Research Statement

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Abstract

My main research interest is **algebraic combinatorics**. I am particularly interested in the combinatorics of **discrete reflection groups** and how it relates to **representation theory**, **partial orders**, **hyperplane arrangements**, and **matroids**.

In 2005 J.L. Waldspurger [18] proved a remarkable statement: given any finite reflection group G, the closed cone over the positive roots is tiled by images of the open weight cone under the action of $(id - g)$ for $g \in G$. A few years later B. Kostant told E. Meinrenken about this surprising result, and Meinrenken was able to prove an analogous version for affine Weyl groups [12]. P.V. Bibikov, and V.S. Zhgoon, after discussing these results with E.B. Vinberg and L. Popov, gave a uniform proof of the analogous theorem for cocompact hyperbolic reflection groups [5]. All of these theorems exhibit tilings with regions indexed by group elements. These tilings are reminiscent of the identification of regions in a Coxeter hyperplane arrangement with group elements, but with less symmetry. For some tilings, the symmetries that remains capture statistics on the group elements.

The combinatorial implications of these theorems are still being unpacked, but in [11] I show that they have connections with alternating sign matrices, abelian ideals in semisimple lie algebras, integer points inside of polytopes, and minimal lattice completions of posets. These connections provide several avenues for future research.

The connection between Waldspurger tilings and alternating sign matrices (ASMs) is highly geometric, and has potential to provide new insight towards long coveted explicit bijection between ASMs and TSSCPPs or DPPs. The lattice of ASMs (partial ordered via monotone triangles) is the smallest lattice containing the Bruhat order on permutations as a subposet. The lattice completions of Bruhat order for other Coxeter groups have been studied by Lascoux and Schützenberger[10], Geck and Kim [8], and Reading [16], but outside of type A no explicit combinatorial description of the elements is known, nor are counting formulas for the number of elements. The combinatorics of the Waldspurger decomposition, being defined for all types, has potential to lend insight here as well. These efforts seem highly related to some recent work of David Anderson [1] which show potential connections to rank conditions defining Schubert varieties. Finally, the combinatorics in the affine cases appears to be related to the theories of arithmetic matroids and toric arrangements, studied by Moci [14], Concini and Procesi [6], Erhenbord, Readdy, Slone [7] and others. There is also a geometric picture of stabilized-interval-free permutations which may lend insight into the theory of connected positroids, studied in [3].

1 The combinatorics of the Waldspurger decomposition

Gian-Carlo Rota, the grandfather of algebraic combinatorics, once described the subject as, "linear algebra, but with bases." In the context of the following theorem, the simple roots are a natural basis to choose. My motivation in [11] was to make the combinatorics explicit.

Theorem 1 *J.L.Waldspurger, 2005 [18] Let G be a Weyl group presented as a reflection group acting on a Euclidean vector space V. Let $C_\omega \subset V$ be the open cone over the fundamental weights and $C_R \subset V$ the* *closed cone spanned by the positive roots. Let the cone associated with group element g be $C_g := (I - g)C_\omega$ (where I is the identity element in G). One has the decomposition*

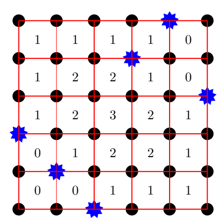
$$C_R = \bigsqcup_{g \in G} C_g.$$

In type A, where the Weyl group is the symmetric group given with the reflection representation, this decomposition identifies each permutation $\pi \in \mathfrak{S}_n$ with a simplicial cone, $C_\pi \subset \mathbb{R}^{n-1}$.

I defined the *Waldspurger Transform* of a permutation $\pi \in \mathfrak{S}_n$ to be the $(n-1) \times (n-1)$ matrix

$$\mathbf{WT}(\pi)_{i,j} := \begin{cases} \sum\limits_{\substack{a \leq i \\ b > j}} \pi_{a,b} & i \leq j \\ \sum\limits_{\substack{a > i \\ b \leq j}} \pi_{a,b} & i \geq j \end{cases}.$$

The non-trivial theorem is that C_π, the simplicial cone associated with the permutation π in Waldspurger's theorem, is equal to the cone over the columns of $\mathbf{WT}(\pi)$ as vectors in simple root coordinates. So these matrices capture all of Waldspurger's geometry.



On the other hand, Waldspurger matrices are very combinatorial objects. The transformation diagram above shows how entries of $\mathbf{WT}(456213)$ are obtained from the permutation matrix. The boxes above the main diagonal count the stars up-and-to-their-right, while the boxes below the main diagonal count the stars down-and-to-their-left. Because permutation matrices are *line-sum-symmetric*, meaning they have their ith row and ith column sums equal, it turns out that the boxes on the diagonal count the stars either way: up-and-to-their-right or down-and-to-their-left. It is not difficult to see that for any permutation π, its transform $\mathbf{WT}(\pi)$ will be forced to have integer entries and unimodal rows and columns. Row and column maximums will be on the diagonal, and each row and column must start and end with a 0 or a 1. If one is given a matrix M satisfying these conditions, must M be the Waldspurger transform of some permutation? No, take for example, the 2×2 identity matrix. How many $n \times n$ matrices satisfy these conditions? An infamous number: $1, 1, 2, 7, 42, 429, 7436, \ldots$

2 Alternating sign matrices

If one sums all of the entries in $\mathbf{WT}(\pi)$ a statistic known as the entropy, or the variance of π is recovered. Entropy is known to give the rank of a permutation in the lattice of alternating sign matrices.

Alternating sign matrices (ASMs) are square matrices with entries in $\{-1, 0, 1\}$ having row and columns sums all equal to one, and non zero entries in each row and column alternate in sign. They are a generalization of permutation matrices that arise naturally when using Dodgson condensation to compute determinants. They were first considered by Mills, Robbins, and Rumsey in 1982 and they observed that they seemed to be counted by the product formula

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}. \tag{1}$$

The proof of this fact eluded combinatorialists for over a decade and was finally put to rest by Zeilberger[19]. A shorter proof was soon given by Kuperberg using an analysis of the six-vertex state model (also called square ice) based on the Yang-Baxter equation [9].

The appearance of the formula 1 actually predates the definition of ASMs and was known to count two other combinatorial objects, namely descending plane partitions (DPPs) and totally symmetric self complementary plane partitions (TSSCPPs). Jessica Striker has exhibited a nice bijection between permutations (as a subset of ASMs) and DPPs with no "special parts" in [17], but explicit bijections between ASMs, DPPs, and TSSCPPs are still not known. Even more suggestive that such bijections "aught to exist," Behrend, Di Francesco and Zinn-Justin [4] have used determinantal formulas to show that the sizes of $\{A \in \mathrm{ASM}(n) | \nu(A) = p, \mu(A) = m, \rho(A) = k\}$ and $\{D \in \mathrm{DPP}(n) | \nu(D) = p, \mu(D) = m, \rho(D) = k\}$ are equal for any n, p, m and k, where ν, μ, and ρ are certain statistics.

Because ASMs, like permutations, are line-sum-symmetric, one may consider their Waldspurger transforms, and get exactly the matrices we observed at the end of the previous section. Each ASM then corresponds to a simplicial cone. What is this geometry like? Does it share a group of symmetries with the DPPs or TSSCPPs?

3 Lattice completions

Lascoux and Schützenberger showed that a natural order on ASMs (via monotone triangles) gives a combinatorial description of the elements of the Dedekind-Macneille completion of Bruhat order– the smallest lattice containing Bruhat order as a subposet [10]. This lattice is distributive and its join-irreducible elements are the bigrassmannian permutations– those with exactly one right descent and one left descent. They are enumerated by the tetrahedral numbers. The Waldspurger transform of ASMs tells this same story quite well: Componentwise comparison of Waldspurger matrices is the same as the ordering from monotone triangles, and bigrasmannian permutations have Waldspurger matrices where one entry is fixed, and the rest are as small as possible, so as to abide by the conditions at the end of section one. One can see Waldspurger matrices as stable configurations of oranges in a tetrahedral orange basket with an edge as its bottom. Bigrassmannian permutations are the configurations of oranges where only one may be removed without causing a tumble.

In type B, Lascoux and Schützenberger showed that the Dedekind-Macneille completion of Bruhat order is again distributive, but here the join-irreducibles are a strict subset of the bigrassmannian elements. They show that the join-irreducibles are enumerated by the octahedral numbers, and give an opaque description of these elements, which is expounded upon by Geck and Kim in [8]. Reading [16], and most recently Anderson [1] have also given combinatorial descriptions of the join-irreducibles. Despite this, there is still no known (or even conjectured) formula for the number of elements of MacNeille completion of Bruhat order for type B. In [11] I defined Waldspurger matrices of type B and showed that their rows and columns must satisfy certain conditions, similar to those for type A. Does the set of all matrices meeting these minimal conditions form a lattice? If so, is it the MacNeille completion of Bruhat order for type B? How many elements does it have? In a recent paper [1] David Anderson defined the essential set of signed permutations. He showed that signed permutations with singleton essential sets are the join-irreducible elements of type B Bruhat order. He also showed that the essential set provides a minimal list of rank conditions defining the Schubert variety or degeneracy locus corresponding to a signed permutation.

Analogous questions to those for type B may be asked for type D, but here the situation is more bleak because Bruhat order lacks the dissective property, meaning that its MacNeille completion will not be distributive.

4 Controlled collapses, toric arrangements, and arithmetic matroids

Meinrenken has an interesting theorem, similar in nature to Waldspurger's original result: For W^a an affine Weyl group, given an endomorphism S from the set $\{S \in \mathrm{End}(V) \mid \det(S - w) \neq 0 \ \forall w \in W\}$, and $V_w^{(S)} := (S - w)A$ where A is the fundamental alcove, the $V_w^{(S)}$ are all disjoint and their closures cover the entire vector space V. When $S \to id$, one recovers the affine version of Waldspurger's original result. On the other hand, taking $S = 0$ gives the toric arrangements defined by root systems, whose combinatorics were studied in [14] and [2].

Toric arrangements, essentially finite families of codimension 1 subtori of a torus, were introduced in 2005 by C. De Concini and C. Procesi as a periodic generalization of hyperplane arrangements [6]. They were able to compute cohomology of the complement of such an arrangement, and defined the characteristic polynomial and the Poincaré polynomial of a toric arrangement. A few years later, Luca Moci introduced a Tutte polynomial for toric arrangements and zonotopes, and showed that specializations gave back the characteristic and Poincaré polynomials (as in the classical case) [13]. Michele D'Adderio and Luca then introduced arithmetic matroids– matroids equipped with the extra structure of a certain multiplicity function. This extra structure allowed them to soup up a stronger invariant– the Arithmetic Tutte polynomial.

For any endomorphism S in the set $\{S \in \mathrm{End}(V) \mid \det(S - w) \neq 0 \ \forall w \in W\}$ each $V_w^{(S)}$ will have full dimension (though its volume will depend on S and w). The complement of the $V_w^{(S)}$'s will be a toric arrangement with the same intersection poset as when $S = 0$, the case studied in [14] and [2]. On the other hand, if one considers an endomorphism T in the *boundary* of $\{S \in \mathrm{End}(V) \mid \det(S - w) \neq 0 \ \forall w \in W\}$ then some of the $V_w^{(S)}$'s will collapse and we will have a new toric arrangement. $T = id$ corresponds to the empty arrangement, and so the complement will have the homology of the torus, but what about other points on the boundary, for example, $T = -id$? Is there an explicit way of taking an endomorphism and associating to it the arithmetic Tutte polynomial of the corresponding toric arrangement?

5 Positroids

One can show that the Waldspurger transform of a permutation will have distinct nonzero columns if and only if the permutation stabilizes no proper subinterval. Such permutations are called stabilized-interval-free, or SIF. In [3], Ardila, Rincón, and Williams showed that SIF permutations index connected positroids. Positroids are matroids which are representable by a point in the Grassmannian with all positive Plücker coordinates and were introduced by Postnikov in [15]. A positroid is connected iff it is connected as a matroid i.e. it cannot be written as a direct sum of matroids. The geometry of the Waldspurger decomposition tells us that SIF permutations are facets on the half open side of a fundamental domain for the action of the coroot lattice. Does this geometry describe a relationship between connected positroids? That is, if two connected positroids correspond to SIF permutations which are "close" in the Waldspurger decomposition, is there a sense in which they are "close" as positroids?

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