Notes on Spivak, Differential Geometry, vol 1.

Chapter 1. Chapter 1 deals with topological manifolds. There is some discussion about more subtle topological aspects (pp. 2–7) which we can gloss over. A topological manifold M (p. 1) of dimension 1 is a metric space which is locally homeomorphic to $\mathbb{R}^n$, i.e.

$$\forall x \in M, \exists U \ni x, \phi : U \to \mathbb{R}^n, \phi \text{ homeomorphism.}$$

A manifold M with boundary (p. 23) is a metric space so that $\forall x \in M, \exists O \ni x$ with O homeomorphic to $\mathbb{R}^n$ or $\mathbb{H}^n$, where $\mathbb{H}^n = \{(x^1, \ldots, x^n \in \mathbb{R}^n, x^n \geq 0\}$.
Examples of manifolds (pp. 8–24) are S^n, cartesian products of manifolds are manifolds, the n–torus $T^n = S^1 \times \cdots \times S^1$, handle = 2–torus $\setminus$ ball.
By gluing manifolds together one can construct new examples, such as 2–holed torus, n–holed torus = sphere with n handles attached. pp. 10–11.
The Möbius strip, and the projective space P^n are examples of manifolds constructed using gluing and identification.
Let M, N be topological manifolds. A topological imbedding of M in N is a map $f : M \to N$ so that $f : M \to f(M) \subset N$ is a homeomorphism w.r.t. the induced topology on $f(M)$. A topological immersion of M in N is a mapping $f : M \to N$ so that f is locally an imbedding.
Note there is no topological imbedding of $\mathbb{P}^2$ in $\mathbb{R}^3$, but there is a topological imbedding of $\mathbb{P}^2$ i $\mathbb{R}^4$. See pp. 15–21 for a detailed discussion of this.

Problems, Chapter 1: 9,12,15,17
Note that many of the problems in Spivak are not just calculational problems, rather the formulation of the problems contain a lot of new information not given in the text itself, therefore the important thing is often to read an meditate on this information.

Chapter 2 Differentiable manifolds are defined in chapter 2. A differentiable C^∞ manifold is a topological manifold which has a maximal differentiable atlas, i.e. a maximal collection $\mathcal{A}$ of charts so that if (x, U) (y, V) are charts then $x \circ y^{-1}$ och $y \circ x^{-1}$ are C^∞ (where they are defined). Spivak calls these C^∞ related charts. A maximal differentiable atlas on M is also called a C^∞ structure on M.
C^∞ manifolds with boundary (p. 42) are defined analogously with the topolgical case. If M is an n–dimensional manfold with boundary, the boundary ∂M inherits a differentiable structure from M and is a C^∞ manifold, cf. problem 12.
The standard examples of manifolds which were discussed in chapter 1 have natural C^∞ structures, for example $\mathbb{R}^n$, S^n, $\mathbb{T}^n$, $\mathbb{P}^n$ etc.
You can cut and paste C^∞ manifolds in a similar way as you can cut and paste topological manifolds, see problem 14, chapter 2.

You can construct "lots" of C^∞ functions on a C^∞ manifold M. Lemma 2 says that if $C \subset U \subset M$ with C compact and U open, then we can find $f : M \to [0,1]$ so that $f\Big|_C = 1$ and $\operatorname{supp}(f) \subset U$.
A diffeomorphism is a C^∞ homeomorphism such that its inverse is C^∞. Note that $f : (M, \mathcal{A}) \to (N, \mathcal{B})$ is a diffeomorphism if and only if $x \circ f \in \mathcal{A} \Leftrightarrow x \in \mathcal{B}$, i.e. $x \circ f$ is a chart (at p) which is in the C^∞ structure of if and only if x is a chart (at $f(p)$) which is in the C^∞ structure of N.
You classify C^∞ manifolds up to diffeomorphism (just like you classify topological manifolds up to homeomorphism). Note that there may be several non–equivalent C^∞ structures on a given topological manifolds, eg. S^7, $\mathbb{R}^4$, cf. p. 39.
If $f : M \to N$ is C^∞ you define rank f at p as the rank of the jacobian of $y \circ f \circ x^{-1}$ where x is a chart at p and y is a chart at $f(p)$.
If $f : M^n \to N^m$ (i.e. $\dim M = n, \dim N = m$) is C^∞ we call $p \in M$ critical if rank $f < n$ at p. The set of critical points is denoted C, $q \in f(C)$ is called a critical value, $q \in N \backslash f(C)$ is called a regular value. Note that $q \in N \backslash f(M)$ is a regular value.
By using charts we can characterize set with measure sero on a manifold in an invariant way. If $A \subset M$ has measure zero and $f : M \to N$ is C^∞ then $f(M)$ has measure zero in N.
Sards theorem says that $f(C)$ has measure zero in N. The reason is that at a critical value the jacobian of f does not have full rank and therefore "one dimension is crushed", and using the smoothness of f we can do a local argument which proves the statement.
$f : M^n \to N^m$ is an immersion if rank $f \equiv n$, note that this holds if and only if $m \geq n$. $f(M)$ is then called an immersed manifold .
If $f : M \to N$ is an immersion and f is a homeomorphism onto $f(M)$ (w.r.t. the induced topology) then f is called an imbedding. Then $f : M \to f(M)$ is a diffeomorphism w.r.t. the C^∞ structure of $f(M)$ inherited from N. Note the examples of pathologies of immersions on pp. 61–63.
Given an open cover $\mathcal{O}$ of M one can always find a locally finite refinement (follows from M paracompact), you can arrange so that the refinement consists of sets diffeomorphic to $\mathbb{R}^n$ (Thm. 13).
Given a locally finite open cover $\mathcal{O}$ you can for $U \in \mathcal{O}$ find $U' \subset U$ so that $\overline{U'} \subset U$ and the collection of U' is a cover (Shrinking Lemma, Thm 14).
Let $\mathcal{O}$ be a cover of M. Then a collection of functions $\{\phi_U\}_{U \in \mathcal{O}}$ is a partition of unity subordinate to $\mathcal{O}$ if it holds that $\operatorname{supp}(\phi_U) \subset U$, for all $p \in M$, $\phi_U(p) \neq 0$ for at most finitely many U, $\phi_U(p) \in [0,1]$,

$$\sum_{U \in \mathcal{O}} \phi_U(p) \equiv 1$$

i.e a partition of unity splits the function 1 into positive functions with small support. This is an enormously important concept in analysis and makes a lot of constructions and arguments possible by localizaton and approximation.
An example of what can be done with this technique is that every manifold can be imbedded in $\mathbb{R}^N$ for some sufficiently large N, (Thm 17). A much more

difficult result is Whitneys imbedding theorem which says that every compact manifold C^∞ manifold of dimension n can be imbedded in $\mathbb{R}^{2n+1}$.

Problems, Chapter 2: 3,7,14,19,26,29,33,34

Chapter 3: Tangent space, tangent bundle

Tangent space of $\mathbb{R}^n$ (pp. 86–88): If we conside a curve in $\mathbb{R}^n$, it is natural to think of $c'(t)$ as a vector based at $p = c(t)$ instead of based at the origin, i.e. at ever $p \in \mathbb{R}^n$, there is a space of vectors "based at p", we call this the tangent space of $\mathbb{R}^n$ at p, denoted $T_p\mathbb{R}^n$.

- *here Spivak uses the notation $\mathbb{R}^n_p$ which is not standard*

It is natural to define $T\mathbb{R}^n = \cup_{p\in\mathbb{R}^n}\mathbb{R}^n_p$, one sees that $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ (homeomorphically) and that there is a natural projection $\pi : T\mathbb{R}^n \to \mathbb{R}^n$ so that $\pi^{-1}(p) \cong \mathbb{R}^n$. Thus we have a $\mathbb{R}^n$–bundle. You can show that $T\mathbb{R}^n$ is a $2n$-dimensionell C^∞ manifold in a natural way. A vectorfield on $\mathbb{R}^n$ is a *section* of $T\mathbb{R}^n$, i.e. a mapping $X : \mathbb{R}^n \to T\mathbb{R}^n$ so that $X(p) \in \mathbb{R}^n_p$, i.e. $\pi \circ X = \text{id}$.

Pushforward (s. 89-92): Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be C^∞, and let $p \in \mathbb{R}^n$. The Frechet derivative Df defines a map $f_* : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$, called *pushforward*. by $f_*v_p = Df(p)v_p$ for $v_p \in \mathbb{R}^n_p$.

Example: let 1_t denote the "standard tangent vector of $\mathbb{R}$", let $c : \mathbb{R} \to \mathbb{R}^n$ be a curve in $\mathbb{R}^n$, then $c'(t) = c_*1_t \in \mathbb{R}^n_{c(t)}$.

One sees that for $f : \mathbb{R}^n \to \mathbb{R}^m, g : \mathbb{R}^m \to \mathbb{R}^k$ we have $(g \circ f)_* = g_* \circ f_*$.

Recall the definition of pullback of functions, $h : \mathbb{R}^m \to \mathbb{R}$, $f^*h : \mathbb{R}^n \to \mathbb{R}$ given by $h \circ f$.

Let X be a vectorfield on $\mathbb{R}^n$. Then we have $X(f^*h) = (f_*X)h$, i.e. f_*X definierar defines a vectorfield on $\mathbb{R}^m$.

Tangent space of imbedding (pp. 92–96):

Let M be an n–dimensional manifold and let $i : M \to \mathbb{R}^N$ be an imbedding. For $p \in M$ and a chart (x, U) at p we have

$$(i \circ x^{-1})_*\Big|_{x(p)} : \mathbb{R}^n_{x(p)} \to \mathbb{R}^M_{i(p)}$$

and one sees that the range of $(i \circ x^{-1})_*\big|_{x(p)}$ has dimension n since x is a chart. Let (y, V) be another chart at p. Due to $(i \circ x^{-1})_* = (i \circ y^{-1})_* \circ (y \circ x^{-1})_*$ and rank $y \circ x^{-1} = n$ we have that the range of $(i \circ x^{-1})_*\big|_{x(p)}$ is independent of the choice of chart. We denote it $T_p(M, i)$ and and define $T(M, i) = \cup_{p\in M}T_p(M, i)$.

Vectorbbundle, bundle map, equivalence, section (pp. 96–114):

Do not be confused by the definition of n–plane bundle, pp. 96–97.

Let V be a vector space. A manifold B together with a map $\pi : B \to M$ (π is called projection, assumed to be continuous and onto) is called a *vector bundle* with total space B, base M and fiber V if it holds that

- $\forall p \in M : \pi^{-1}(p) \cong V$ (as vectorspace).
- $\forall p \in M\ \exists U \ni p : \pi^{-1}(U) \cong U \times V$.

One checks easily that these conditions are satisfied for $T(M, i)$, which thus is a vector bundle with fiber $\mathbb{R}^n$. $T(M, i)$ is the prototype of TM *the tangent*

bundle of M. Thm. 1, p. 101 shows that there is a bundle TM assiciated to every C^∞ manifold M and that to every C^∞ map, there is a natural bundle map given by pushforward.
The theorem uses the definition in point 1 below. I prefer to use the definition given by 3 below, cf. Theorem 3.
You often use the notation $\xi = \pi : B \to M$ and refer to the bundle as ξ.
Let $\xi_i = \pi_i : B_i \to M_i$ be vector bundles.
A map $\tilde{f} : B_1 \to B_2$ is a *bundle map* if there is a map $f : M_1 \to M_2$ so that $f \circ \pi_1 = \pi_2 \circ \tilde{f}$ (i.e. $\tilde{f}$ maps fibers to fibers) and $\tilde{f} : \pi_1^{-1}(p) \to \pi_2^{-1}(f(p))$ is linera.
A bundle map $\tilde{f} : B_1 \to B_2$ is called a bundle equivalence if the corresponding $f : M_1 \to M_2$ is a homeomorphism and $\tilde{f} : \pi_1^{-1}(p) \to \pi_2^{-1}(f(p))$ is an isomorphism. We write $\xi_1 \cong \xi_2$.
A vector bundle $\xi = \pi : B \to M$ with fiber $\mathbb{R}^k$, (k is called the *fiberdimension*) is called *trivial* if $B \cong M \times \mathbb{R}^n$.
Let $\xi = \pi : B \to M$ be a vector bundle. A map $s : M \to B$ is called a *section* of ξ if $\forall p \in M, s(p) \in \pi^{-1}(p)$, i.e. if $\pi \circ s = \text{id}$.
A vector bundle with fiber $\mathbb{R}^k$ is trivial if and only if there is a global choice of k linearly independent sections.
All the above has its natural differentiable correspondence.
TM is in general not (globally) trivial, for example on S^2, every vector field has a zero, on the M̈bius strip there is one nonvanishing vector field but not two linearly independet ones.
You can show TS^n is trivial if and only if $n = 1,\ n = 3$ or $n = 7$.
Tangent vectors of M as derivations (pp. 106–111, Thm 3): Let M be an n–dimensional C^∞ manifold. We want to get at $TM = \cup_{p \in M} T_pM$ in a more direct way. There is (at least) 3 different ways to define T_pM without referring to an imbedding.

1. (1) equivialence classes of vectors: let $p \in M$, let (x, U), (y, V) be charts at p, define an equivalenc relation by $(p, x, v) \sim (p, y, v')$ iff $(y \circ x^{-1})_* v = v'$. Equivalence classes of charts and vectors w.r.t. $\sim$ can be shown to be n–dimensional vector spaces, this gives T_pM¿.
2. (2) Equivalence classes of curves: let c_1, c_2 be curves through p, can assume $c_1(0) = c_2(0) = p$. Define $c_1 \sim c_2$ if $(x \circ c_1)'(0) = (x \circ c_2)'(0)$ for all charts x at p. This gives an n–dimensional vector space T_pM.
3. (3) The set of all derivations at $p \in M$ of C^∞ functions on M can be shown to be isomorphic to $\mathbb{R}^n$, this gives T_pM.

The (in my opinion) most interesting way of defining the tangent space T_pM is by point 3 above.
Let $\mathcal{F}$ denote the space of (locally defined) C^∞ functions on M. $\mathcal{F}$ r en "sheaf", the so–called structure sheaf, this is a concept that is important also in the theory of complex manifolds and in algebraic geometry. If you know $\mathcal{F}$, then you know the C^∞ (complex-analytic, algebraic) structure on M.
A derivation of $\mathcal{F}$ is a linear map $\ell : \mathcal{F} \to \mathcal{F}$ so that $\ell(fg) = (\ell f)g + f(\ell g)$ (Leibniz' rule). One shows that if $\mathbf{1}(p) = 1$ denotes the constant function 1, then $\ell \mathbf{1} = 0$ and that if $f = g$ in an open neighborhood U, then $\ell f = \ell g$ i U.

Now you can define T_pM, the tangent space at $p \in M$, as the set of derivations at p of $C^\infty(M)$. One shows this becomes an n–dimensional vector space and we have a local expression $\ell = \sum_{i=1}^n \ell(x^i)\partial_{x^i}$. Thus a basis for T_pM is given by the coordinate derivatives $\{\partial_{x^i}\}_{i=1}^n$.
Now we define the tangent bundle of M as $TM = \cup_{p\in M} M_p$. Then we have a naturall projection $\pi : TM \to M$. One shows that TM is a C^∞ manifold of dimension $2n$ and that TM is locally trivial.
The C^∞ structure on p TM:
A brief decription of the C^∞ structure of TM is given by the following: Let $X \in TM, p = \pi(X), (x, U)$ be a chart at p. For $Y \in \pi^{-1}(U)$, define functions $Y^1, \dots Y^n$ by $Y = \sum_{i=1}^n Y^i \partial_{x^i}$. Then we can define a map (local trivialization) $\psi_x : \pi^{-1}(U) \to U \times \mathbb{R}^n \cong \mathbb{R}^{2n}$ by $\psi_x(Y) = (\pi(Y), Y^1, \dots, Y^n) \in U \times \mathbb{R}^n$. These local trivializations together with the C^∞ structure of M define a C^∞ structure on TM. One sees that if x, y are charts at p, then $\psi_x \circ \psi_y^{-1} = \mathrm{id} \times \psi_{xy}$ where $\psi_{xy}(p) = (x \circ y^{-1})_*(p) \in Gl(n)$ are C^∞. Further, $\psi_{xy} = \psi_{yx}^{-1}$ samt $\psi_{xy}\psi_{yz} = \psi_{xz}$ (the cocycle condition)
Pushforward on manifolds: Now we can move the definition of pushforward to manifolds. If $f : M \to N$ is C^∞ we define a C^∞ bundle map $f_* : TM \to TN$ by using the definition of pushforward from $\mathbb{R}^n$ and work locally in charts.
Vector fields: C^∞ vektorfields are C^∞ sections of TM (this is well defined as TM has a C^∞ structure). The space of C^∞ sections of TM can be denoted $C^\infty(TM)$.
Let $X, Y \in C^\infty(TM)$ and $f \in \mathcal{F}$. Then we have naturally defined $X + Y \in C^\infty(TM)$, $fX \in C^\infty(TM)$. In terms of local coordinates, we have $X = \sum_i X^i \partial_{x^i}, Y = \sum_i Y^i \partial_{y^i}$, $X + Y = \sum_i (X^i + Y^i)\partial_{x^i}$, $fX = \sum_i fX^i \partial_{x^i}$.
Note that vector fields act as derivations on $\mathcal{F}$: if $f, g \in \mathcal{F}, X \in C^\infty(TM)$ then $X(fg) = (Xf)g + f(Xg)$. Every derivation (or partial differential operator of order 1) on $\mathcal{F}$ can be identified with a vector field.
We will return to the concept orientation of TM in connection with differential forms.

Problems, Chapter 3: 14,15,17,18,19,24,25,26,32