# Math 113 Finite Math with a Special Emphasis on Math & Art

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# 1 Geometry notes

## 1.1 Angles and Parallel Lines

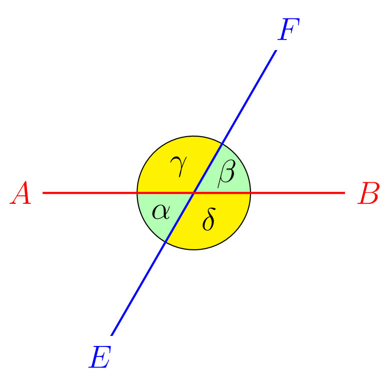
**Definition 1.1.** We defined the *complement* of an acute angle and the *supplement* of an angle (less than $180°$). Then had some examples of problems that use these words.

**Example 1.2.** The measure of an angle is $56°$ greater than its complement. Find the measure of the two angles.

**Group Work 1.3.** The measure of an angle is $33°$ less than twice that of its supplement. Find the measure of the two angles.

**Group Work 1.4.** Let $\alpha$ and $\beta$ be complementary angles. If $\alpha = x + 7$ and $\beta = 2x - 4$, find the measures of the two angles.

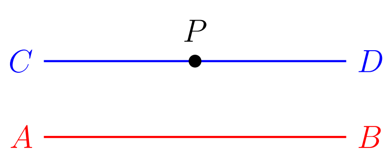
**Definition 1.5.** Vertical angles.




When a line $AB$ is transversed by a line $EF$, then the *vertical angles* formed ($\alpha$ and $\beta$; or $\gamma$ and $\delta$) are equal, i.e., $\alpha = \beta$ and $\gamma = \delta$.

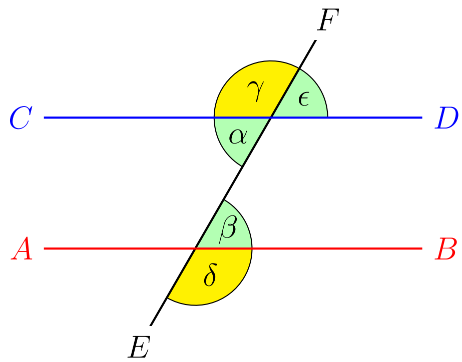
**Definition 1.6.** Two distinct lines are *parallel* if they have no points in common. (Otherwise, they are *intersecting* lines and have a point in common.)

*Remark* 1.7. The following definition/proposition assumes implicitly *Euclid's Parallel Postulate*. This postulate says that given a line $AB$ and a point $P$ not on this line, then there exists exactly one line $CD$ through $P$ and parallel to $AB$. We shall see later in the course how this postulate may be changed to give us alternative and more interesting geometries!




Given a line $AB$ and a point $P$ not on this line, then there exists exactly one line $CD$ through $P$ and parallel to $AB$.

**Definition 1.8.** In the diagram below, $AB$ and $CD$ are parallel and $EF$ is a transversal. Then $\alpha$ and $\beta$ are *alternating interior angles*; $\gamma$ and $\delta$ are *alternating exterior angles*; and $\beta$ and $\epsilon$ are *corresponding angles*.

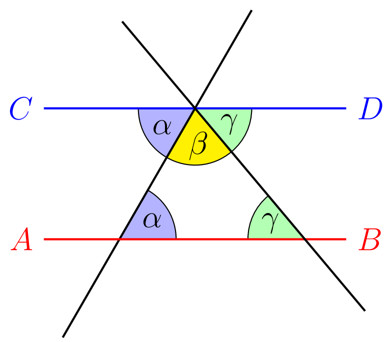



When we assume that $AB$ and $CD$ are parallel, i. e., $AB \parallel CD$, then $\alpha = \beta$; $\gamma = \delta$; and $\beta = \epsilon$.

**Proposition 1.9.** *Alternating interior angles are equal. Alternating exterior angles are equal. And corresponding angles are equal.*

Then we worked on several different examples of problems involving two parallel lines with a transversal. (I'll try to make the diagrams soon.)

**Proposition 1.10.** *The sum of the angles of a triangle is* $180°$*.*

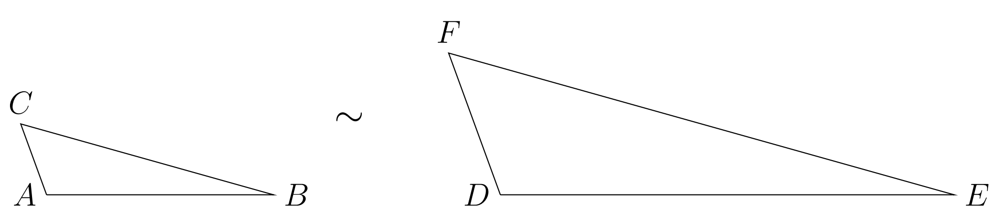



Any given triangle can be situated between two parallel lines; let its base be on the line $AB$, then by Euclid's Parallel Postulate, there exists a unique parallel line $CD$ through the triangle's top vertex. By the proposition above, alternating interior angles are equal; therefore, the sum of the angles of a triangle is $\alpha + \beta + \gamma$, which is $180°$, the angle measure of a straight line.

Now we did some examples and group work that use this fact to find missing angles.

## 1.2 Similar Triangles

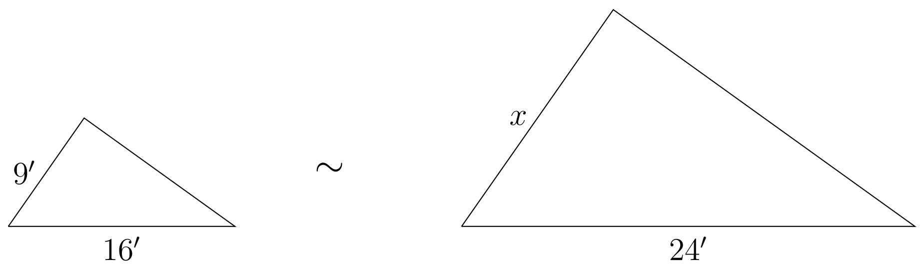
**Definition 1.11.** Two triangles are *similar*, denoted by the symbol $\sim$, if their corresponding angles are equal.



**Proposition 1.12.** *In similar triangles, the lengths of corresponding sides are proportional.*

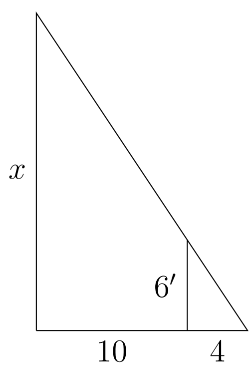
$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

**Example 1.13.** Given that the two triangles are similar, find the missing length $x$.




The next example is trickier as the similar triangles are somewhat hidden, i.e., nested one is inside the other. The thing is that people often set up the ratios wrong and get the wrong answer. So be careful! This kind of similar triangles problem is rather important for us as they'll reappear when we study linear perspective–in fact, we use them to find some key formulas.

**Example 1.14.** I stand $10ft$ from the base of a lamp post. If my shadow is $4ft$, how tall is the lamp post given that I am $6ft$ tall?




As the lengths of the corresponding sides are proportional,

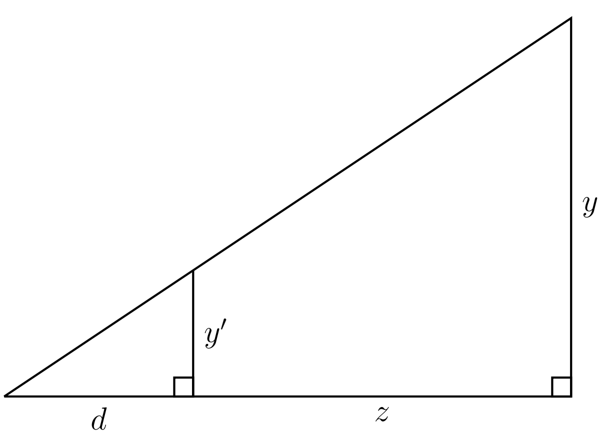
$$\frac{x}{6} = \frac{10+4}{4} = \frac{14}{4} \quad and$$

$$x = 6 \cdot \frac{14}{4} = 3 \cdot 7 = 21.$$

Thus, the lamp post is $21ft$ tall.

The next example is the exact formulation of the nested similar triangles problem that we will need later.

**Example 1.15.** Express $y'$ in terms of $d, y$, and $z$.




By proportionality, we have:

$$\frac{y'}{y} = \frac{d}{z+d}$$

and so,

$$y' = \frac{d \cdot y}{z+d}$$

## 1.3 Polygons & Tessellations

**Definition 1.16.** An *$n$-sided polygon*, or more simply an *$n$-gon*, is a closed shape with $n$ straight sides where $n \geq 3$.

Clearly, we need at least three sides to make a polygon; so a triangle is the simplest polygon. If $n = 4$, we have what's called a *quadrilateral*, a four-sided polygon. Squares and rectangles are special cases of $4$-gons. We showed that the sum of the interior angles of a triangle is $180°$. It's a natural question then to ask what the sum of the interior angles of a $4$-gon is, then the sum for a $5$-gon, $6$-gon,..., or an $n$-gon.

Consider the sum of the interior angles of a $4$-gon; naturally, $360°$ comes to mind since a square or rectangle (both $4$-gons) have four $90°$ angles. But is this true for quadrilaterals without right angles like the one below? If we draw a line segment from two non-adjacent (opposite) corners, we see that the $4$-gon is made up of two triangles.

Notice that if we look at the angles of these triangles, two of them are the same as those of the original $4$-gon, while the others make up the remaining angles of the quadrilateral. In other words, if we add up all the angles of the two triangles we would get the sum of all the angles of the $4$-gon. But since we know the sum of the interior angles of a triangle is $180°$, the sum of the interior angles of two triangles must be $360°$

We may use this method of cutting up into triangles, called *"triangulating"*, to find the sum of the interior angles of any $n$-gon. So the question now amounts to: how many triangles are in a given $n$-gon?

**Group Work.** Fill in the table below and see if you see a pattern that allows you to guess a formula for any n-sided polygon.

| # of sides | sum of interior angles |
|---|---|
| $3$ | $180°$ |
| $4$ | $360°$ |
| $5$ | |
| $6$ | |
| $7$ | |
| $n$ | |

After triangulating the $5, 6, 7$-gons in the above table, it's pretty clear that the formula for the sum of the interior angles of an $n$-sided polygon is $(n-2) \times 180°$ since it'll have $(n-2)$ triangles in it.

**Definition 1.17.** A *regular $n$-gon* is an $n$-gon all of whose sides are of equal length (equilateral); and all of whose interior angles have equal measure (equiangular).

**Examples.**
A regular $3$-gon is better known as an equilateral triangle, which happens to also be equiangular. A regular $4$-gon is a square; notice that an equilateral $4$-gon is not necessarily a square as it could also be a rhombus; and an equiangular $4$-gon could be a rectangle. For this reason our definition of a regular $n$-gon requires that it be both equilateral and equiangular. A regular $5$-gon is known as a pentagon. We

will look at the pentagon some more when we study symmetry; in particular, we'll see why it's not really a good idea to design a building that has the shape of a pentagon.

It makes sense to talk about the measure of the interior angle of a regular $n$-gon as they are all the same. An equilateral triangle has interior angle measure $60°$ since it has three angles of the same measure and their sum is $180°$; thus, $180 \div 3 = 60°$. Similarly, a square has angle measure $90°$ as $360° \div 4 = 90°$. To find the interior angle measure of a pentagon, we take the sum of its interior angles and divide by five, the number of its angles: $540° \div 5 = 108°$

**Group Work.** Fill in the table below and guess the formula for any **regular** n-sided polygon.

| # of sides | interior angle of a regular polygon |
|---|---|
| $3$ | $60°$ |
| $4$ | $90°$ |
| $5$ | $108°$ |
| $6$ | |
| $7$ | |
| $8$ | |
| $9$ | |
| $n$ | |

**Definition 1.18.** A *tessellation* is a pattern made up of repeated use of the **same** geometric shape(s) to completely cover the plane without gaps or overlaps. A *strict tessellation* uses only one regular geometric shape.

**Question.** Can we get a strict tessellation with equilateral triangles? How about with squares?
The answer in both cases is obviously yes, but why? Notice that the interior angle measure of an equilateral triangle is $60°$, which is a factor of $360°$, i.e., $360° = 6 \times 60°$. Likewise the angle measure of a square is a factor of $360°$.

**Group Work.** Can we get a strict tessellation wit a pentagon? a hexagon? a heptagon (regular $7$-gon)? or an octogon (regular $8$-gon)?

**Tessellating with more than one geometric shape.**
If we relax a bit and allow two or more shapes, we get more interesting tessellations.

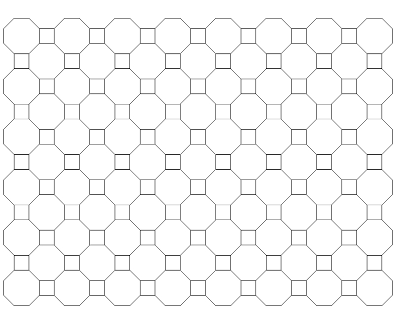


**fig.1** Tessellation by octagons and squares.

**Example.** Is it possible to get a tessellation with octogons and squares? Since we have more than one shape, we need to do more than just see if the interior angles are factors of $360°$; we need to see if there is some combination of them that will add up to $360°$. By our previous work, we know that the interior angle of an octogon is $(8-2)180°/8 = 135°$. If we have two octogons side-by-side then at a corner the sum of the two angles is $270°$, which leaves $90°$, which is exactly enough room to slide in a square. Thus, from the math, we can say that a tessellation by octagons and squares is possible. Of course, one look at the picture shows that this is indeed a familiar pattern found in bathrooms and kitchens the world over. In fact, we see it around the UM campus on tiled areas like the walkways up to the fountain in front of Cox building.

**Group work.** Can we tessellate with
1) Squares and hexagons? 2) Squares, hexagons, and equilateral triangles? 3) Squares and equilateral triangles? 4) Hexagons and equilateral triangles? 5) $12$-gons and equilateral triangles? 6) $12$-gons, hexagons, and squares?

## 1.4 Area and the Pythagorean Theorem

### 1.4.1 Area

The natural place to start would seem to be to pose the question: “What is area?” However, it turns out that it’s a lot easier to answer a different question, and that is “What does area do?” Whatever it is (we could argue this philosophical question for an entire semester) we would all have to agree that it does the following things. [1]

**Axiom 1.** If you slide a shape around, its area does not change.

**Axiom 2.** If one shape is entirely contained in another, then the area of the first is no larger than the second.

**Axiom 3.** The area of a rectangle is obtained by multiplying its two side lengths.

**Axiom 4.** If you cut a shape into a few pieces, then the areas of the pieces add up to the area of the original shape.

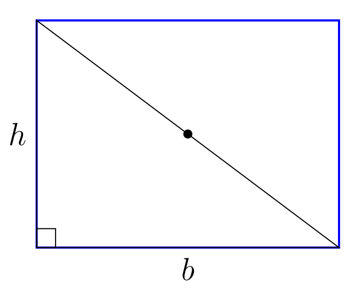
**Axiom 5.** If you expand a shape by a factor of 2 (respectively, by a factor of $\alpha$) in every direction, then its area increases by a factor of 4 (respectively, by a factor of $\alpha^2$).

From these axioms we can deduce the area of a triangle.

**Proposition 1.19.** *The area, $A$, of a right triangle is given by*

$$A = \frac{1}{2}\, b \cdot h$$

*where $b$ is the length of its base and $h$ is its height.*




PROOF: Clearly, a right triangle with base $b$ and height $h$ is contained in a rectangle of the same dimensions. Furthermore, this triangle can be slid (by a $180°$ rotation about the center of the rectangle) onto the top triangle of the rectangle. This shows that the area of the rectangle must be twice the area of the triangle. The formula follows by Axiom 3. □

In class, we then showed that using this proposition, we can demonstrate that the area of any triangle (not just a right triangle) is $\frac{1}{2}\, b \cdot h$. We did so by considering two cases: 1) where both base angles were

[1] For a more in-depth account of area together with the notion of *limit* see Timothy Gower’s *Mathematics: A Very Short Introduction*. I have adapted his presentation for our purposes.

acute (less than $90°$); and 2) where one of the base angles was obtuse (greater than $90°$. In both cases, we drop an altitude from the top angle to the base; and then consider the areas of the right triangles that appear. Thus we have the following general statement.

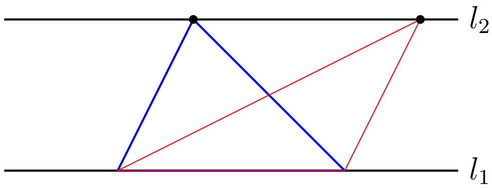
**Proposition 1.20.** *The area, $A$, of any triangle is given by*

$$A = \frac{1}{2}\, b \cdot h$$

*where $b$ is the length of its base and $h$ is its height.*

This has an immediate corollary that is extremely useful.

**Corollary 1.21.** *The area of any two triangles sharing a fixed base on a line $l_1$ with top angles on a line $l_2$ parallel to $l_1$ have the same area.*




The areas of the red and blue triangles are the equal.

### 1.4.2 The Pythagorean Theorem

Now we have the tools to show one of the most beautiful, profound, and meaningful theorems in all of mathematics!

**Theorem 1.22.** *In a right triangle, the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides. In mathematical symbols,*

$$a^2 + b^2 = c^2$$

*where $a$ and $b$ are the lengths of the sides adjacent to the right angle, and $c$ is the length of the hypotenuse.*

*Remark* 1.23*.* The statement of the theorem is about the lengths of the sides of a right triangle. People usually remember the $a^2 + b^2 = c^2$ part, but forget the *right* triangle part. Without this, $a^2 + b^2 = c^2$ is totally untrue. I mean, is "$1^2 + 2^2 = 3^2$"? No way.

Now there are many ways to show how this theorem is true, which is a good indication that people have thought about this one for millenia. Actually, people knew the statement of the theorem long before Pythagorus, so how he got *his* name attached to it is a good question. I suppose it has something to do with his awesome marketing skills[2] and that he was Greek. Since Greece is the cradle of Western civilization and since modern mathematics and science are the extension of Greek math and natural philosophy, it kind of makes sense that the lion's share of results from antiquity are attributed to Greek thinkers.

[2] Pythagorus started a cult that traveled around by ship and besides needing to be good at math, its members had to swear not to eat beans. To learn more about this interesting character and other Greek thinkers have a look at Bertrand Russel's *History of Western Philosophy*. It's quite entertaining.

**fig.2** Lun-Yi Tsai, *Change*, 2008, encaustic on plywood, 41.5 x 38.5in

Naturally, I couldn't resist showing you the proof of the Pythagorean Theorem using an encaustic[3] painting that I made in 2008. As I talked about in class, the argument repeated uses the corollary and transforms the triangles in the upper squares into parts of the larger lower square thereby showing that the area of the larger square in equal to the sum of the areas of the two smaller squares. But the squares sit on the sides of a right triangle in the center; so their areas are $a^2$, $b^2$, and $c^2$. And we have demonstrated the Pythagorean Theorem.

Now, this was Euclid's proof. It isn't the prettiest (by far), but the importance of it is that it is built up from his axioms. It doesn't just pop out of nowhere. By contrast, the other demonstration I showed you by the great Indian mathematician Bhaskara is a lot cuter but seems rather mysterious and magical.

I'll make a painting or sculpture of it someday, but for now here's a stand-in.

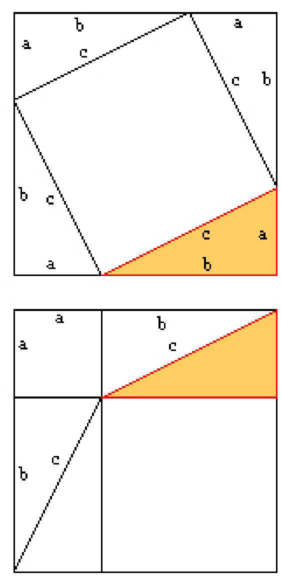



**fig.3** Bhaskara's Proof of the Pythagorean Theorem.

[3] This ancient technique of painting with pigment suspended in melted beeswax was used by the Egyptians. As long as you keep it cool, it's guaranteed to last thousands of years. The well known Fayum mummy portraits were painted in encaustic. Hopefully, long after we're all dead, gone, and forgotten, this painting of this eternal theorem (called *Change*) will still be around; and if not, there will definitely be intelligent life forms that will contemplate the Pythagorean Theorem and its meaning.