(1) (a) Find the determinant of the matrix
\[ A = \begin{bmatrix}
3 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 2 & 1 & 0 \\
3 & 1 & 0 & -1
\end{bmatrix}. \]
(b) Is \( A \) invertible? If so, what is \( \det(A^{-1}) \)? If not, why not?
(2) Quickly, what is the determinant of
\[ A = \begin{bmatrix}
0 & 0 & 2 & 1 & -2 & 2 & 9 \\
0 & 0 & 1 & 10 & -7 & 0 \\
0 & 0 & 0 & 0 & 1 & 93 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}? \]
(3) What numbers arise as determinants of orthogonal matrices? Justify your answer.
(4) (a) How can you easily conclude that the matrix
\[ B = \begin{bmatrix}
3 & 1 & 3 & 8 \\
9 & -2 & 9 & 1 \\
-7 & 5 & -77 & 11 \\
2 & 1 & 22 & -8
\end{bmatrix} \] is singular?
(b) Why does this tell you that 4 is an eigenvalue of
\[ A = \begin{bmatrix}
7 & 1 & 33 & 8 \\
9 & 2 & 99 & 1 \\
-7 & 5 & -73 & 11 \\
2 & 1 & 22 & -4
\end{bmatrix}? \]
(c) Use this to find an eigenvector of \( A \) with eigenvalue 4. Then find a unit eigenvector of \( A \) with
eigenvalue 4.
(5) (a) What are the eigenvalues of
\[ A = \begin{bmatrix}
3 & 2 \\
3 & -2
\end{bmatrix}? \]
(b) Find an eigenvector for each eigenvalue; describe the eigenspace for each eigenvalue.
(c) Use this information to diagonalize \( A \).
(d) Repeat this for the matrix
\[ M = \begin{bmatrix}
3 & 2 & 0 \\
3 & -2 & 0 \\
0 & 0 & 4
\end{bmatrix}. \]
(6) The matrix
\[ A = \begin{bmatrix}
1 & 0 & 3 \\
0 & -2 & 0 \\
3 & 0 & 1
\end{bmatrix} \]
is symmetric. Find an orthonormal basis for \( \mathbb{R}^3 \) consisting of
eigenvectors of \( A \). Use this to construct a diagonalization \( A = Q\Lambda Q^T \).
(7) The Singular Value Decomposition of
\[ A = \begin{bmatrix}
1 & 0 & -1 & -1 \\
2 & -2 & 0 & 2
\end{bmatrix} \]
is:
\[ A = U \Sigma V^T \]
\[ \begin{bmatrix}
1 & 0 & -1 & -1 \\
2 & -2 & 0 & 2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
2\sqrt{3} & 0 & 0 & 0 \\
0 & \sqrt{3} & 0 & 0
\end{bmatrix} \begin{bmatrix}
1/\sqrt{3} & -1/\sqrt{3} & 0 & 1/\sqrt{3} \\
1/\sqrt{3} & 0 & -1/\sqrt{3} & -1/\sqrt{3} \\
1/\sqrt{6} & 2/\sqrt{6} & 0 & 1/\sqrt{6} \\
1/\sqrt{6} & 0 & 2/\sqrt{6} & -1/\sqrt{6}
\end{bmatrix} \]
(a) What are the eigenvalues of \( A^T A \)? What are the eigenvalues of \( AA^T \)?
(b) What is the closest rank 1 approximation to \( A \)?
(c) Using this SVD, give an orthonormal basis for the null space of \( A \).
(d) What is the pseudoinverse \( A^+ \) of \( A \)?
(8) Let \( P_2 \) be the vector space of polynomials of degree at most 2. That is, \( P_2 = \{ ax^2 + bx + c | a, b, c \in \mathbb{R} \} \).
(a) Show that the transformation \( T: P_2 \to P_2 \) that takes a polynomial \( p(x) \) to the polynomial \( p(x-1) \) is a linear transformation.
(b) Choose a basis for \( P_2 \) and find the associated matrix of this linear transformation.