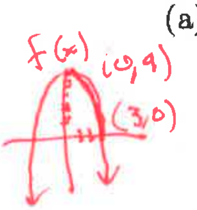
1. (1 point) Finish the statement of the inverse function theorem. If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

2. (5 points) Applying the inverse function theorem:

   (a) (1 point) Suppose that $f(x) = 9 - x^2$ on the restricted domain $0 \leq x \leq 3$. Write an expression for its compositional inverse, f^{-1} and specify its domain.



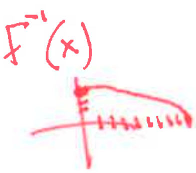
$x = 9 - y^2$

$y^2 = 9 - x$

$y = \pm\sqrt{9-x}$

$f^{-1}(x) = \sqrt{9-x}$ domain: $0 \leq x \leq 9$

   (b) (1 point) Write an expression for the derivative of this inverse, $(f^{-1})'$.



$$(f^{-1})'(x) = \tfrac{1}{2}(9-x)^{-1/2}(-1) = \frac{-1}{2\sqrt{9-x}}$$

   (c) (1 point) What is $(f^{-1})'(8)$?

$$\frac{-1}{2\sqrt{9-8}} = \frac{-1}{2\cdot 1} = -\frac{1}{2}$$

   (d) (2 points) Use the inverse function theorem to find $(f^{-1})'(8)$.

$f'(x) = -2x$ $f^{-1}(8) = \sqrt{9-8} = 1$

$$(f^{-1})'(8) = \frac{1}{f'(f^{-1}(8))} = \frac{1}{-2(f^{-1}(8))} = \frac{1}{-2\cdot 1} = -1/2$$

3. (2 points) Differentiate $y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$ (hint: don't use quotient rule and don't over-simplify... use ln.)

$$\ln(y) = \ln\left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right) = \frac{3}{4}\ln(x) + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

$\downarrow \frac{d}{dx}$ $\quad \frac{d}{dx}$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4x} + \frac{1}{2(x^2+1)}\cdot 2x - \frac{5}{3x+2}\cdot 3$$

$$\frac{dy}{dx} = \left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right)\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right]$$

4. (1 point) Find the most generic anti-derivative: $\frac{1}{3}\int 3x^2 e^{x^3}\,dx = \frac{1}{3}\int e^u\,du$

$u = x^3$

$du = 3x^2\,dx$

$= \frac{1}{3}e^u + c$

$= \frac{1}{3}e^{x^3} + c$

5. (2 points) For the purposes of this problem, make the (unrealistic) assumption that the change in the price of bitcoin at any given point in time is directly proportional to the price of bitcoin at that time.
A few years ago I invested \$100 in bitcoin. After ln(9) years (approximately 2.19722457734 years) my investment had grown to \$300. How much can I expect my bitcoin to be worth after ln(81) years (approximately 4.39444915467 years)? (hint: you shouldn't use a calculator and your answer should be a nice number.)

$P(t) = 100e^{kt}$

$P(\ln(9)) = 100e^{\ln(9)\cdot k} = 300$

$9^k = (e^{\ln(9)})^k = e^{\ln(9)\cdot k} = 3$

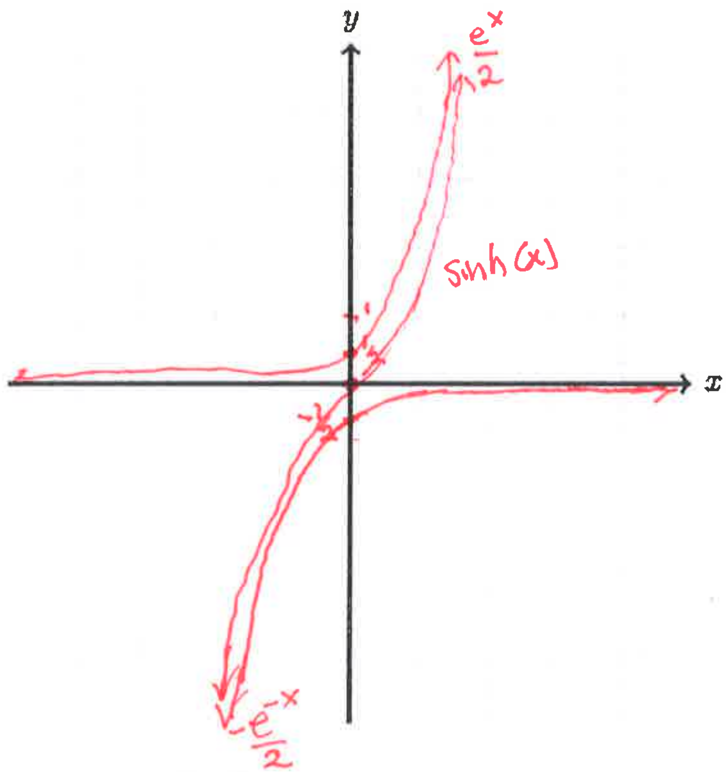
$\boxed{k = 1/2}$

$P(\ln(81)) = 100e^{\frac{1}{2}\cdot\ln(81)}$

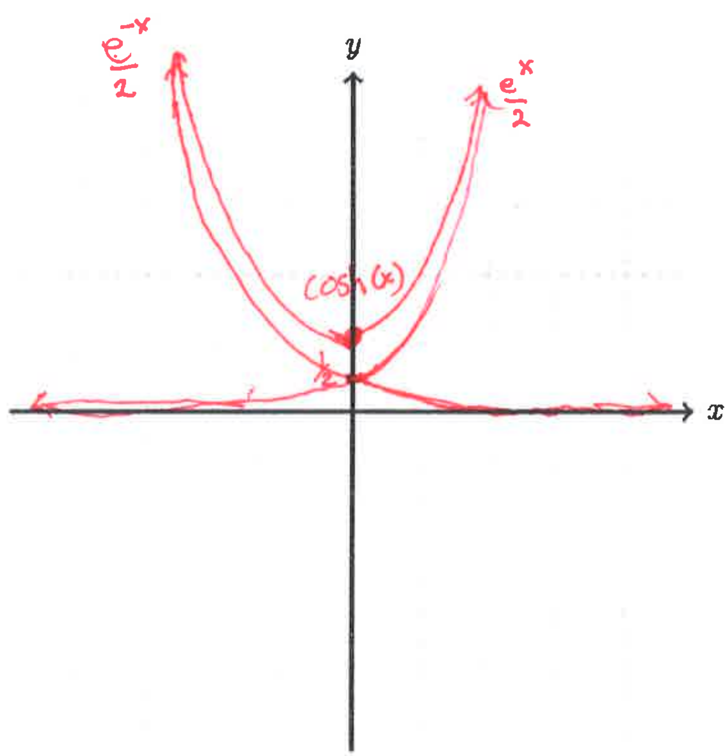
$= 100\left(e^{\ln(81)}\right)^{1/2}$

$= 100\,(81)^{1/2} = 100\cdot 9 = \boxed{\$900}$

6. (2 points) Sketch the curves $\frac{e^x}{2}, \frac{-e^{-x}}{2}$, and $\sinh(x)$. Label y-intercepts and functions.



7. (2 points) Sketch the curves $\frac{e^x}{2}, \frac{e^{-x}}{2}$, and $\cosh(x)$. Label y-intercepts and functions.



Call this W

8. (3 points) $\int \underbrace{e^{2\theta}}_{u}\underbrace{\sin(3\theta)d\theta}_{dv} = \frac{e^{2\theta}\cos(3\theta)}{-3} + \frac{2}{3}\underbrace{\int e^{2\theta}\cos(3\theta)d\theta}_{*}$

$du = 2e^{2\theta}d\theta \quad v = -\frac{\cos(3\theta)}{3}$

$* = \int \underbrace{e^{2\theta}}_{u}\underbrace{\cos(3\theta)\,d\theta}_{dv} = \frac{e^{2\theta}\sin(3\theta)}{3} - \frac{2}{3}\int e^{2\theta}\sin(3\theta)d\theta$

$du = 2e^{2\theta}d\theta \quad v = \frac{\sin(3\theta)}{3}$

$W = \frac{e^{2\theta}\cos(3\theta)}{-3} + \frac{2}{3}\left[\frac{e^{2\theta}\sin(3\theta)}{3} - \frac{2}{3}W\right] + C \Rightarrow \frac{13}{9}W = \frac{e^{2\theta}\cos(3\theta)}{-3} + \frac{2e^{2\theta}\sin(3\theta)}{9} +$

$W = \frac{3e^{2\theta}\cos(3\theta)}{-13} + \frac{2e^{2\theta}\sin(3\theta)}{13} + C$

$W = \frac{e^{2\theta}(2\sin(3\theta) - 3\cos(3\theta))}{13} + C$

9. (2 points) $\int \sin^5(\theta)\cos^2(\theta)d\theta =$

$= \int (1-\cos^2(\theta))^2 \cdot \sin(\theta)\cos^2(\theta)\,d\theta$

$= \int [1 - 2\cos^2(\theta) + \cos^4(\theta)]\sin(\theta)\cos^2(\theta)\,d\theta$

$= \int \cos^2(\theta)\sin(\theta)d\theta - 2\int \cos^4(\theta)\sin(\theta)d\theta + \int \cos^6(\theta)\sin(\theta)d\theta$

$u = \cos\theta,\ du = -\sin\theta\,d\theta$

$= -\frac{\cos^3(\theta)}{3} + \frac{2\cos^5(\theta)}{5} - \frac{\cos^7(\theta)}{7} + C$

10. (3 points) $\int \frac{1}{\sqrt{x^2-81}}dx = \int \frac{1}{9\sqrt{\left(\frac{x}{9}\right)^2 - \frac{81}{81}}}dx = \frac{1}{9}\int \frac{1}{\sqrt{\left(\frac{x}{9}\right)^2 - 1}}dx$

$\sec^2\theta - 1 = \tan^2\theta$

let $\sec\theta = \frac{x}{9}$

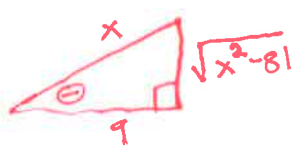
so $x = 9\sec\theta$

$dx = 9\sec\theta\tan\theta\,d\theta$

$= \frac{1}{9}\int \frac{1}{\sqrt{\sec^2\theta - 1}} \cdot 9\sec\theta\tan\theta\,d\theta$

$= \int \frac{\sec\theta\tan\theta}{\tan\theta}d\theta$

$= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$



$= \ln\left|\frac{x}{9} + \frac{\sqrt{x^2-81}}{9}\right| + C = \ln|x + \sqrt{x^2-81}| + C$

11. (2 points) $\int \frac{x^2+2x+2}{x^3+2x}dx = \int \frac{x^2+2x+2}{x(x^2+2)}dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+2}dx$

$= \int \frac{1}{x} + \frac{2}{x^2+2}dx$

$\boxed{= \ln|x| + \sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C}$

$x^2+2x+2 = A(x^2+2) + (Bx+C)x$

$= Ax^2 + 2A + Bx^2 + Cx$

$1x^2 + 2x + 2 = (A+B)x^2 + Cx + 2A$

$1 = A+B$

$2 = C$

$2 = 2A$

$\boxed{\begin{array}{l} B=0 \\ C=2 \\ A=1 \end{array}}$

$\int \frac{2}{x^2+2}dx = \int \frac{2}{2\left[\left(\frac{x}{\sqrt{2}}\right)^2+\frac{2}{2}\right]}dx$

$= \int \frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}dx = \frac{1}{\sqrt{2}}\int \frac{1}{u^2+1}du$

$u = \frac{x}{\sqrt{2}}$

$du = \frac{1}{\sqrt{2}}dx$

$= \frac{1}{\sqrt{2}}\tan^{-1}(u)+C$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$

EC) Find the values of p for which $\int_e^\infty \frac{1}{x(\ln x)^p}\,dx$ converges and evaluate the integral for those values of p.

- If $p=0$ then the integral diverges. Indeed, $\int_e^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \ln(t) - \ln(e)$
$= \infty - 1 = \infty$.

- If $p \neq 0$ and $p \neq 1$ let $u = \ln x$, $du = \frac{1}{x}dx$

$$\int_e^\infty \frac{1}{x(\ln x)^p}dx = \lim_{t\to\infty} \int_1^t u^{-p}\,du$$

$$= \lim_{t\to\infty} \left.\frac{u^{-p+1}}{-p+1}\right|_1^t$$

$$= \lim_{t\to\infty} \frac{t^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}$$

$$\stackrel{?}{=} \underbrace{\lim_{t\to\infty} \frac{1}{1-p}\cdot\frac{1}{t^{p-1}}} - \lim_{t\to\infty}\frac{1}{-p+1}$$

if $p>1$ this is 0, and so the whole integral $= \boxed{\frac{1}{p-1}}$

if $p<1$ this diverges.

When $p=1$,

let $u = \ln x$

$du = \frac{1}{x}dx$

$$\int_e^\infty \frac{1}{x\ln x}dx = \lim_{t\to\infty}\int_1^t \frac{1}{u}\,du = \lim_{t\to\infty} \ln|u|\Big|_1^t$$

$$= \lim_{t\to\infty} \ln|t| - \ln|1| = \infty$$

so the integral diverges here too.

So the integral converges $\forall p>1$ to $\frac{1}{p-1}$ and diverges $\forall p \leq 1$.