3 \times 3$ Waldspurger matrices of type A:

$$
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad
\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}
$$
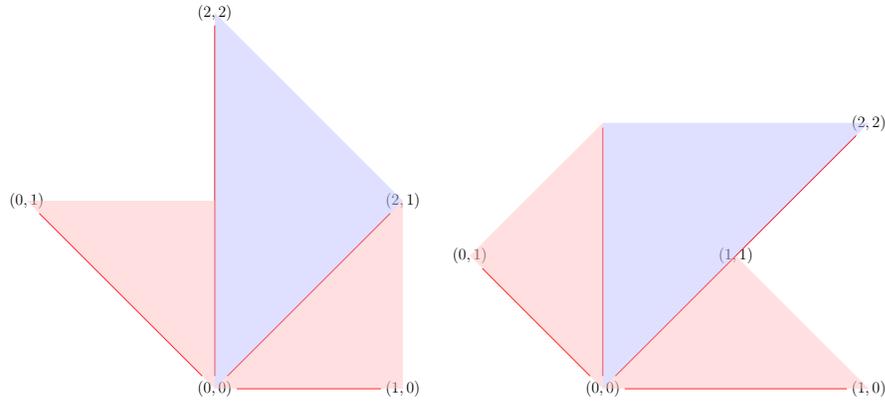
Figure 2.5: The Meinrenken tiles for $C_2$ and $B_2$ respectively

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

We may fold them vertically to get type C Waldspurger matrices, or horizontally to get type B:

$$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \right\}.$$

Recall that, in type A, the dimensions of each of the simplices was determined by the number of cycles of the corresponding permutation, and so the number of simplices of a given dimension was a Stirling number of the first kind. In type B, we see "type B Stirling numbers of the first kind" [42] with our 1 point, 4 edges, and 3 triangles for $B_2$ and $C_2$. In this dimension there are two HTASMs which are not permutations,

22
12
22   12
11   12
10   12
11   01
10   00
00   01
00
00

22
12
22   12
11   12
12
11
10   12
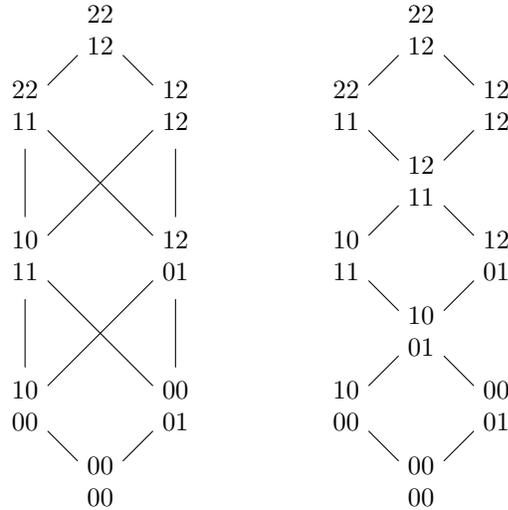11   01
10
01
10   00
00   01
00
00

Figure 2.6: In the case of $C_2$, (and $B_2$, though it is not shown here) Waldspurger order is exactly Bruhat order, and componentwise comparison of folded centrally symmetric ASMs is exactly its Dedekind-MacNeille completion.

with type A Waldspurger matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. They fold vertically to give us the two extra matrices pictured in the right hand side of Figure 2.6.

One is given hope from Figure 2.4 and other low dimensional examples that Bruhat order might, as in type A, be merely componentwise comparison of Waldspurger matrices. Indeed, it is true for $C_2$ and $C_3$ and in both cases, the Dedekind-MacNeille completion comes from simply folding centrally symmetric ASMs. We will see, in Section 2.6 that it fails for $C_n$ when $n \geq 4$.

## 2.5   UM Vectors for types B and C

One may wonder if the column vectors and row vectors of type B and type C Waldspurger matrices admit nice classifications, similar to UM vectors in the type A case. By Proposition 2.3 the set of columns (respectively rows) coming from $\mathbf{WT}_B$ will be

the same as the set of rows (respectively columns) coming from $\mathbf{WT}_C$. Unlike in type A, however, it is not true that the set of columns coming from $\mathbf{WT}_B$ will be the same as the set of rows coming from $\mathbf{WT}_B$. For example, the vector $(2, 1)$ appears as a row of a $B_2$ Waldspurger matrix, but not as a column. (See the list of all type $B_2$ Waldspurger matrices on page 92).

This subtlety should not mask the fact that both column and row vectors (for both $\mathbf{WT}_B$ and $\mathbf{WT}_C$) are essentially folded UM vectors, and we desire to handle them as such. Notice where the column/row disparity arises: When folding centrally symmetric Waldspurger matrices vertically via the map $\mathcal{F}$ (to get type C Waldspurger matrices), the middle column gets doubled, but the middle row does not. Similarly, when folding horizontally (to get type B Waldspurger matrices) the middle row gets doubled but the middle column does not.

Desiring to give a unified definition for UM vectors of types B and C, we conflate these two notions by considering a new vertical folding map $\widetilde{\mathcal{F}}$ which folds the middle row onto itself, doubling it.

$$
\widetilde{\mathcal{F}}(M)_{i,j} := \begin{cases} M_{i,j} + M_{2n-i+1,j} \text{ for all } 1 \le i, j < n \\ 2M_{i,j} \text{ for all } i = n, j \le n \end{cases}.
$$

**Definition 11.** A **UM vector of type B or C** is any vector which appears as a column of $\widetilde{\mathcal{F}}(\mathbf{WT}(\pi))$ for $\pi \in \mathfrak{CS}_{2n}$.

The inequality description of UM vectors from Theorem 1.4 immediately gives an inequality description for "UM vectors for types B and C".

**Proposition 2.10.** *UM vectors of type B or C must start with entries $0, 1, 2$, increase by $0, 1,$ or $2$ up to the diagonal, and increase by $-1, 0,$ or $1$ after the diagonal, ending with an even number.*

The proof involves showing that for all $i \in [n-1]$, the $i+1$st entry of a UM vector of type B or C depends on the the $i$th entry, and showing that, moreover, every vector satisfying this condition appears as a column of $\widetilde{\mathcal{F}}(\mathbf{WT}(\pi))$ for some $\pi \in \mathfrak{CS}_{2n}$. It is quite similar to the proof of Theorem 1.2 and so we omit it.

Given $v = (v_0, v_1, \ldots, v_n)$ a UM vector of type $B_n$ or $C_n$, one can show that $(v_0, v_1, \ldots, \frac{v_n}{2})$ will be a column vector in some type C Waldspurger matrix, (and a row vector in some type B Waldspurger matrix). The vector $v$ will also be a row vector in some type C Waldspurger matrix (and column vector in some type B Waldspurger matrix) unless all of its entries are even. In that case, $(\frac{v_0}{2}, \frac{v_1}{2}, \ldots, \frac{v_n}{2})$ will be. In all these cases, the enumeration is the same and follows from the previous proposition.

**Corollary.** *There are $2 \cdot 3^{n-1}$ UM vectors of type B or C.*

We also omit this proof as it is similar to the type A case.

We saw in Section 1.2 that UM vectors correspond to abelian ideals in the Lie algebra $\mathfrak{sl}_n$ and that there are $2^{n-1}$ of them. It is known [20] that the number of abelian ideals is a power of two for both of the Lie algebras $\mathfrak{so}_{2n+1}$ and $\mathfrak{sp}_{2n}$ (corresponding to types B and C). Our enumeration from the corollary above is a bit disappointing because it implies that the type A (UM vectors $\leftrightarrow$ abelian ideals) phenomenon does not generalize to types B and C.

**Question 8.** Do UM vectors of type B and C have Lie theoretic significance? Do they characterize certain order filters in the poset of positive roots?

## 2.6 Folding the Base and Waldspurger Order

We now return to the poset of join irreducibles $P_2 \subset BASM_n \subset \mathbf{WT}(HTASM_{2n})$ from Section 2.2 and consider its image under the vertical folding map $\mathcal{F}$. Because of the combinatorial equivalence established in Proposition 2.3 we will restrict our attention to the type C case.

Recall from Theorem 2.2 that every element of the base (that is, a join irreducible in $BASM_n \subset \mathbf{WT}(HTASM_{2n})$ or element of $P_2$) is uniquely minimal with respect to some $(i,j,k)$ triple in the fundamental octahedron. In contrast, minimal type C Waldspurger matrices, with respect to a single fixed entry, need not be unique. The smallest such example arises from the group $\pm\mathfrak{S}_4$ of $4\times4$ signed permutation matrices. There are two incomparable elements of the base whose type C Waldspurger matrices are minimal after fixing the $(2,2)$ entry to be a two:

$$
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad \text{vs} \quad
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 2 & 2 & 1 \\
0 & 2 & 2 & 1 \\
0 & 2 & 2 & 1
\end{bmatrix}.
$$

These two matrices may be "unfolded" to the centrally symmetric type A Waldspurger matrices:

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

If we write the $(2,2)$ entry as $2 = 2 + 0$ and we "fall down" in the type A way, we get the matrix on the left. If we write $2 = 1 + 1$ and "fall down" in the type A way, we get the matrix on the right.

Folding seems to take yet another step away from the underlying octahedral geometry. The poset $P_1$ (of join-irreducible elements of $\mathbf{WT}(HTASM_{2n})$) perfectly

encapsulated this geometry; stable configurations of oranges in an octahedral orange basket governed by gravity. The poset $P_2$, preserved the direction of gravity and even some of the octahedral geometry, but somehow a tetrahedron of oranges got "punched" upwards. In terms of oranges and gravity, the poset $\mathcal{F}(P_2)$ is best described as a strong mimosa, and I will say nothing more for fear that this thesis might be rejected.

Not only this, folding destroys Bruhat order. Among the bigrassmannian elements of the Weyl group for type $C_4$, there are exactly two cover relations in type C Waldspurger order which are not cover relations in Bruhat order:

$$
\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} < \begin{bmatrix} 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 2 & 2 & 1 \end{bmatrix} < \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & 2 \\ 0 & 2 & 4 & 2 \end{bmatrix}.
$$

Folding $\mathbf{WT}(HTASM_8)$ also breaks the lattice structure with respect to componentwise order. The same sort of failures were recognized for signed monotone triangles by Reading in Section 10, Question 4 of [34].

## 2.7   Further Questions

1. Is there a description of Waldspurger order in terms of words in the Coxeter group?

2. For type A, Waldspurger order *equals* Bruhat order. For types B and C, Waldspurger order *extends* Bruhat order. How does Waldspurger order behave in other types?

3. In the beginning of Chapter 3 we defined the Waldspurger transform for general crystallographic root systems in terms of the inverse Cartan matrix, $C_\Phi^{-1}$, as

$\mathbf{WT}_\Phi(w) := (id - w)C_\Phi^{-1}$. This was somewhat less natural than letting $A$ be the matrix whose column vectors give the non-zero vertices of the fundamental alcove in root coordinates and defining $\mathbf{WT}_\Phi(w) \underset{\text{alt}}{=} (id - w)A$. In choosing to leave the scaling of columns by the "marks" (see page 17) as an afterthought, we were able make type A, B and C, Waldspurger matrices all have non-negative integer entries. Is this concession in definitions sufficient to make $\mathbf{WT}_\Phi(w)_{i,j} \in \mathbb{Z}_{\geq 0}$ for all crystallographic types?

4. How many elements are there in the Dedekind-MacNeille completion of Bruhat order for type B?

5. In [25], Meinrenken has another intriguing theorem: Let $W$ an affine Weyl group with $A$ a fundamental alcove. Then for any endomorphism $S$, in the same connected component as 0 in the set $\{S \in \text{End}(V) \mid \det(S - w) \neq 0 \forall w \in W\}$, the simplices $(S - w)A$ for $w \in W$ are all disjoint and their closures cover the entire vector space $V$.

This theorem seems to provide an interesting interpolation between the affine hyperplane arrangement, or Stiefel diagram, and the Meinrenken tile. Does any nice combinatorics arise from selecting nice endomorphisms? Is there an intrinsic characterization of the types of tilings that arise in this way? (See Figure 5 in the appendix, or [2].)

# Chapter 3

# Other Combinatorics in the Meinrenken Tile

Thus far, we have shown that the Waldspurger transform of permutations and alternating sign matrices is intimately connected with the type A and type B Bruhat orders and their Dedekind-MacNeille completions. The motivation for defining the **WT**, however, was to better understand the geometry of the Waldspurger and Meinrenken theorems. In this chapter we return to the geometry primarily in the type A case, now better equipped with a combinatorial understanding of **WT**. We rely heavily on the notation introduced in Section 0.8. In particular, for $\pi \in \mathfrak{S}_n$, let $c_i$ denote the $i$th column vector of $\mathbf{WT}(\pi)$ and recall that we associated one cone and two simplices with $\pi$:

$$C_\pi := \left\{ \sum_{i=1}^{n-1} a_i c_i \quad | \quad a_i \in \mathbb{R}_{\geq 0} \right\}$$

$$A_\pi := \left\{ \sum_{i=1}^{n-1} a_i c_i \quad | \quad a_i \in \mathbb{R}_{\geq 0} \text{ and } \sum a_i \leq 1 \right\}$$

$$\Delta_\pi := \left\{ \sum_{i=1}^{n-1} a_i c_i \quad | \quad a_i \in \mathbb{R}_{\geq 0} \text{ and } \sum a_i = 1 \right\}.$$

We start by describing basic symmetries of the Meinrenken tile $\mathfrak{M} := \bigsqcup_{w \in W} A_w$ in terms of operations on the permutations and their transformation diagrams.

## 3.1 Symmetries of the Waldspurger Decomposition and Meinrenken Tile

Recall from Figure 9 and Figure 10 that the dimension of the cone $C_w$ in the Waldspurger decomposition and the dimension of the simplex $A_w$ in the Meinrenken tile depend on the element $w \in W$. In general, the codimension of $A_w$ in $\mathfrak{M}$ is exactly the dimension of the space fixed by $w$ acting on $V$. When $W = \mathfrak{S}_n$, this is equivalent to the following:

**Proposition 3.1.** *The number of cycles of $\pi \in \mathfrak{S}_n$ equals the codimension of the cone $C_\pi$ and the simplex $A_\pi$ in $\mathbb{R}_0^n$. That is,*

$$dim(C_\pi) = dim(A_\pi) = n - \#cycles \ of \ \pi.$$

Indeed, in Figure 1.4 the four-cycles are the triangles, the three-cycles and products of two disjoint two-cycles are the edges, the transpositions are vertices, and the identity permutation is the empty face. This was also noted in [7] and can be seen as a corollary of the Chevalley-Shephard-Todd theorem [10].

Notice also in Figure 1.4 that there is a left-right symmetry to the Waldspurger decomposition. This corresponds to 180° rotation of the transformation diagram, or conjugation by the longest element (the permutation sending $k$ to $n-k$ for all $k \in [n]$).

We will see that multiplication by the transposition $(1, n)$ (on both the left and right) also plays a fundamental role in the geometry:

**Theorem 3.2.** *Let $R$ denote reflection through the affine hyperplane orthogonal to the longest positive root, $\theta$, at height one. Then $R$ is an involution on the set of $\Delta_\pi$'s. At*
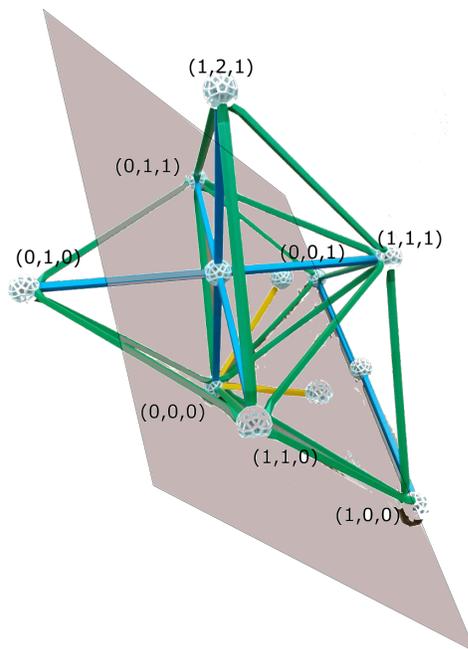
Figure 3.1: The Meinrenken tile for $\mathfrak{S}_3$ and the hyperplane corresponding to reflection $R$ from Theorem 3.2.

*the level of permutations, this involution is multiplication by the transposition $(1, n)$ on the left:*

$$R(\Delta_\pi) = \Delta_{(1,n)\pi}.$$

*In contrast, applying the transposition $(1, n)$ on the right to $\pi$ gives the unique $\psi \in \mathfrak{S}_\mathfrak{n}$ for which $\Delta_\psi$ and $\Delta_\pi$ are simple coroot translates:*

$$\Delta_\pi + c = \Delta_{\pi(1,n)} \quad \text{for some simple coroot c.}$$

*Proof.* Consider how the transformation diagram changes when one applies the transposition $(1, n)$ on the left (respectively right). Only two of the stars in the diagram will move– those on the left and right (respectively top and bottom) of the diagram. The two moving stars will cause $\theta = \underbrace{(1, 1, \ldots, 1)}_{n-1}$ to be subtracted from all columns

(respectively rows) starting and ending with 1's, and to be added to all columns (respectively rows) starting and ending with 0's.

Adding $\theta$'s to columns is the reflection $R$ across the hyperplane orthogonal to $\theta$ at height one. Indeed, consider where $R$ sends column vectors. If we let $P$ denote projection onto $\theta$, then $R$ sends $v \mapsto (id - 2P)v + \theta$. In root coordinates, $2P$ is described by the matrix $\frac{2\theta\theta^T C}{\theta^T C \theta} = JC$ where $J$ is the matrix of all ones and $C$ is the Cartan matrix. One may verify that

$$JC = \begin{bmatrix} 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 1 \end{bmatrix}$$

and thus $R$ sends column vector $v$ to

$$v \mapsto (id - JC)v + \theta = v - (v_1 + v_{n-1})\theta + \theta = \begin{cases} v & \text{if } v_1 + v_{n-1} = 1 \\ v + \theta & \text{if } v_1 + v_{n-1} = 0 \\ v - \theta & \text{if } v_1 + v_{n-1} = -1. \end{cases}$$

In contrast, adding or subtracting $\theta$'s from rows is acting by translation by a co-root on the $\Delta_\pi$'s. Since coroot translation (disjointly) tiles space with the Meinrenken tile and this transformation preserves being a Waldspurger matrix, it must be unique.

$\square$

## 3.2    SIF Permutations and Combinatorial Dimension

There are three types of dimension associated with each $\pi \in \mathfrak{S}_n$ via the Waldspurger transform:

- The **linear dimension** $:= \dim(A_\pi)$.

- The **affine dimension** $:= \dim(\Delta_\pi)$.

- The **combinatorial dimension** $CD(\pi) := \#$ distinct non-zero columns of $\mathbf{WT}(\pi)$.

Proposition 3.1 tells us that the linear dimension of $\pi$ is exactly $n - \#$cycles of $\pi$. This is a well understood phenomenon.

Affine dimension is closely related to linear dimension, but can be at most $n - 2$. If $\mathbf{WT}(\pi)$ has an all zeros column, then by definition $A_\pi = \Delta_\pi$ and the affine dimension and linear dimension of $\pi$ coincide. If, however, $\mathbf{WT}(\pi)$ does not have a zero column then $\dim(\Delta_\pi) = \dim(A_\pi) - 1$. Suppose the all zeros vector is not a column of $\mathbf{WT}(\pi)$. By Theorem 3.2, there is a unique $\psi \in \mathfrak{S}_n$ such that $\Delta_\pi$ and $\Delta_\psi$ are coroot translates and $0 \in \Delta_\psi$. Meinrenken's theorem (Theorem 0.8) implies that $\Delta_\pi \cap \mathfrak{M} = \varnothing$ (even though $\Delta_\pi \cap \overline{\mathfrak{M}} = \Delta_\pi$) and $\Delta_\psi \cap \mathfrak{M} = \Delta_\psi$. One may think of $\mathfrak{M}$ as being topologically "half-open" in $\mathbb{R}_0^n$, and we use this matching to enumerate permutations by their affine dimension.

**Theorem 3.3.** *Let $a(n,k)$ denote the number of $\pi \in \mathfrak{S}_n$ with $\dim(\Delta_\pi) = k$. Then*

$$\sum_{k=0}^{n-1} a(n,k)x^{n-1-k} = 2 \prod_{i=2}^{n-1} (x+i).$$

*Proof.* Let $c(n,k)$ denote the unsigned Stirling numbers of the first kind, i.e. the number of permutations in $\mathfrak{S}_n$ with exactly $k$ disjoint cycles (counting fixed points as cycles). Proposition 1.3.7 of EC1 [37] tells us that

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)(x+2)\dots(x+n-1).$$

We know from Proposition 3.1 that $c(n,k) = \#\{\pi \in \mathfrak{S}_n \mid \dim(A_\pi) = n - k\}$ and because $\dim(A_\pi) \approx \dim(\Delta_\pi)$ it is at least reasonable that the two formulas look

similar. We proceed using the "buddy system." Say that $\pi$ and $\psi$ are *buddies* if $\pi \cdot (1,n) = \psi$, and call $\dim(\Delta_\pi)$ the *affine dimension* of $\pi$. It follows from Proposition 3.1 that if $\pi$ is an $n$-cycle, then $\dim(\Delta_\pi) = n - 1$. By Theorem 3.2, we know that know $\Delta_\pi$ and $\Delta_{\pi \cdot (1,n)}$ are coroot translates thus $\dim(\Delta_{\pi \cdot (1,n)}) = n - 1$ as well. We also know that $\pi \cdot (1,n)$ must have exactly two cycles. This accounts for, $n! = c(n,1)$ of the permutations with exactly two cycles, and we see that there are $2c(n,1)$ permutations of affine dimension $n - 1$.

The remaining $c(n,2) - c(n,1)$ permutations with two disjoint cycles have affine dimension one lower. Their buddies will have exactly three cycles and will again have the same affine dimension.

The remaining $c(n,3) - (c(n-2) - c(n-1))$ permutations with three cycles have affine dimension one lower still, as will their buddies, etc.

This sequence has the generation function

$$2 \sum_{k=1}^{n} \Big( \sum_{i=0}^{k} c(n,i)(-1)^{i+k} \Big) x^{k-1} \tag{3.1}$$

We will prove that this equals $2(x+2)(x+3)\ldots(x+n-1)$ by induction.

$$2 \sum_{k=1}^{n+1} \Big( \sum_{i=1}^{k} c(n+1,i)(-1)^{i+k} \Big) x^{k-1}$$

$$= 2 \sum_{k=1}^{n+1} \sum_{i=1}^{k} \big[ n \cdot c(n,i) + c(n,i-1) \big] (-1)^{i+k} x^{k-1}$$

$$= 2 \sum_{k=1}^{n+1} n \sum_{i=1}^{k} c(n,i)(-1)^{i+k} x^{k-1} + 2 \sum_{k=1}^{n+1} \sum_{i=1}^{k} c(n,i-1)(-1)^{i+k} x^{k-1}$$

$$= 2n(x+2)(x+3)\ldots(x+n-1) + 2x \sum_{k=1}^{n+1} \sum_{i=2}^{k+1} c(n,i-1)(-1)^{i+k+2} x^{k-1}$$

$$= 2n(x+2)(x+3)\ldots(x+n-1) + 2x(x+2)\ldots(x+n-1)$$

$$= 2(x+2)(x+3)\ldots(x+n-1)(x+n).$$

$\square$

We see that linear dimension and affine dimension are closely related and would like to see how combinatorial dimension fits into the picture. For all $\pi \in \mathfrak{S}_n$ it is immediate that $\dim(A_\pi) - 1 \le CD(\pi) \le n - 1$ since $\mathbf{WT}(\pi)$ has at most $n - 1$ distinct non-zero columns and since the linearly independent column vectors are necessarily distinct and non-zero. Unfortunately, it seems that combinatorial dimension is a much more complicated notion and that even enumerating the permutations in $\mathfrak{S}_n$ with combinatorial dimension $n - 1$ is non-trivial.

**Definition 12.** A permutation on $[n] = \{1, 2, \ldots, n\}$ is *stabilized-interval-free* (SIF) if it does not stabilize any proper subinterval of $[n]$.

For example $(3, 6, 5, 4)(1, 7, 2)$ in cycle notation, fails to be SIF because it stabilizes the interval $[3, 6] = \{3, 4, 5, 6\}$. SIF permutations were recently shown by Ardila, Rincón, and Williams to be in bijection with connected positroids [4], a subclass of matroids introduced by Postnikov to study the totally nonnegative grassmannian.

Let $f_n$ denote the number of SIF permutations in $\mathfrak{S}_n$, define a generating function $A(x) = \sum_{n=0}^{\infty} f_n x^n$, and let $[x^n]$ be an operator that extracts the coefficient of $x^n$ from a power series. Callan [13] showed that $[x^{n-1}]A(x)^n = n!$. He also obtained a recurrence for enumerating SIF permutations:

$$f_0 = f_1 = 1, \qquad f_n = (n-1)f_{n-1} \sum_{j=2}^{n-2} (j-1) f_j f_{n-j} \quad \text{for all } n \ge 2.$$

**Theorem 3.4.** *For any permutation $\pi \in \mathfrak{S}_n$,*

$$CD(\pi) = n - 1 \Longleftrightarrow \pi \text{ is SIF.}$$

*Proof.* For notational brevity, let $M_\sigma = \mathbf{WT}(\sigma)$ and suppose that $M_\sigma$ has column $i$ all zeros. Then row $i$ must also be all zeros. Indeed, recall that Waldspurger matrices are unimodal in both row and column, with both row and column maxima on the

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 1 | | | | | | | |
| 3 | 1 | 3 | 2 | | | | | | |
| 4 | 1 | 6 | 10 | 7 | | | | | |
| 5 | 1 | 10 | 30 | 45 | 34 | | | | |
| 6 | 1 | 15 | 70 | 170 | 258 | 206 | | | |
| 7 | 1 | 21 | 140 | 490 | 1134 | 1778 | 1476 | | |
| 8 | 1 | 28 | 252 | 1190 | 3766 | 8764 | 14196 | 12123 | |
| 9 | 1 | 36 | 420 | 2562 | 10458 | 32340 | 76956 | 128241 | 111866 |

Figure 3.2: It is an open problem to enumerate permutations in $\mathfrak{S}_n$ with combinatorial dimension $k$. Note that the trailing sequence $f(n, n-1) = 2, 7, 34, 206, \ldots$ enumerates SIF permutations.

diagonal. If $M_{i,i} = 0$, then both row $i$ and column $i$ must be all zeros. This means that $\sigma$ stabilizes the intervals $[1, i]$ and $[i+1, n]$. Thus, $\sigma$ SIF implies $0 \notin \Delta_\pi$.

Now suppose $M_\sigma$ has identical columns $i$ and $j$. I claim that $\sigma$ stabilizes $[i+1, j]$. Indeed, since $M_{i,i} = M_{i,j}$, all stars above and to the right of $(i, i)$ in the transformation diagram must be above and to the right of $(i, j)$ making the region in between and above a "no ~~man's~~ star's zone." Likewise, if $M_{j,i} = M_{j,j}$, all stars below and to the left of $(j, j)$ in the transformation diagram must be below and to the left of $(j, i)$ making the region in between and below another "no star's zone." We are forced to conclude that $\sigma(i+1), \sigma(i+2), \ldots, \sigma(j) \in [i+1, j]$. The claim follows from the fact that $M_\sigma$ is a permutation matrix. Thus $\sigma$ SIF implies $M_\sigma$ has distinct columns.

The converse now follows easily. If $\sigma$ is not SIF, then either is stabilizes an interval of the form $[1, i]$ (and hence also $[i+1, n]$) forcing $M_\sigma$ to have $i$th column zero, or $\sigma$ stabilizes an interval $[i, j]$ forcing $M_\sigma$ to have columns $i-1$ and $j$ equal. $\square$

**Question 9.** Define $f(n, k) := \#\{\pi \in \mathfrak{S}_n \mid CD(\pi) = k\}$ so that $f(n, n-1) = \#$SIF pemutations $\in \mathfrak{S}_n$. (See Figure 3.2.) One can show that $f(n, 0) = 1$ and $f(n, 1) = \binom{n}{2}$ and $f(n, 2) = 2\binom{n+1}{4}$. Find $f(n, k)$ for general $n$ and $k$.

Notice in Figure 3.2 that for fixed $n$, the numbers $f(n, k)$ appear to be unimodal with maximum $f(n, n-2)$. We are able to show that $f(n, n-2) > f(n, n-1)$ whenever

$n > 2$ and obtain a few more enumerative results by breaking up the set

$$\{\pi \in \mathfrak{S}_n \quad | \quad CD(\pi) = n - 2\}.$$

This set consists of the permutations $\pi$ for which $\mathbf{WT}(\pi)$ has a single zero column, and those for which $\mathbf{WT}(\pi)$ has exactly one repeated column and no zero column. To that end we make the following definition:

**Definition 13.** Call a permutation on $[n]$ h-SIF (*half-stabilized-interval-free*) if it stabilizes exactly one pair of intervals $1, \ldots, k$ and $k + 1, \ldots, n$.

One may alternatively define h-SIF permutations to be the set of $\pi \in \mathfrak{S}_n$ for which $\mathbf{WT}(\pi)$ has distinct columns, one of which is the all zeros vector. While (by Theorem 3.2) the set of SIF permutations equals $\{\pi \in \mathfrak{S}_n \quad | \quad CD(\pi) = n - 1\}$, note that the set of h-SIF permutations is a strict subset of $\{\pi \in \mathfrak{S}_n \quad | \quad CD(\pi) = n - 2\}$. For example, $CD(4321) = 2$, but $\mathbf{WT}(4321)$ does not have a zero column.

**Theorem 3.5.** *The number of h-SIF permutations in $\mathfrak{S}_n$ with affine dimension $n - 2$ is given by the formula*

$$\sum_{k=0}^{n-2} k!(n - 2 - k)!$$

*Proof.* Note that this is less than $2(n - 1)! = \#\{\pi \in \mathfrak{S}_n \quad | \quad \dim(\Delta_\pi) = n - 2\}$. By definition, h-SIF permutations have Waldspurger matrices with all distinct columns exactly one of which is the all zeros vector. Having a zero column forces a Waldspurger matrix to have a zero row and forces the permutation to have at least two cycles. Having more than two cycles would reduce the linear dimension which equals the affine dimension (because of the zero column). Thus, we are counting all permutations with exactly two cycles, the first of which contains $1, \ldots, k$. From here the formula is immediate. $\square$

**Theorem 3.6.** *Let $o_n$ denote the number of h-SIF permutations of $n$ and $s_n$ denote the number of SIF permutations of $n$. Then*

$$o_n = \sum_{k=0}^{n} s_k s_{n-k}.$$

*Proof.* The proof is entirely analogous to that of Theorem 3.5. □

It is immediate form Theorem 3.6, that the number of h-SIF permutations is greater than the number of SIF permutations (whenever n>2) and because

$$\{\pi \in \mathfrak{S}_n \quad | \quad \pi \text{ is h-SIF}\} \subset \{\pi \in \mathfrak{S}_n \quad | \quad CD(\pi) = n - 2\}$$

we may conclude that $f(n, n-2) > f(n, n-1)$. Because of the restrictions on where columns of a Waldspurger matrix may appear, it seems more difficult to enumerate

$$\{\pi \in \mathfrak{S}_n \quad | \quad CD(\pi) = n - 2\} \smallsetminus \{\pi \in \mathfrak{S}_n \quad | \quad \pi \text{ is h-SIF}\}.$$

**Question 10.** In an attempt to answer Question 9, one may consider separately the two statistics, #non-zero columns of $\mathbf{WT}(\pi)$ and # distinct columns of $\mathbf{WT}(\pi)$. The first statistic is equivalent to the number of "components" of the permutation, and has been studied [30]. The second statistic does not appear to have been studied, but appears to have nice unimodality properties. (See Figure 6 in the appendix for some data.) Find an enumerative formula.

**Question 11.** Linear, affine, and combinatorial dimension may be defined for any crystallographic type $\Phi$ using the $\mathbf{WT}_\Phi$. One may define SIF elements abstractly as elements of the Weyl group with maximum combinatorial dimension. Can one find a combinatorial interpretation for these analogs of SIF permutations?
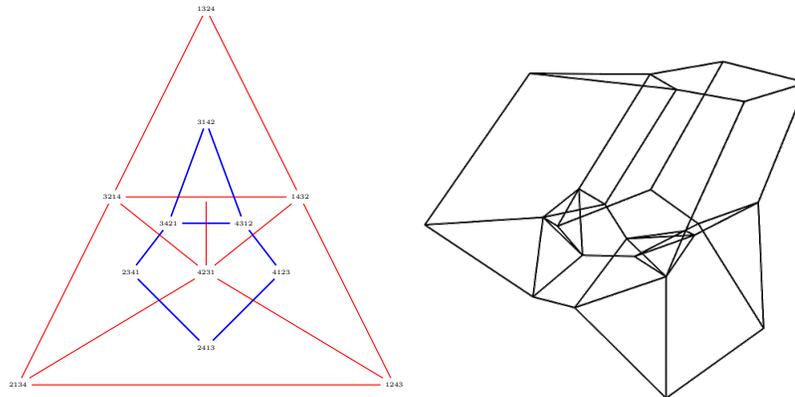
Figure 3.3: the dual graph on $n$-cycles for $\mathfrak{S}_3$ and $\mathfrak{S}_4$

## 3.3   A Dual Graph on $n$-cycles

**Theorem 3.7.** *(Bibikov and Zhgoon 2009 [7]) Given $(W, S)$ a finite Coxeter system, two cones $C_\pi$ and $C_\sigma$ of the same dimension in the Waldspurger decomposition share a codimension one boundary iff $\pi s_i = \sigma s_j$ for some pair of simple generators $s_i, s_j \in S$.*

Restricting our attention to type A, this gives rise to a graph structure $G(n)$ on $n$-cycles of $\mathfrak{S}_n$. Two $n$-cycles $c_1$ and $c_2$ are adjacent iff there exist adjacent transpositions $s_i$ and $s_j$ such that $c_1 s_i s_j = c_2$. This section is dedicated to the study of this graph. The following lemma will establish bounds on degrees of vertices in $G(n)$.

**Lemma 3.** *Let $c_1$ be an $n$-cycle and $s_i$ be the transposition switching $i$ and $i + 1$. Either $c_1 s_i s_{i+1}$ is an $n$-cycle, or $c_1 s_{i+1} s_i$ is an $n$-cycle. Not both.*

*Proof.* Suppose your $n$-cycle is

$$(i, a_1, ..., a_k, i + 1, b_1, ..., b_j, i + 2, c_1, ..., c_l).$$

Multiplying by $(i, i + 1)(i + 1, i + 2) = (i, i + 1, i + 2)$ on the right gives the $n$-cycle

$$(i, b_1, ..., b_j, i + 2, a_1, ..., a_k, i + 1, c_1, ..., c_l).$$

but multiplying by $(i+1, i+2)(i, i+1) = (i, i+2, i+1)$ on the right gives

$$(i, c_1, ..., c_l)(i+1, a_1, ..., a_k)(i+2, b_1, ..., b_j).$$

On the other hand, suppose your $n$-cycle is

$$(i, a_1, ..., a_k, i+2, b_1, ..., b_j, i+1, c_1, ..., c_l).$$

Multiplying by $(i, i+1)(i+1, i+2) = (i, i+1, i+2)$ on the right gives

$$(i, c_1, ..., c_l)(i+1, b_1, ..., b_j)(i+2, a_1, ..., a_k).$$

but multiplying by $(i+1, i+2)(i, i+1) = (i, i+2, i+1)$ on the right gives the $n$-cycle

$$(i, c_1, ..., c_l, i+1, b_1, ..., b_j, i+2, a_1, ..., a_k).$$

$\square$

**Corollary.** *Vertices in $G(n)$ must have at least $n-2$ neighbors and may have at most $\binom{n-1}{2}$ neighbors.*

Observe that if $n$ is odd, the $n$-cycle $(1, 3, 5, \ldots, n, 2, 4, 6, \ldots, n-1)$ and its inverse both obtain the $\binom{n-1}{2}$ maximum number of neighbors. The same is true for $(1, 3, 5, \ldots, n-1, 2, 4, 6, \ldots, n)$ and its inverse when $n$ is even.

**Lemma 4.** *If $j > i+1$, then there are two classes of $n$-cycles related by $(i, i+1)(j, j+1)$*

$$(i, a_1, \ldots, a_k, j, b_1, \ldots, b_l, i+1, c_1, \ldots, c_m, j+1, d_1, \ldots, d_n) \tag{3.2}$$

*and*

$$(i, c_1, \ldots, c_m, j+1, b_1, \ldots, b_l, i+1, a_1, \ldots, a_k, j, d_1, \ldots, d_n). \tag{3.3}$$

*Proof.* The proof is a straightforward exercise in multiplying permutations, much like the proof of Lemma 3 and we omit it. □

This lemma implies that there are two types of edges in our Waldspurger dual graph; those coming from "adjacent adjacent transpositions" $s_i s_{i+1}$ and those coming from "non-adjacent adjacent transpositions" $s_i s_j$ where $j > i + 1$. It is helpful to study them separately. To that end we make the following definitions:

**Definition 14.** Let $G_{adj}(n)$ be a graph on $n$-cycles where $c_1$ and $c_2$ share an edge iff there exists an $i \in [n-2]$ such that $c_1 s_i s_{i+1} = c_2$

**Definition 15.** Let $G_{nadj}(n)$ be a graph on $n$-cycles where $c_1$ and $c_2$ share an edge iff there exists $i, j \in [n-2]$ with $j > i + 1$ such that $c_1 s_i s_j = c_2$.

Lemma 3 implies that $G_{adj}(n)$ is a regular graph. We will use this fact to prove the following:

**Theorem 3.8.**
$$\#E(G(n)) = \frac{(n+3)(n-2)(n-1)!}{12}$$

*Proof.* There are two types of edges in $G(n)$, edges from $G_{adj}$ and edges from $G_{nadj}$. In $G_{adj}$ there are $(n-1)!$ vertices each with degree $n-2$ for a total of

$$\frac{(n-2)(n-1)!}{2} \text{edges in } G_{adj}(n).$$

In $G_{nadj}$ vertices of forms

$$(\mathbf{i}, a_1, \ldots, a_k, \mathbf{j}, b_1, \ldots, b_l, \mathbf{i+1}, c_1, \ldots, c_m, \mathbf{j+1}, d_1, \ldots, d_n) \tag{3.4}$$

and

$$(\mathbf{i}, c_1, \ldots, c_m, \mathbf{j+1}, b_1, \ldots, b_l, \mathbf{i+1}, a_1, \ldots, a_k, \mathbf{j}, d_1, \ldots, d_n). \tag{3.5}$$

are adjacent via the edge labeled $s_i s_j$. These are the only edges in $G_{nadj}$. To count such edges we may pick the pair $s_i s_j$ in $\binom{n-2}{2}$ ways, order the numbers of $[n]-\{i, i+1, j, j+1\}$ in $(n-4)!$ ways, and break up that ordering into $a$'s, $b$'s, $c$'s, and $d$'s in $\binom{n-1}{3}$ ways. That is, there are

$$\binom{n-2}{2}\binom{n-1}{3}(n-4)! = \frac{(n-3)(n-2)(n-1)!}{12} \text{ edges in } G_{nadj}(n).$$

Adding the edges from $G_{adj}$ and $G_{nadj}$ we see that $G(n)$ has

$$\frac{6(n-2)(n-1)!}{2} + \frac{(n-3)(n-2)(n-1)!}{12} = \frac{(n+3)(n-2)(n-1)!}{12} \text{ edges.}$$

□

*Proof.* A bijective proof:

Consider tableaux of shape $(3, 1^{n-2})$. It is a result of Campbell [31] that there are $\frac{(n+3)(n-2)(n-1)!}{12}$ ways to fill this shape using each element of $[n-2]$ once, and infinities twice with the row decreasing and containing at most one infinity. We will biject these tableaux with pairs of $n$-cycles and the $s_i s_j$ connecting them in $G$.

**Case 1:**

There is an infinity in the top row. This signifies that $j = i + 1$. The placement of the second infinity determines what $i$ is. In the example below since the second infinity is in the fourth block of size one, $i = 4$. By Lemma 3, we know that

$$(i, a_1, ..., a_k, i+1, b_1, ..., b_j, i+2, c_1, ..., c_l)s_i s_{i+1} = (i, b_1, ..., b_j, i+2, a_1, ..., a_k, i+1, c_1, ..., c_l)$$

thus all that is left is to construct our sequence of $a$'s, $b$'s and $c$'s. The integer entries in the first column form a permutation in one line notation. Write it in the letters of the alphabet $[n] - \{i, i+1, i+2\}$ In the example below, 3214 written in the alphabet

$\{1237\}$ is 3217. We now use the numbers in the top row to break this word up into $a$'s, $b$'s, and $c$'s. The last block tells us the number of $a$'s plus one and the second block minus the last block tells us the number of $b$'s plus one. In the example, the 5 tells us there are four $a$'s and the 6 tells us there are no $b$'s (which there better not be since we used all four of our letters already). Any numbers remaining from our word will become $c$'s.

| $\infty$ | 6 | 5 |
|---|---|---|
| 3 | | |
| 2 | | |
| 1 | | |
| $\infty$ | | |
| 4 | | |

$\longmapsto$

$S_4 S_5$

$(\mathbf{4}321\mathbf{756})$

$(\mathbf{46}32\mathbf{175})$

**Case 2:**

The two infinities are in the $i$th and $j$th rows of size one. Then our $n$-cycles will be related by $s_i s_{j+1}$. From Lemma 4 we know that

$$(\mathbf{i}, a_1, \ldots, a_k, \mathbf{j}, b_1, \ldots, b_l, \mathbf{i+1}, c_1, \ldots, c_m, \mathbf{j+1}, d_1, \ldots, d_n)s_i s_j =$$
$$(\mathbf{i}, c_1, \ldots, c_m, \mathbf{j+1}, b_1, \ldots, b_l, \mathbf{i+1}, a_1, \ldots, a_k, \mathbf{j}, d_1, \ldots, d_n).$$

As in case one, we use the integer entries of the first column to make a word in the alphabet $[n] - \{i, i+1, j, j+1\}$ and the entries in the first column to decide how to break up the word into $a$'s $b$'s $c$'s and $d$'s. In the example we have $S_2 S_5$ coming from the positions of the infinities. We rewrite the permutation 163 in the alphabet $\{147\}$ to get 174 and break it using the first row. There is $2 - 1 = 1$ $a$ (our 1), $4 - 2 - 1 = 1$ $b$ (our 7), and no $c$'s. This leaves one $d$ (our 4).

| 5 | 4 | 2 |
|---|---|---|
| 1 | | |
| $\infty$ | | |
| 6 | | |
| $\infty$ | | |
| 3 | | |

$\longmapsto$

$S_2 S_5$

$(\mathbf{2}157\mathbf{364})$

$(\mathbf{267}3\mathbf{154})$

In both cases, the algorithm is reversible. Given a pair of adjacent transpositions we know how to place our infinities. We can count our number of $a$'s, $b$'s, $c$'s (and $d$'s if we are in case two) and use them to determine the entries of our top row. We can then concatenate the $a$'s, $b$'s, $c$'s (and $d$'s) to make a permutation word which we can write in the alphabet of whatever letters we have left to place in our tableau and we are done. $\square$

The following table gives the number of vertices of given degree for $G(n)$:

| $n\backslash deg$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 2 | | | | | | | | | | | | | | |
| 4 | | 4 | 2 | | | | | | | | | | | | |
| 5 | | | 8 | 10 | 4 | 2 | | | | | | | | | |
| 6 | | | | 16 | 34 | 30 | 24 | 8 | 6 | 2 | | | | | |
| 7 | | | | | 32 | 98 | 138 | 158 | 106 | 80 | 58 | 28 | 16 | 4 | 2 |

Since $G_{adj}(n)$ is regular, it suffices to study the degree sequence for $G_{nadj}(n)$:

| $n\backslash deg$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 2 | | | | | | | | | | | | | | | |
| 4 | 4 | 2 | | | | | | | | | | | | | | |
| 5 | 8 | 10 | 4 | 2 | | | | | | | | | | | | |
| 6 | 16 | 34 | 30 | 24 | 8 | 6 | 2 | | | | | | | | | |
| 7 | 32 | 98 | 138 | 158 | 106 | 80 | 58 | 28 | 16 | 4 | 2 | | | | | |
| 8 | 64 | 258 | 504 | 764 | 774 | 692 | 632 | 492 | 380 | 220 | 122 | 74 | 46 | 10 | 6 | 2 |

For notational convenience, define $B_{n,k} := \{v \in G_{nadj}(n) \mid deg(v) = k\}$

**Theorem 3.9.**

$$\#B_{n,0} = 2^{n-2}$$

*Proof.* If $n$ and $n-1$ are not to be entangled then they must appear next to each other in our $n$-cycle. If not, we would have a partition of $[n-2]$ into two blocks, one of which would have to contain some $i < n-2$ with $i+1$ in the other block, meaning that our $n$-cycle was entangled via $s_i s_{n-1}$. Now view $n-1$ and $n$ as a single block and note that in a similar way, $n-2$ must be appear next to this block. Continuing in this way, we have one choice at each step: do we place $i$ on the left or right of $n$? Once we place 1 we close our cycle. Notice that our placement of 1 and $n$ was forced, meaning we made $2^{n-2}$ choices. $\qquad\square$

**Theorem 3.10.**

$$\#B_{n,1} = (n-4)2^{n-2} + 2$$

*Proof.* Induction on $n$: When $n = 4$ there are two such vertices and when $n = 5$ there are 10 such vertices (as seen in the table). Suppose for induction that $\#B_{n-1,1} = (n-5)2^{n-3} + 2$ For the remainder of the proof we will consider our permutations written in cycle notation starting with a one.

There is a two natural injective maps $\hat{\phi} : B_{n-1,1} \longrightarrow B_{n,1}$ and $\tilde{\phi} : B_{n-1,1} \longrightarrow B_{n,1}$ that inserts an $n$ directly to the left (respectively right) of $n-1$. That is to say, for any given

$$v = (1, a_1, \ldots, a_k, n-1, b_1, \ldots, b_{n-k-2}) \in B_{n-1,1}$$

I claim that both

$$\hat{v} := (1, a_1, \ldots, a_k, n-1, n, b_1, \ldots, b_{n-k-2}) \in B_{n,1}$$

$$\text{and } \tilde{v} := (1, a_1, \ldots, a_k, n, n-1, b_1, \ldots, b_{n-k-2}) \in B_{n,1}$$

Indeed, $v \in B_{n-1,1}$ means there exists a unique pair $s_i s_j$ $j > i+1$ so that $i, i+1, j, j+1$ appear in an entangled order as in Lemma 4.

No entanglements will be created or destroyed by inserting $n$ into $v$ to make $\hat{v}$ or $\tilde{v}$.

Thus, there are $2((n-5)2^{n-2} + 2)$ vertices in $B_{n,1}$ such that $n-1$ and $n$ are next to each other.

We must now figure out what other $n$-cycles $v$ are in $B_{n,1}$. Since $v$ will be of the form

$$(n, \underline{\qquad}, n-1, \underline{\qquad})$$

we must consider how to place the remaining $n-2$ elements in the two blanks so that we only have one entanglement. We must have the following properties:

1. Only one pair $i, i+1$ can be in separate blanks.

2. $1, 2, \ldots, i-1$ must be in the same blank as $i$ and $i+2, i+3, \ldots, n-2$ must be in the same blank as $i+1$

3. Entries in the blank containing one must be "anti-unimodal" (decreasing and then increasing).

The first two properties are obviously necessary to avoid extra entanglements. The third property follows from Theorem 3.9.

Consider how we may insert $n-2$ into the pattern above. It must be adjacent to $n-1$. (Otherwise $1, 2, \ldots, n-3$ must be between $n-2$ and $n-1$ meaning the other blank is empty.) Thus $n-2$ can be placed in two ways. $n-3$ can either be placed next to $n-2$ (sandwiching $n-2$ between $n-1$ and $n-3$) or in the opposite blank from $n-2$. Once we place a number in the second blank, by property 1 there is no going back. If we do want to stay in the first blank, by the same argument as above, $i$ must always be placed next to $i+1$. Once we place a number in the second blank,

every number after it can be placed either to the right or to the left of where one will be. Thus there are $2^{n-2} - 2$ ways of placing $1, 2, \ldots, n - 2$ since we cannot place all of them in the first blank or all in the second, but otherwise we a binary choice to make each time we place a number. This gives us the desired number of vertices:

$$2^{n-2} - 2 + 2((n - 5)2^{n-2} + 2) = (n - 4)2^{n-2} + 2$$

$\square$

There seems little hope for enumerating $B_{n,k}$ in general. For $k$ not equal to zero or one, the entanglements become quite complected. There is one more interesting observation about $G_{nadj}$, however which seems worth noting.

**Definition 16.** Define the $n$th **Pell number**, $P(n)$ defined recursively from $P(0) = 0, P(1) = 1$ and for $n > 1$,

$$P(n) = 2P(n - 1) + P(n - 2).$$

**Conjecture 1.** *Let $G_{nadj}$ be the graph on $n$-cycles where the $n$-cycles $c_1$ and $c_2$ share and edge iff there exist adjacent transpositions $s_i$ and $s_j$ for $j \notin \{i - 1, i, i + 1\}$ such that $c_1 s_i = c_2 s_j$. Then $G_{nadj}$ has $P(n)$ connected components, where $P(n)$ is the $n$th Pell number.*

Recall that Theorem 3.3 is quite general, holding for any finite Coxeter system (not necessarily even crystallographic!) It turns out that $n$-cycles are to $\mathfrak{S}_n$ as "Coxeter elements" are to Coxeter system $(W, S)$. A **Coxeter element** is a product of all simple reflections. The product depends on the order in which they are taken, but different orderings produce conjugate elements, which have the same (group theoretic) order in $W$.

**Theorem 3.11.** *(Bibikov and Zhgoon 2009 [7]) Given a finite real reflection group $W$ with a set of simple generators $S$, the top dimensional cones $C_{w_1}$ and $C_{w_2}$ share a codimension one boundary in the Waldspurger decomposition iff $w_1 s_i = w_2 s_j$ for some $s_i, s_j \in S$.*

One may then, define a Waldspurger dual graph with respect to any $(W, S)$.

**Definition 17.** Given a Coxeter system $(W, S)$ define three graphs with vertex set, the Coxeter elements.

1. $G((W, S))$ where two Coxeter elements share an edge iff $w_1 s_i = w_2 s_j$ for any $s_i, s_j \in S$ with $i \neq j$

2. $G_{adj}((W, S))$ where two Coxeter elements share an edge iff $w_1 s_i = w_2 s_j$ for any $s_i, s_j \in S$ with $i \neq j$ and $s_i s_j = s_j s_i$

3. $G_{nadj}((W, S))$ where two Coxeter elements share an edge iff $w_1 s_i = w_2 s_j$ for any $s_i, s_j \in S$ with $i \neq j$ and $s_i s_j \neq s_j s_i$

Computer evidence supports the following two conjectures:

**Conjecture 2.** *The number of connected components in the graph $G_{nadj}(B_n)$ is $4^n - n2^n$ and satisfies the recurrence*

$$a(n) = 4a(n-1) + 2a(n-2).$$

**Conjecture 3.** *Let $P(n)$ denote the nth Pell number. Then the number of connected components in the graph $G_{nadj}(D_n)$ is*

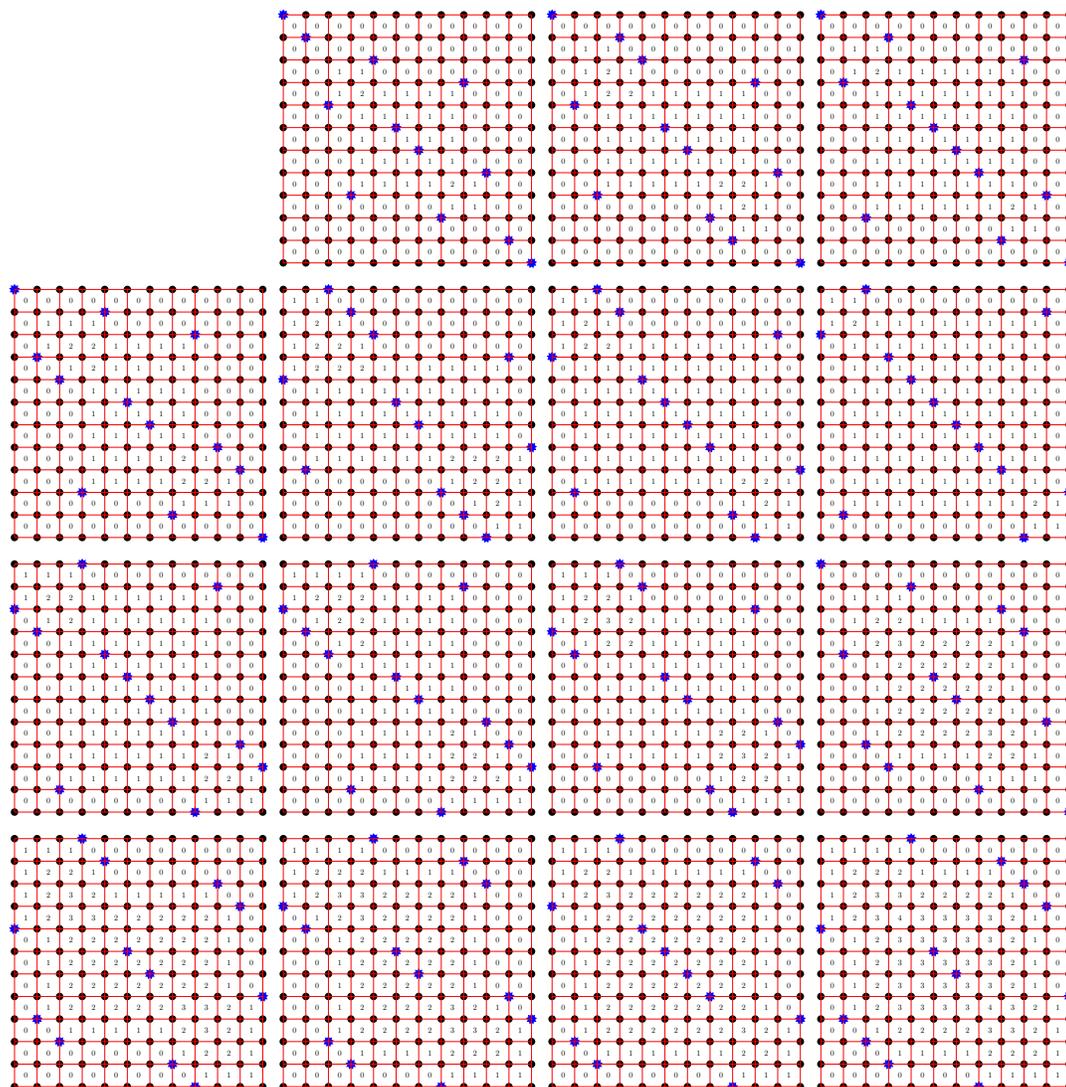$$n P(n) \sum_{d|n} \frac{1}{d P(d)}.$$

# Appendices

Figure 4: There are $\binom{6}{4} = 15$ Bigrassmannian elements in the Weyl group $B_6$ which are not join-irreducible in the Dedekind-MacNeille completion of Bruhat order. Here are the transformation diagrams for the corresponding centrally symmetric permutations in $\mathfrak{CS}_{12}$.
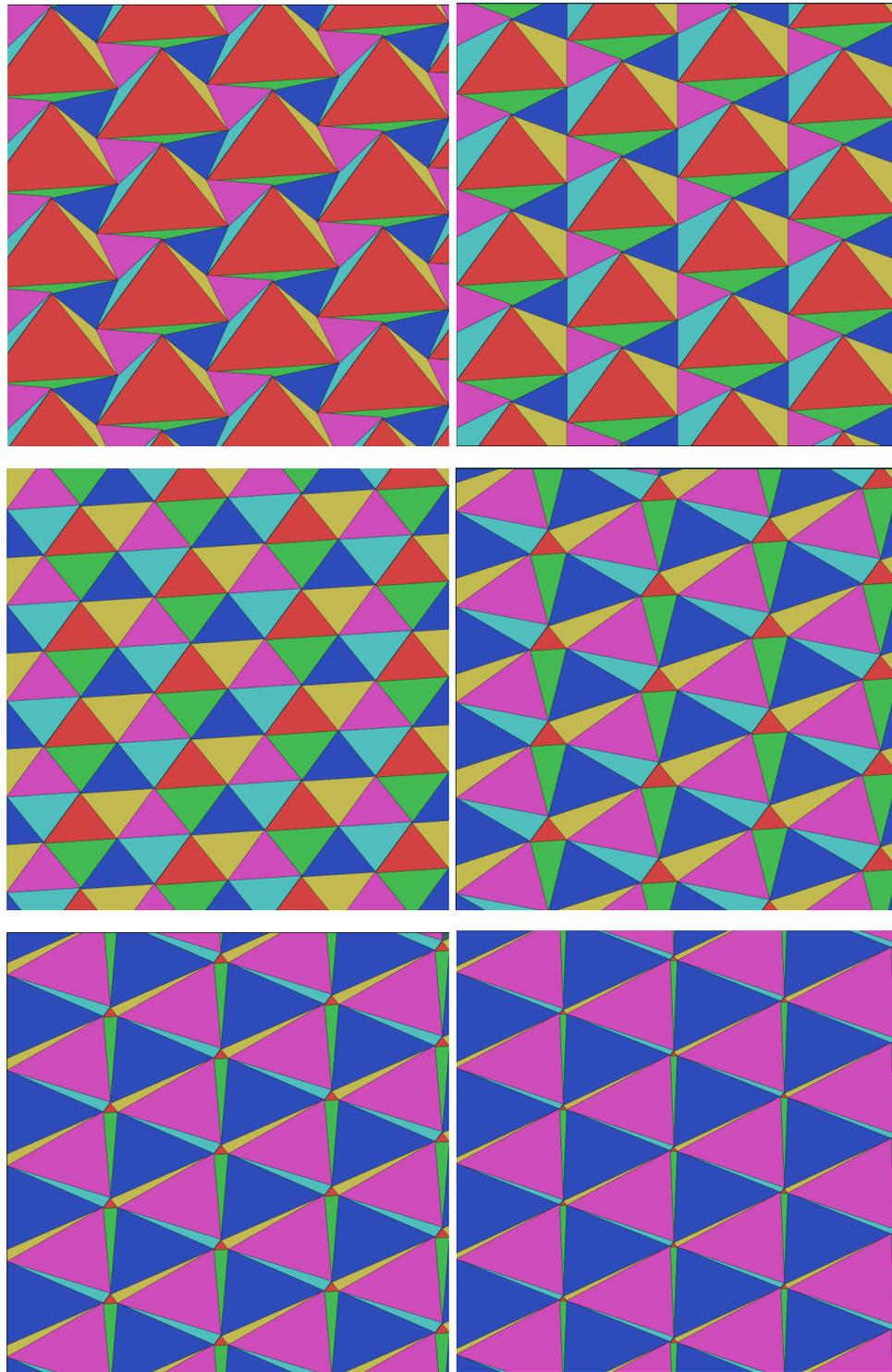
Figure 5: Each triangle in each figure is $(S - w)A$ for $A$ the fundamental alcove and $w$ some element of the affine symmetric group, $\widetilde{\mathfrak{S}}_3$. The six figures correspond to six different $S$, each a scalar times the identity. Reading left to right and top to bottom, the six scalar values are $-0.8, -0.5, 0, 0.5, 0.8, 0.9$.

| $n\backslash k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 2 | 2 | | | | | | |
| 3 | 2 | 4 | | | | | |
| 4 | 2 | 10 | 12 | | | | |
| 5 | 2 | 18 | 48 | 52 | | | |
| 6 | 2 | 28 | 124 | 274 | 292 | | |
| 7 | 2 | 40 | 260 | 890 | 1864 | 1984 | |
| 8 | 2 | 54 | 480 | 2280 | 7116 | 14716 | 15672 |

Figure 6: It is an open problem to count the $\pi \in \mathfrak{S}_n$ with for which $\mathbf{WT}(\pi)$ has $k$ distinct columns.

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