Vertex-magic edge Z_{2nm}-labeling of $C_n \square C_m$

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Abstract

A vertex-magic edge Γ-labeling of a graph $G(V, E)$ with $|E| = k$ is a bijection from E to an Abelian group Γ of order k such that the sum of labels of all incident edges of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$. We present a vertex-magic edge Z_{2nm}-labeling of Cartesian product of two cycles, $C_n \square C_m$ for n odd. This along with an earlier result by Ivančo proves that a vertex-magic edge Z_{2nm}-labeling of $C_n \square C_m$ exists for every $n, m \geq 3$.

Keywords: Magic-type labeling, vertex-magic labeling, group edge labeling, Cartesian product of cycles
2000 Mathematics Subject Classification: 05C78

1 Motivation

The Cartesian product of cycles $C_{n_1}, C_{n_2}, \cdots, C_{n_s}$, denoted $C_{n_1} \square C_{n_2} \square \cdots \square C_{n_s}$ can be viewed as the Cayley graph of Abelian group $Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_s}$ generated by group elements $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$. It is an intriguing question whether we can label the elements (that is, edges, vertices, or both) of such a graph with elements of the group (or another Abelian group of an appropriate order) so that the sum of the labels of the elements incident or adjacent to every edge or vertex is the same group element μ, called a *magic constant*. We provide exact definitions of the above notions in Section 2.

It seems natural to label just edges and sum the edge labels incident with each vertex. Another natural approach is to only label vertices and look at the sum of labels of vertices adjacent to each vertex. The latter notion has been studied in several papers already. Froncek [3] studied Z_{nm}-labeling of $C_n \square C_m$, and Cichacz [1] proved more results for some other Abelian groups. Cichacz and Froncek [2] investigated circulant graphs in this context.

This paper is the first attempt to study the edge labeling version. We present a method for a vertex-magic edge Z_{2nm}-labeling of Cartesian product of two cycles, $C_n \square C_m$ for n odd, and prove that such labeling exists for every magic constant, $\mu = 4\nu$.

2 Definitions

First we define the Cartesian product of graphs. Because we focus on products of two cycles in this paper, we restrict our definition to that case. A more general definition for s graphs can be obtained recursively.

Definition 1. The *Cartesian product* $G = G_1 \square G_2$ of graphs G_1 and G_2 with disjoint vertex and edge sets V_1, V_2, and E_1, E_2 respectively, is the graph with vertex set $V = V_1 \times V_2$ where any two vertices $u = (u_1, u_2) \in G$ and $v = (v_1, v_2) \in G$ are adjacent in G if and only if either $u_1 = v_1$ and u_2 is adjacent with v_2 in G_2 or, $u_2 = v_2$ and u_1 is adjacent with v_1 in G_1.

Now we define the labelings we are investigating.

Definition 2. A *vertex-magic edge* Γ*-labeling* of a graph $G(V, E)$ with $|E| = k$ is a bijection f from E to an Abelian group Γ of order k such that the sum of labels of all incident edges of every vertex $x \in V$, called the *weight* of x and denoted $w(x)$, is equal to the same element $\mu \in \Gamma$, called the *magic constant*. That is,

$$w(x) = \sum_{y:xy \in E} f(xy) = \mu$$

for every vertex $x \in V$.

In fact, our definition is a generalization of a previously studied notion of *vertex-magic edge labeling*, where the labels are just consecutive positive integers. This type of labeling is also often called *supermagic labeling*.

Definition 3. A *vertex-magic edge labeling* or a *supermagic labeling* of a graph $G(V, E)$ with $|E| = k$ is a bijection g from E to the set $\{1, 2, \ldots, k\}$ such that the sum of labels of all incident edges of every vertex $x \in V$, called the *weight* of x and denoted $w(x)$, is equal to the same integer c, called the *magic constant*. That is,

$$w(x) = \sum_{y:xy \in E} g(xy) = c$$

for every vertex $x \in V$.

As we also briefly discuss analogous results obtained earlier for distance magic labeling, we define it here as well.

Definition 4. A Γ-*distance magic labeling* of a graph $G(V, E)$ with $|V| = p$ is a bijection f from V to and Abelian group Γ of order p such that the sum of labels of all adjacent vertices of every vertex $x \in V$, called the *weight* of x and denoted $w(x)$, is equal to the same element $\mu \in \Gamma$, called the *magic constant*. That is,

$$w(x) = \sum_{y:xy \in E} f(y) = \mu$$

for every vertex $x \in V$.

As in the case of edge labelings, even here results on labeling with positive integers preceded those on labeling with group elements.

Definition 5. A *distance magic labeling* of a graph $G(V,E)$ with $|V| = p$ is a bijection g from V to the set $\{1, 2, \ldots, p\}$ such that the sum of labels of all adjacent vertices of every vertex $x \in V$, called the *weight* of x and denoted $w(x)$, is equal to the same integer c, called the *magic constant.* That is,

$$w(x) = \sum_{y:xy\in E} g(y) = c$$

for every vertex $x \in V$.

3 Known results

We first list results on distance magic and Γ-distance magic labelings of Cartesian cycle products, as they have been a motivation of our research.

Rao, Singh, and Parameswaran in [5] proved the following.

Theorem 6. *The graph $C_n \square C_m$ has a distance magic labeling if and only if $n = m \geq 6$ and $n, m \equiv 2 \pmod 4$.*

Based on this notion of distance magic graphs, Froncek [3] introduced the concept of Γ-distance magic labeling and proved a complete result on Γ-distance magic labeling of Cartesian product of two cycles with cyclic groups.

Theorem 7. *The Cartesian product $C_n \square C_m$ has a Z_{nm}-distance magic labeling if and only if $n, m \geq 3$ and nm is even.*

Cichacz and Froncek [2] proved the following.

Theorem 8. *Let G be an r-regular graph on n vertices, where r is odd. Then there does not exist an Abelian group Γ of order n having exactly one involution (an element that is its own inverse) admitting a Γ-distance magic labeling of G.*

Cichacz [1] proved a more general result for other Abelian groups.

Theorem 9. *Let n, m, t, s be positive integers, $n, m \geq 3$ and $l = \operatorname{lcm}(n, m)$. Let $\Gamma = Z_{lt} \times A$, where A is an Abelian group of order s and $nm = lts$. Then the Cartesian product $C_n \square C_m$ has a Γ-distance magic labeling.*

Results analogous to Theorem 6 for vertex-magic edge labeling were proved by Ivančo [4].

Theorem 10. *$C_n \square C_n$ has a vertex-magic edge labeling for any $n \geq 3$.*

Theorem 11. *Let $n, m \geq 2$ be integers. Then $C_{2n} \square C_{2m}$ has a vertex-magic edge labeling.*

Ivančo also conjectured that the Cartesian product $C_n \square C_m$ allows a vertex-magic edge labeling for any $n, m \geq 3$.

In the following sections, we prove that a group edge labeling equivalent of the conjecture is true.

4 Construction

First we present a construction for a product of two odd cycles. We denote the horizontal m-cycles by $B^0, B^1, \ldots, B^{n-1}$ and the vertical n-cycles by $C^0, C^1, \ldots, C^{m-1}$. Then a vertex x_{ij} belongs to B^i and C^j.

We start by labeling the edges of B^0 (going from left to right and skipping every other edge) by consecutive even numbers, $0, 2, 4, \ldots, 2m-2$. Because m is odd, all edges receive labels. Then we continue labeling B^1 with the next m even numbers, $2m, 2m+2, \ldots, 4m-2$. Again, as the number of cycles is even, we label edges of all n horizontal m-cycles while using all even elements of Z_{2nm}. In general, edges in B^{2s} are labeled $2s, 2s+2, 2s+4, \ldots, 2s+2m-2$, where the superscript is taken mod n.

The layout of the labels is shown in Figure 1.

	$x_{i0}x_{i1}$	$x_{i1}x_{i2}$	$x_{i2}x_{i3}$	$x_{i3}x_{i4}$	$\cdots$	$x_{i(m-1)}x_{i0}$
B^0	0	$m+1$	2	$m+3$	$\cdots$	$m-1$
B^1	$2m$	$3m+1$	$2m+2$	$3m+3$	$\cdots$	$3m-1$
B^2	$4m$	$5m+1$	$4m+2$	$5m+3$	$\cdots$	$5m-1$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
B^{n-2}	$-4m$	$-3m+1$	$-4m+2$	$-3m+3$	$\cdots$	$-3m-1$
B^{n-1}	$-2m$	$-m+1$	$-2m+2$	$-m+3$	$\cdots$	$-m-1$

Figure 1: Labeled horizontal cycles, n, m odd

We will call the sum of labels of the horizontal edges incident with a vertex x_{ij} the *horizontal partial weight of* x_{ij} and denote it by $w_h(x_{ij})$. Similarly, the sum of labels of the vertical edges incident with x_{ij} will be called the *vertical partial weight of* x_{ij} and denoted by $w_v(x_{ij})$. More precisely,

$$w_h(x_{ij}) = f(x_{i(j-1)}x_{ij}) + f(x_{ij}x_{i(j+1)})$$

and

$$w_v(x_{ij}) = f(x_{(i-1)j}x_{ij}) + f(x_{ij}x_{(i+1)j}).$$

The partial weights $w_h(x_{ij})$ of vertices are listed in Figure 2. Notice that the partial weights in each column form a coset of Z_{2nm} induced by the subgroup

$\langle 4m \rangle$. Therefore, we need to label the vertical cycles so that each cycle has partial weights w_v forming an appropriate coset as well. We will call the cosets of type $\langle 4m \rangle + 2t$ *even* and those of type $\langle 2m \rangle + 2t + 1$ *odd*.

Namely, for a column with partial weights in a coset $\langle 4m \rangle + 2t$, we want the vertical cycle with partial weights w_v forming the coset $\langle 4m \rangle - 2t$, listed in the opposite order. Then we have $w_h(x_{ij}) = 4mi + 2t$ and $w_v(x_{ij}) = 4m(n-i) - 2t$ which yields $w(x_{ij}) = 4mi + 2t + 4nm - 4mi - 2t = 4nm = 0$.

	x_{i0}	x_{i1}	x_{i2}	x_{i3}	$\cdots$	$x_{i(m-1)}$
B^0	$m-1$	$m+1$	$m+3$	$m+5$	$\cdots$	$3m-3$
B^1	$5m-1$	$5m+1$	$5m+3$	$5m+5$	$\cdots$	$7m-3$
B^2	$9m-1$	$9m+1$	$9m+3$	$9m+5$	$\cdots$	$11m-3$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
B^{n-2}	$-7m-1$	$-7m+1$	$-7m+3$	$-7m+5$	$\cdots$	$-5m-3$
B^{n-1}	$-3m-1$	$-3m+1$	$-3m+3$	$-3m+5$	$\cdots$	$-m-3$

Figure 2: Partial weights in horizontal cycles, n, m odd

We achieve this goal by labeling edges of each C^j consecutively with elements of a coset induced by the subgroup $\langle 4m \rangle$. However, because n, m are both odd, we have $\langle 4m \rangle = \langle 2m \rangle$. We also set $m' = (m-1)/2$ to simplify notation in the labeling presented below in Figure 3.

There are two cases, depending on whether m' is odd or even. They both have the same partial weights, because they only differ by the placement of element nm in their labels. The labels in each column (that is, a coset) either all contain that element, or none do. Moreover, for any two consecutive columns, exactly one of them has nm added in each term. Thus, the sum of two neighboring edges making up the partial weight in every other cycle contains $2nm$, which in Z_{2nm} is indeed equal to zero. We need to do that to ensure that the labels form *odd* cosets, as we have used all even ones for the horizontal cycles.

The partial weights for vertical cycles are presented in Figure 4. Adding the partial weights in Figures 2 and 4, we can see that the total weights all equal zero and thus the labeling is vertex-magic. We provide a more rigorous proof below.

Theorem 12. *For n, m both odd, $C_n \square C_m$ can be labeled with group elements from Z_{2nm} to form a vertex-magic edge Z_{2nm}-labeling with magic constant $\mu = 0$.*

m' **even**	C_0	C_1	C_2	$\cdots$	C_{m-1}
$x_{0j}x_{1j}$	$-m-m'$	$-m-nm-m'-1$	$-m-m'-2$	$\cdots$	$-m-m'-(m-1)$
$x_{1j}x_{2j}$	$-3m-m'$	$-3m-nm-m'-1$	$-3m-m'-2$	$\cdots$	$-3m-m'-(m-1)$
$x_{2j}x_{3j}$	$-5m-m'$	$-5m-nm-m'-1$	$-5m-m'-2$	$\cdots$	$-5m-m'-(m-1)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$x_{(n-2)j}x_{(n-1)j}$	$3m-m'$	$3m-nm-m'-1$	$3m-m'-2$	$\cdots$	$3m-m'-(m-1)$
$x_{(n-1)j}x_{0j}$	$m-m'$	$m-nm-m'-1$	$m-m'-2$	$\cdots$	$m-m'-(m-1)$

m' **odd**	C_0	C_1	C_2	$\cdots$	C_{m-1}
$x_{0j}x_{1j}$	$-m-nm-m'$	$-m-m'-1$	$-m-nm-m'-2$	$\cdots$	$-m-nm-m'-(m-1)$
$x_{1j}x_{2j}$	$-3m-nm-m'$	$-3m-m'-1$	$-3m-nm-m'-2$	$\cdots$	$-3m-nm-m'-(m-1)$
$x_{2j}x_{3j}$	$-5m-nm-m'$	$-5m-m'-1$	$-5m-nm-m'-2$	$\cdots$	$-5m-nm-m'-(m-1)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$x_{(n-2)j}x_{(n-1)j}$	$3m-nm-m'$	$3m-m'-1$	$3m-nm-m'-2$	$\cdots$	$3m-nm-m'-(m-1)$
$x_{(n-1)j}x_{0j}$	$m-nm-m'$	$m-m'-1$	$m-nm-m'-2$	$\cdots$	$m-nm-m'-(m-1)$

Figure 3: Labeled vertical cycles, n, m odd

	C_0	C_1	C_2	C_3	$\cdots$	C_{m-1}
x_{0j}	$-m+1$	$-m-1$	$-m-3$	$-m-5$	$\cdots$	$-3m+3$
x_{1j}	$-5m+1$	$-5m-1$	$-5m-3$	$-5m-5$	$\cdots$	$-7m+3$
x_{2j}	$-9m+1$	$-9m-1$	$-9m-3$	$-9m-5$	$\cdots$	$-11m+3$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$x_{(n-2)j}$	$7m+1$	$7m-1$	$7m-3$	$7m-5$	$\cdots$	$5m+3$
$x_{(n-1)j}$	$3m+1$	$3m-1$	$3m-3$	$3m-5$	$\cdots$	$m+3$

Figure 4: Partial weights in vertical cycles, n, m odd

Proof. Let again $m = 2m' + 1$. First we look at the horizontal labels

$$f(x_{ij}x_{i(j+1)}) = \begin{cases} 2mi + j & \text{for } j \text{ even} \\ 2mi + m + j & \text{for } j \text{ odd.} \end{cases}$$

Therefore, for j even we get

$$\begin{aligned} w_h(x_{ij}) &= f(x_{i(j-1)}x_{ij}) + f(x_{ij}x_{i(j+1)}) \\ &= (2mi + m + j - 1) + (2mi + j) \\ &= (4i + 1)m + 2j - 1 \end{aligned}$$

and for j odd,

$$\begin{aligned} w_h(x_{ij}) &= f(x_{i(j-1)}x_{ij}) + f(x_{ij}x_{i(j+1)}) \\ &= (2mi + j - 1) + (2mi + m + j) \\ &= (4i + 1)m + 2j - 1. \end{aligned}$$

So, in both cases the partial weight is

$$w_h(x_{ij}) = (4i + 1)m + 2j - 1. \tag{1}$$

For the vertical labels, we need to distinguish two cases. When m' is even, we have

$$f(x_{ij}x_{(i+1)j}) = \begin{cases} -m(2i + 1) - m' - j & \text{for } j \text{ even} \\ -m(2i + 1) - nm - m' - j & \text{for } j \text{ odd.} \end{cases}$$

Then for j even we have

$$\begin{aligned} w_v(x_{ij}) &= f(x_{(i-1)j}x_{ij}) + f(x_{ij}x_{(i+1)j}) \\ &= (-m(2i - 1) - m' - j) + (-m(2i + 1) - m' - j) \\ &= -4mi - 2m' - 2j \end{aligned}$$

and for j odd we have

$$\begin{aligned} w_v(x_{ij}) &= f(x_{(i-1)j}x_{ij}) + f(x_{ij}x_{(i+1)j}) \\ &= (-m(2i-1) - nm - m' - j) + (-m(2i+1) - nm - m' - j) \\ &= -4mi - 4nm - 2m' - 2j \\ &= -4mi - 2m' - 2j \end{aligned}$$

as for j even. Now substituting back $m = 2m' + 1$, we obtain

$$w_v(x_{ij}) = -4mi - 2m' - 2j = -4mi - (m-1) - 2j = -(4i+1)m - 2j + 1. \quad (2)$$

It now follows from (1) and (2) that

$$w(x_{ij}) = w_h(x_{ij}) + w_v(x_{ij}) = ((4i+1)m + 2j - 1) + (-(4i+1)m - 2j + 1) = 0,$$

as desired.

When m' is odd, we have

$$f(x_{ij}x_{(i+1)j}) = \begin{cases} -m(2i+1) - nm - m' - j & \text{for } j \text{ even} \\ -m(2i+1) - m' - j & \text{for } j \text{ odd.} \end{cases}$$

Then for j even we have

$$\begin{aligned} w_v(x_{ij}) &= f(x_{(i-1)j}x_{ij}) + f(x_{ij}x_{(i+1)j}) \\ &= (-m(2i-1) - nm - m' - j) + (-m(2i+1) - nm - m' - j) \\ &= -4mi - 4nm - 2m' - 2j \\ &= -4mi - 2m' - 2j \end{aligned}$$

and for j odd we have

$$\begin{aligned} w_v(x_{ij}) &= f(x_{(i-1)j}x_{ij}) + f(x_{ij}x_{(i+1)j}) \\ &= (-m(2i-1) - m' - j) + (-m(2i+1) - m' - j) \\ &= -4mi - 2m' - 2j \end{aligned}$$

as well. We observe that the vertical partial weight is the same as for m' even, so we again obtain

$$w_v(x_{ij}) = -(4i+1)m - 2j + 1. \quad (3)$$

This is the same temporary weight as for m' even in (2), so adding (1) and (3) we have again

$$w(x_{ij}) = w_h(x_{ij}) + w_v(x_{ij}) = 0,$$

which completes the proof. □

The construction for n odd and m even is similar. Using the same notation, we set $n = 2n' + 1, m = 2m'$ and label edges of B_0 consecutively with elements $0, 2, \ldots, 2m - 2$. Then we continue with B_1 starting at $2m$ and so on, utilizing all even elements of Z_{2nm}. The labeling is presented in Figure 5.

The partial weights in this case are not all different, as they only use elements from the cosets $\langle 2m\rangle + 4t + 2$. In particular, coset $\langle 2m\rangle + 4t + 2$ appears in columns t and $m' + t$. However, while the cosets are in the same position for $t = 0$, for other values of t the values in column $m' + t$ are cyclically shifted up by $n' + 1$ positions (or by n' down, which is indeed the same) compared with column t. The partial weights are also shown in Figure 5.

For vertical cycles, we again use all odd cosets, namely for C_j the coset $\langle 2m\rangle - (2j - 1)$. Looking at Figure 6 the labels may be a bit confusing, starting with $C_{m'}$. This is because we made some simplifications in the above formula. For instance, since $m = 2m'$, we have

$$f(x_{0m'}x_{1m'}) = (n-1)m-(2(m'+1)-1) = (n-1)m-(m-1) = (n-2)m+1.$$

A detailed proof follows.

Theorem 13. *For n odd and m even, $C_n \Box C_m$ can be labeled with group elements from Z_{2nm} to form a vertex-magic edge Z_{2nm}-labeling with magic constant $\mu = 0$.*

Proof. Let $n = 2n' + 1$ and $m = 2m'$. The horizontal labels are defined as

$$f(x_{ij}x_{i(j+1)}) = 2mi + 2j.$$

Hence, for $j \neq 0$ we have the horizontal partial weights

$$\begin{aligned} w_h(x_{ij}) &= f(x_{i(j-1)}x_{ij}) + f(x_{ij}x_{i(j+1)}) \\ &= (2mi + 2(j-1)) + (2mi + 2j) \\ &= 4mi + 4j - 2 \end{aligned} \tag{4}$$

and for $j = 0$ we have

$$\begin{aligned} w_h(x_{i0}) &= f(x_{i(m-1)}x_{i0}) + f(x_{i0}x_{i1}) \\ &= (2mi + 2(m-1)) + 2mi \\ &= 4mi + 2m - 2 \end{aligned} \tag{5}$$

The vertical edges are labeled

$$f(x_{ij}x_{(i+1)j}) = m(n-1) - 2mi - (2j-1) = m(n-2i-1) - 2j + 1$$

for $j \neq 0$ and

$$f(x_{i0}x_{(i+1)0}) = -2m(i+1) + 1$$

	$x_{i0}x_{i1}$	$x_{i1}x_{i2}$	$x_{i2}x_{i3}$	$\cdots$	$x_{im'}x_{i(m'+1)}$	$x_{i(m'+1)}x_{i(m'+2)}$	$\cdots$	$x_{i(m-1)}x_{i0}$
B^0	0	2	4	$\cdots$	m	$m+2$	$\cdots$	$2m-2$
B^1	$2m$	$2m+2$	$2m+4$	$\cdots$	$3m$	$3m+2$	$\cdots$	$4m-2$
B^2	$4m$	$4m+2$	$4m+4$	$\cdots$	$5m$	$5m+2$	$\cdots$	$6m-2$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$B^{n'}$	$(n-1)m$	$(n-1)m+2$	$(n-1)m+4$	$\cdots$	nm	$nm+2$	$\cdots$	$(n+1)m-2$
$B^{n'+1}$	$(n+1)m$	$(n+1)m+2$	$(n+1)m+4$	$\cdots$	$(n+2)m$	$(n+2)m+2$	$\cdots$	$(n+3)m-2$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
B^{n-2}	$-4m$	$-4m+2$	$-4m+4$	$\cdots$	$-3m$	$-3m+2$	$\cdots$	$-2m-2$
B^{n-1}	$-2m$	$-2m+2$	$-2m+4$	$\cdots$	$-m$	$-m+2$	$\cdots$	-2

	x_{i0}	x_{i1}	x_{i2}	$\cdots$	$x_{im'}$	$x_{i(m'+1)}$	$\cdots$	$x_{i(m-1)}$
B^0	$2m-2$	2	6	$\cdots$	$2m-2$	$2m+2$	$\cdots$	$4m-6$
B^1	$6m-2$	$4m+2$	$4m+6$	$\cdots$	$6m-2$	$6m+2$	$\cdots$	$8m-6$
B^2	$10m-2$	$8m+2$	$8m+6$	$\cdots$	$10m-2$	$10m+2$	$\cdots$	$12m-6$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$B^{n'}$	-2	$-2m+2$	$-2m+6$	$\cdots$	-2	2	$\cdots$	$2m-6$
$B^{n'+1}$	$4m-2$	$2m+2$	$2m+6$	$\cdots$	$4m-2$	$4m+2$	$\cdots$	$6m-6$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
B^{n-2}	$-6m-2$	$-8m+2$	$-8m+6$	$\cdots$	$-6m-2$	$-6m+2$	$\cdots$	$-4m-6$
B^{n-1}	$-2m-2$	$-4m+2$	$-4m+6$	$\cdots$	$-2m-2$	$-2m+2$	$\cdots$	-6

Figure 5: Labeled horizontal cycles and partial weights, n odd, m even

	C_0	C_1	C_2	$\cdots$	$C_{m'}$	$C_{m'+1}$	$\cdots$	C_{m-1}
$x_{0j}x_{1j}$	$-2m+1$	$(n-1)m-1$	$(n-1)m-3$	$\cdots$	$(n-2)m+1$	$(n-2)m-1$	$\cdots$	$(n-3)m+3$
$x_{1j}x_{2j}$	$-4m+1$	$(n-3)m-1$	$(n-3)m-3$	$\cdots$	$(n-4)m+1$	$(n-4)m-1$	$\cdots$	$(n-5)m+3$
$x_{2j}x_{3j}$	$-6m+1$	$(n-5)m-1$	$(n-5)m-3$	$\cdots$	$(n-6)m+1$	$(n-6)m-1$	$\cdots$	$(n-9)m+3$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$x_{n'j}x_{(n'+1)j}$	$-(n'+1)m+1$	-1	-3	$\cdots$	$-m+1$	$-m-1$	$\cdots$	$-2m+3$
$x_{(n'+1)j}x_{(n'+2)j}$	$-(n'+3)m+1$	$-2m-1$	$-2m-3$	$\cdots$	$-3m+1$	$-3m-1$	$\cdots$	$-4m+3$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$x_{(n-2)j}x_{(n-1)j}$	$2m+1$	$(n+3)m-1$	$(n+3)m-3$	$\cdots$	$(n+2)m+1$	$(n+2)m-1$	$\cdots$	$(n+1)m+3$
$x_{(n-1)j}x_{0j}$	1	$(n+1)m-1$	$(n+1)m-3$	$\cdots$	$nm+1$	$nm-1$	$\cdots$	$(n-1)m+3$

	C_0	C_1	C_2	$\cdots$	$C_{m'}$	$C_{m'+1}$	$\cdots$	C_{m-1}
x_{0j}	$-2m+2$	-2	-6	$\cdots$	$-2m+2$	$-2m-2$	$\cdots$	$-4m+6$
x_{1j}	$-6m+2$	$-4m-2$	$-4m-6$	$\cdots$	$-6m+2$	$-6m-2$	$\cdots$	$-8m+6$
x_{2j}	$-10m+2$	$-8m-2$	$-8m-6$	$\cdots$	$-10m+2$	$-10m-2$	$\cdots$	$-12m+6$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$x_{n'j}$	2	$2m-2$	$2m-6$	$\cdots$	2	-2	$\cdots$	$-2m+6$
$x_{(n'+1)j}$	$-4m+2$	$-2m-2$	$-2m-6$	$\cdots$	$-4m+2$	$-4m-2$	$\cdots$	$-6m+6$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_{n-2}	$6m+2$	$8m-2$	$8m-6$	$\cdots$	$6m+2$	$6m-2$	$\cdots$	$4m+6$
x_{n-1}	$2m+2$	$4m-2$	$4m-6$	$\cdots$	$2m+2$	$2m-2$	$\cdots$	6

Figure 6: Labeled vertical cycles and partial weights, n odd, m even

otherwise. Therefore, the partial weights in the vertical cycles are

$$\begin{aligned} w_v(x_{ij}) &= f(x_{(i-1)j}x_{ij}) + f(x_{ij}x_{(i+1)j}) \\ &= (m(n-2i+1)-2j+1) + (m(n-2i-1)-2j+1) \\ &= 2nm - 4mi - 4j + 2 \\ &= -4mi - 4j + 2 \end{aligned} \tag{6}$$

and for $j = 0$ we have

$$\begin{aligned} w_h(x_{i0}) &= f(x_{(i-1)0}x_{i0}) + f(x_{i0}x_{(i+1)0}) \\ &= (-2mi+1) + (-2m(i+1)+1) \\ &= -4mi - 2m + 2. \end{aligned} \tag{7}$$

Adding (4) and (6), we get

$$\begin{aligned} w(x_{ij}) &= w_h(x_{ij}) + w_v(x_{ij}) \\ &= 4mi + 4j - 2 + (-4mi - 4j + 2) \\ &= 0 \end{aligned}$$

and adding (5) and (7), we get

$$\begin{aligned} w(x_{i0}) &= w_h(x_{i0}) + w_v(x_{i0}) \\ &= 4mi + 2m - 2 + (-4mi - 4m + 2) \\ &= 0, \end{aligned}$$

which completes the proof. □

Now we are ready to state our main result.

Theorem 14. *The Cartesian product $C_n \square C_m$ admits a vertex-magic edge Z_{2nm}-labeling for all $n, m \geq 3$.*

Proof. For n odd, the proof follows from Theorems 12 and 13. For both n, m even it follows from Ivančo's result in Theorem 11. Obviously, if the weight of every vertex is the same positive integer c, then by performing addition in Z_{2nm} rather than in $\mathbb{Z}$ we obtain a vertex-magic edge Z_{2nm}-labeling with $\mu = c \bmod 2nm$. □

Although both labelings in Theorems 12 and 13 result in magic constant $\mu = 0$, it is easy to observe that when we have any labeling with $\mu = 0$, then we can also find labelings for any $\mu = 4\nu$, where ν is an element of Z_{2nm}. We prove a slightly more general result.

Observation 15. *When a 4-regular graph G of order p admits a vertex-magic edge Z_{2p}-labeling with magic constant μ, then there is a vertex-magic edge Z_{2p}-labeling of G with $\mu + 4\nu$ for any ν in Z_{2p}. On the other hand, no such labeling with μ odd can exist.*

Proof. We start with a labeling f inducing a magic constant μ, and define f_ν as $f_\nu(xy) = f(xy) + \nu$. This is indeed again a bijection from E to Z_{2p} and $w_\nu(x) = w(x) + 4\nu$ for every x in V.

On the other hand, there is no labeling with an odd magic constant. When $\mu = 2\nu + 1$, then

$$\sum_{x \in V} w_\nu(x) = p(2\nu + 1) = 2p\nu + p = p.$$

Since every edge label contributes to the weights of two vertices, we also have

$$\sum_{x \in V} w_\nu(x) = 2 \sum_{xy \in E} f_\nu(xy) = 2p(2p - 1) = 0,$$

which is a contradiction. ☐

We combine the previous claims into one as follows.

Theorem 16. *The Cartesian product $C_n \square C_m$ admits a vertex-magic edge Z_{2nm}-labeling with magic constant μ*

(i) if and only if $\mu \equiv 0 \pmod 2$ when n, m are both odd,

(ii) if and only if $\mu \equiv 2 \pmod 4$ when n, m are both even,

(iii) for every $\mu \equiv 0 \pmod 4$ when n is odd and m is even,

whenever $n, m \geq 3$. Moreover, no such labeling with μ odd exists for n odd and m even.

Proof. It follows from Observation 15 that $\mu \equiv 0 \pmod 2$ for all three cases.

For n, m both odd, the result follows from Theorem 12 and Observation 15 and the fact that in this case, Z_{2nm} is of order $2nm \equiv 2 \pmod 4$ and hence $4\nu \neq 0$ generates the subgroup $\langle 2 \rangle$.

For n, m both even, the labeling with positive integers in Theorem 11 gives the magic constant $c = 4nm + 2$. To see that, we observe that

$$\sum_{x \in V} w(x) = 2 \sum_{xy \in E} g(xy),$$

and because $|V| = nm$ and $w(x) = c$ for every x in V, we have

$$nmc = 2 \sum_{t=1}^{2nm} t = 2nm(2nm + 1)$$

and hence

$$c = 2(2nm + 1).$$

Reducing c modulo $2nm$, we get $\mu = 2$. It follows from Observation 15 that a desired labeling exists for any $\mu \equiv 2 \pmod 4$. To see that we cannot have $\mu \equiv 0 \pmod 4$, we use the fact that $C_n \square C_m$ for n and m even is bipartite with partite sets V_0 and V_1. Because every edge has one end-vertex in V_0, we have

$$\sum_{x \in V_0} w(x) = |V_0|\mu = \sum_{xy \in E} f(xy),$$

and because $|V_0| = 2n'm'$, we have

$$2n'm'\mu = \sum_{a \in Z_{2nm}} a = nm(2nm-1) = nm, \tag{8}$$

since nm is even and thus $(nm)(2nm) = 0$ and $-nm = nm$. However, when $\mu \equiv 0 \pmod 4$, say $\mu = 4\nu$, the left-hand side in (8) is

$$2n'm'4\nu = 2nm\nu = 0,$$

because the multiplication is performed in Z_{2nm}. This is a contradiction showing that $\mu \not\equiv 0 \pmod 4$.

Finally, for n odd and m even, Z_{2nm} is of order $2nm \equiv 0 \pmod 4$ and hence $4\nu \neq 0$ generates the subgroup $\langle 4 \rangle$. The result then follows from Theorem 13 and 15. $\square$

We currently do not know any labeling with $\mu \equiv 2 \pmod 4$ for case (iii) in Theorem 16. Hence, we pose an open problem.

Open Problem. *Does there exist a vertex-magic edge Z_{2nm}-labeling of the Cartesian product $C_n \square C_m$ with magic constant $\mu \equiv 2 \pmod 4$ for n odd and m even?*

There are two other obvious directions in which one could investigate vertex-magic edge Z_{2nm}-labelings of Cartesian products of cycles. One is edge Γ-labelings of products of s cycles for $s \geq 3$. Another one is labeling of $C_n \square C_m$ with other Abelian groups of order $2nm$. The ultimate goal is to completely characterize all Abelian groups and cycle lengths such that there exists a vertex-magic edge Γ-labeling of $C_{n_1} \square C_{n_2} \square \cdots \square C_{n_s}$ where Γ is of order $sn_1n_2 \ldots n_s$.

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