Math 113 Finite Math with a Special Emphasis on Math & Art

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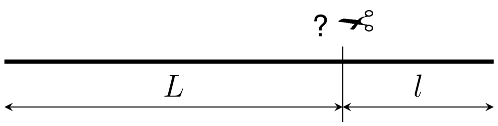
3 Hello Goodbye Golden Ratio

For the sake of completeness, we include this very brief section[1] on the Golden Ratio, which is also known as the Golden Section, the Golden Mean, etc.

fig.1 Lun-Yi Tsai, *Goodbye Golden Ratio*, 2004, oil on acrylic on canvas.

3.1 The Golden Section

Given a line segment of some arbitrary length, the question may arise as to what the most natural and most pleasing way to section (or cut) it into two unequal pieces.



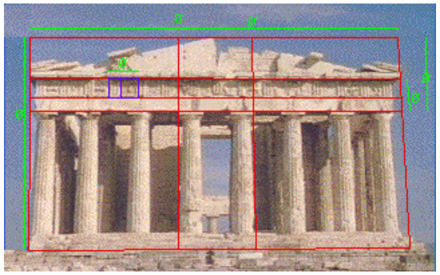


fig.2 The Parthenon in Athens, Greece.

I suppose the Greeks came up with the idea that one way might be to cut it in such a way that "the ratio of the longer piece to the shorter piece should be the same as the ratio of the original length (of the uncut piece) to the longer piece." This idea can immediately be translated into math:

$$\frac{L}{l} = \frac{L + l}{L}$$

Now we just need to solve for L. If we set $l = 1$, we not only simplify our equation, but we also get our ratio to be $L = \frac{L}{1}$. And after we solve for L, we will define it to be the Golden Section, φ (the Greek letter *phi*, which is pronounced "fee").

[1] Whenever people hear that I do math and art and they know a bit about math, their immediate reaction is: "So, you must use the Golden Ratio, φ, right? I just love the Fibonacci Sequence!" This is all well and good, but somehow I think that what happens is that these people think that that's all there is to the intersection of math and art; and though there is indeed a great wealth of fascinating and aesthetically pleasing things coming from the Golden Ratio, there is so much more to math and art than just this relatively small patch. But having said this, I must admit that I do incorporate the Golden Ratio φ in my work albeit indirectly–usually in the dimensions of a drawing, canvas, or sculpture. My only work explicitly using the Golden Ratio is called "Goodbye Golden Ratio."

Recall from high school algebra, we now multiply both sides of

$$\frac{L}{1} = \frac{L+1}{L}$$

by L to get

$$L^2 = L + 1,$$

which is a quadratic equation that we can put into *standard form*

$$L^2 - L - 1 = 0.$$

Now, to solve a quadratic, we can try to factor, complete the square, or use the quadratic formula. A moment's thought will convince us that our equation cannot be factored. So, let's just use everyone's favorite, the quadratic formula,[2] which gives us

$$L = \varphi = \frac{1 \pm \sqrt{5}}{2}.$$

Since φ is a ratio and a length, we want it to be a *positive* number; for this reason, we choose the *positive* solution $\varphi = \frac{1+\sqrt{5}}{2}$. This very special number, $\varphi \approx 1.618$, is the one that mathematically-minded poets and artists will talk about forever and ever.

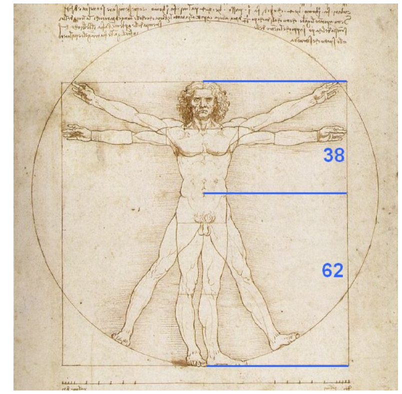
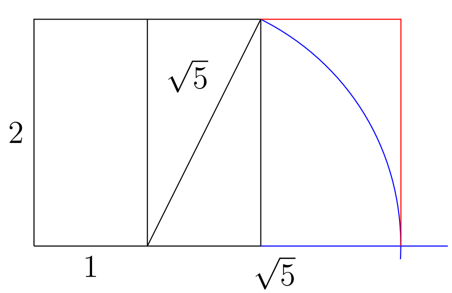


fig.3 Da Vinci's *Vitruvian Man.*

3.2 The Golden Rectangle

Let's take a square with side length 2. If we cut it down the middle, we get two vertical rectangles with side length 2 and width 1. By our old friend *The Pythagorean Theorem*, each diagonal of these rectangles has length $\sqrt{5}$. Now we we extend the base of the square to the right using a blue line segment. By drawing an arc with center the midpoint of the bottom side of the square



we get a point of intersection of the arc with the extended base. If we use this point of intersection to draw a rectangle by adding the red line segments, we get a rectangle that has length $1 + \sqrt{5}$ and width 2. Because the ratio of its sides is $\varphi = \frac{1+\sqrt{5}}{2}$, it's known as The Golden Rectangle.

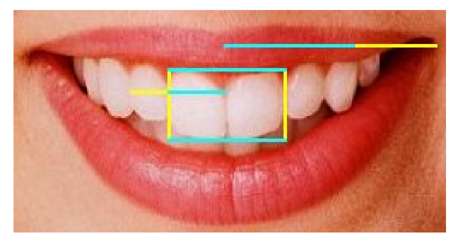


fig.4 Elements of a great smile.

[2]The *Quadratic Formula* tells us that a quadratic equation in standard form: $ax^2 + bx + c = 0$, where a, b, and c are real numbers, has solutions given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

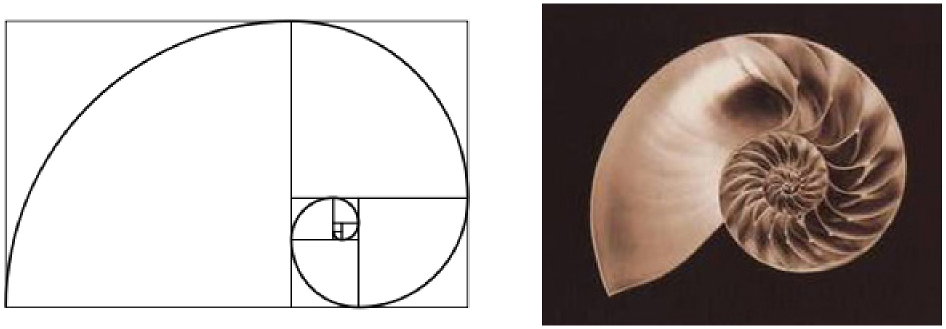


fig.5 The Golden Spiral and Nautilus Shell.

3.3 The Golden Spiral

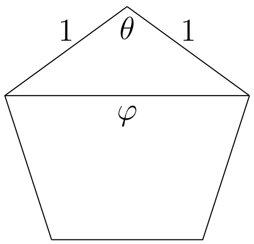
From the Golden Rectangle, we can get the "Golden Spiral," which is a special case of the "Spira Mirabilis," the marvelous spiral, which was first investigated by René Descartes, and then by Jacob Bernouilli who gave it its special name. The "Spira Mirabilis" can be found extensively in nature from the arrangement of sunflower seeds and the chambers of nautilus shells to the spiral arms of galaxies and hurricanes. It even turns out that a hawk approaches its prey along a marvelous spiral, which makes one think that perhaps we should be looking for such a path to get the solution of other life problems.

A modification of the procedure for getting the Golden Spiral from the Golden Rectangle led to my painting "Goodbye Golden Ration," which is displayed at the beginning of this chapter.

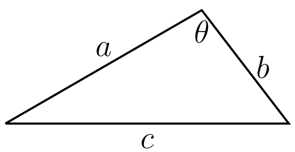
fig.6 Sunflower Seeds.

3.4 The Pentagon

As a last example of the Golden Ratio, let's take a look at the *pentagon*. We know from our study of symmetry that it has the group of symmetries $D_5 = \{I, R = R_{72^\circ}, R^2, R^3, R^4, F, FR, FR^2, FR^3, FR^4\}$ and we can write down its multiplication table as well using the formula $RF = FR^4$. These symmetries alone make the pentagon interesting, but what makes it especially magical is the presence of the Golden Ratio in it.



Let's see how φ makes its appearance. Suppose the sides of the pentagon are of length 1, then it turns out that any diagonal has length φ. To show this let's recall a theorem from trigonometry, the *Law of Cosines*[3]:



> Given a triangle with sides a, b, c with an angle θ between a and b, then $c^2 = a^2 + b^2 - 2ab\cos\theta$.

We're going to use this fact on the top triangle in the pentagon to find out the length of the diagonal. Let $a = b = 1$ and from our study of geometry we know that the interior angle of a pentagon is given by $\frac{180^\circ \times (5-2)}{5} = 108^\circ$. And thus, $c = \sqrt{2 - 2\cos 108^\circ} = \varphi \approx 1.618$.

Furthermore, each diagonal cuts (or sections) another diagonal into two pieces that have ratio again equal to φ. It should now be no wonder that the pentagon and the *pentagram* generated by it's diagonals have been such powerful symbols throughout history.

fig.7 The Pentagram comes from the diagonals of a pentagon. It has powerful religious significance.

[3] Don't worry–you don't need to remember your trigonometry!