AK-invariant of affine domains

Sh. Kaliman and L. Makar-Limanov *

Dedicated to Professor Miyanishi

Abstract

The AK-invariant of an affine domain is the intersection of kernels of locally nilpotent derivations on this domain. It allows often to distinguish exotic structures from Euclidean spaces, especially in dimension 3. In this paper we develop methodology for computation of AK-invariant and give examples of its applications.

2000 Mathematics Subject Classification: 1405, 14R10, 14R20.
Key words: $\mathbf{C}_+$ -action, locally nilpotent derivation, contractible affine hypersurface

1 Introduction

Let A be an affine domain over $\mathbf{C}$ and $\mathrm{LND}(A)$ be the set of all locally nilpotent derivations of A . We study methods of computation of the following invariant

$$\mathrm{AK}(A) = \bigcap_{\partial \in \mathrm{LND}(A)} \mathrm{Ker}\ \partial.$$

This invariant was introduced in [M-L96] where it was used to distinguish the Russell cubic (which is the smooth contractible hypersurface $x + x^2y + z^2 + t^3 = 0$ in $\mathbf{C}^4$) from $\mathbf{C}^3$ while old invariants failed to do so. It turned out that the AK-invariant (or its modification [De97]) is the only known tool to show that some contractible affine algebraic manifolds are exotic algebraic structures on $\mathbf{C}^n$ (i.e. they are diffeomorphic to $\mathbf{R}^{2n}$ but not isomorphic to $\mathbf{C}^n$) [KaM-L97b, KaZa00, Ka02]. Since the task of recognizing exotic structures appears naturally in several classical problems of affine algebraic geometry [Za97] the AK -invariant has some nice applications. The most essential of them is the contribution [KaM-L97b] to the positive solution of the Linearization Conjecture which was eventually achieved by M. Koras and P. Russell [KoRu99, KaKoM-LRu97]. Another result obtained with the help of the AK-invariant is the classification [Ka02] of smooth

*Both authors were partially supported by NSA grants.

contractible affine threefolds with a non-degenerate $\mathbf{C}_+^2$-action which in combination with [KaZa01] yields the following theorem: every polynomial on $\mathbf{C}^3$ with general $\mathbf{C}^2$-fibers is a variable. Unfortunately, none of these papers contain a coherent technique of computation of $\mathrm{AK}(A)$ for a sufficiently large class of affine domains A. Our aim here is to fix this situation.

Here is the scheme of computation of $\mathrm{AK}(A)$. An associated algebra $\widehat{A}$ with a natural mapping $\mathrm{gr}\ : A \to \widehat{A}$ is constructed in such a way that for every nonzero $\partial \in \mathrm{LND}(A)$ there is a canonically defined associated nonzero $\widehat{\partial} \in \mathrm{LND}(\widehat{A})$ with the following property

$$\deg_\partial(a) \geq \deg_{\widehat{\partial}}(\mathrm{gr}\ (a))\ \ \forall a \in A$$

where $\deg_\partial$ (resp. $\deg_{\widehat{\partial}}$) is the degree function on A (resp. $\widehat{A}$) generated by ∂ (resp. $\widehat{\partial}$) [FLN92]. In many cases it is easier to study $\mathrm{LND}(\widehat{A})$ than $\mathrm{LND}(A)$. Knowing $\mathrm{LND}(\widehat{A})$ and using the inequality on degrees it is sometimes possible to compute $\mathrm{AK}(A)$. In our previous papers $\widehat{A}$ appeared as the associated graded algebra of a filtered algebra $(A, \mathcal{F})$ where the filtration $\mathcal{F}$ was generated by a degree function on A. Here we show that in fact $\widehat{A}$ can be viewed as the algebra of regular function on an associated affine algebraic variety $\widehat{X}$. This approach enables us to give an explicit construction of the associated locally nilpotent derivation $\widehat{\partial}$ after replacing ∂ by an equivalent derivation (i.e. by a locally nilpotent derivation with the same kernel). We suggest also an alternative more geometrical way of constructing $\widehat{A}$. Suppose that $X := \mathrm{spec} A$ is normal and Γ is a germ of a smooth curve at point o. We consider the germ of a normal affine algebraic variety $\mathcal{X}$ with morphism $\rho : \mathcal{X} \to \Gamma$ so that $\mathcal{X}^* = \mathcal{X} \setminus \rho^{-1}(o)$ is naturally isomorphic to $X \times \Gamma^*$ with $\Gamma^* = \Gamma \setminus o$. We shall show that if the fiber $\rho^*(o)$ is reduced then the algebra of regular functions on it may serve as $\widehat{A}$. It is worth mentioning that in [Po86] associated objects appear in a somewhat similar manner. However our construction is essentially different.

We conclude the paper with several examples. In section 10 we consider a smooth surface with Kodaira logarithmic dimension 1. It was shown in [KaM-L97a] that every such a surface is isomorphic to a hypersurface in $\mathbf{C}^3_{x,y,z}$ given by a “standard” equation $P(x, y, z) = 0$. Consider hypersurfaces in $\mathbf{C}^4_{x,y,z,t}$ given by $t^l - P(x, y, z) = 0,\ l \geq 2$. Some of them (but not all) are contractible and, therefore, diffeomorphic to $\mathbf{R}^6$ [ChDi94]. We show that the AK-invariant of such a contractible hypersurface is nontrivial, i.e. it is an exotic structure on $\mathbf{C}^3$. In section 11 we prove a similar result for the hypersurface $x + x^2y + z^2 + t^3 + u^5 = 0$ in $\mathbf{C}^5_{x,y,z,t,u}$ which is diffeomorphic to $\mathbf{R}^8$. (We attribute the question about this hypersurface to M. Masuda who asked the first author about it long ago. But this question is so natural that it occurred certainly to other people.) One can find more complicated examples (different from hypersurfaces) in [Ka02] but we avoid them in this paper since computations are a bit tedious.

2 Preliminaries

Notation. In this paper A will be always the algebra of regular functions $\mathbf{C}[X]$ on a closed reduced affine algebraic subvariety X of $\mathbf{C}^n$. The defining ideal of X will be denoted by I , i.e. $A = \mathbf{C}^{[n]}/I$. Beginning from Lemma 2.2 we shall suppose that X is irreducible, i.e. A is an affine domain.

Definition 2.1. Recall that a derivation ∂ on A is a linear endomorphism which satisfies the Leibnitz rule, i.e. $\partial(ab) = a\partial(b) + b\partial(a)$. This derivation is called locally nilpotent if for each $a \in A$ there exists an $m = m(a)$ such that $\partial^m(a) = 0$. The set of all locally nilpotent derivations of A will be denoted by $\mathrm{LND}(A)$. We shall denote the kernel of ∂ by A^∂ .

The exponent of $t\partial$ where $\partial \in \mathrm{LND}(A)$, $t \in \mathbf{C}$ can be viewed as an (associated) algebraic $\mathbf{C}_+$ -action on X , and every algebraic $\mathbf{C}_+$ -action on X arises in this way (e.g., see [Re68]). Note that $a \in A^\partial$ iff a is fixed under the associated $\mathbf{C}_+$ -action. In particular, we have a more geometrical description of the AK-invariant: it is the subring of A whose elements are the regular functions that are fixed under all algebraic $\mathbf{C}_+$ -actions. The associated action is nontrivial iff ∂ is nonzero. Hence we have

Lemma 2.1. *Let $\partial \in \mathrm{LND}(A)$ be nonzero and $\varphi = (a_1, \ldots, a_k) : X - \to \mathbf{C}^k$ be a rational map with each a_i ($i = 1, \ldots, k$) in the fraction field $\mathrm{Frac}(A^\partial)$. Then the fibers $\varphi^{-1}(c), c \in \mathrm{Im}\,\varphi \subset \mathbf{C}^k$ are invariant under the associated $\mathbf{C}_+$ -action and for a general $c \in \varphi(X)$ the restriction of this action to $\varphi^{-1}(c)$ is nontrivial.*

Definition 2.2. We shall call such a restriction of the action to a general fiber $\varphi^{-1}(c)$ *a specialization* of $a_1, \ldots, a_k$.

If A is a domain every locally nilpotent derivation defines a degree function $\deg_\partial$ on A with natural values [FLN92] given by the formula $\deg_\partial(a) = \max\{m \mid \partial^m(a) \neq 0\}$ for every nonzero $a \in A$. This implies

Lemma 2.2. *Let the assumption of Lemma 2.1 hold, A_c be the algebra of regular functions on $X_c = \varphi^{-1}(c)$ for $c \in \mathrm{Im}\,\varphi$, and $a_c = a|_{X_c}$. Denote by ∂_c the locally nilpotent derivation on A_c generated by ∂ . Then $\deg_\partial(a) = \deg_{\partial_c}(a_c)$ for general $c \in \mathrm{Im}\,\varphi$.*

Proposition 2.1. [M-L96, KaM-L97b, Ka02] *Let $\partial \in \mathrm{LND}(A)$ be nonzero.*

(1) *A has transcendence degree one over A^∂ . The field $\mathrm{Frac}(A)$ of fractions of A is a purely transcendental extension of $\mathrm{Frac}(A^\partial)$, and A^∂ is algebraically closed in A .*

(2) *Let $b \in A$ and $\deg_\partial(b) = 1$. Then for every $a \in A$ such that $\deg_\partial(a) = m$ there exist $a', a_0, a_1, \ldots, a_m \in A^\partial$ for which $a', a_m \neq 0$ and $a'a = \sum_{j=0}^m a_j b^j$.*

(3) *Suppose $a_1, a_2 \in A$. Then $a_1 a_2 \in A^\partial \setminus \{0\}$ implies $a_1, a_2 \in A^\partial$. In particular, every unit $u \in A$ belongs to A^∂ .*

(4) *Let $F = (f_1, \ldots, f_s) : X \to Y \subset \mathbf{C}^s$ and $G : Y \to Z \subset \mathbf{C}^j$ be dominant morphisms of reduced affine algebraic varieties. Put $H = G \circ F = (h_1, \ldots, h_j) : X \to Z$. Suppose that for general point $\xi \in Z$ there exists a (Zariski) dense subset T_ξ of $G^{-1}(\xi)$ such that the image of any non-constant morphism from $\mathbf{C}$ to $G^{-1}(\xi)$ does not meet T_ξ . If $h_1, \ldots, h_j \in A^\partial$ then $f_1, \ldots, f_s \in A^\partial$.*

The next fact is an immediate consequence of (4).

Corollary 2.1. *Suppose that an irreducible polynomial $p \in \mathbf{C}[x, y]$ is not a variable in any polynomial coordinate system on $\mathbf{C}^2$ (resp. none of a nonzero fibers of p contains* *a polynomial curve) and morphism* $(a_1, a_2) : X \to \mathbf{C}^2$ *is dominant for* $a_1, a_2 \in A$ *(resp.* $p(a_1, a_2) \neq 0$ *). Let* $p(a_1, a_2) \in A^\partial$ *. Then* $a_1, a_2 \in A^\partial$ *. In particular, if* $a_1^2 + a_2^3 \in A^\partial \setminus \{0\}$ *then* $a_1, a_2 \in A^\partial$ *.*

Definition 2.3. Two derivations are called *equivalent* if they have the same kernel.

Lemma 2.3. (cf. Prop. 1.2 from [Da97]) *Every two locally nilpotent derivations* ∂ *and* δ *on* A *are equivalent iff they generate the same degree function. Furthermore, there exist* $\alpha, \beta \in A^\partial \setminus 0$ *such that* $\alpha\partial = \beta\delta$ *.*

Proof. If ∂ and δ generate the same degree function they have the same kernel which consists of elements of degree zero. Let ∂ and δ be equivalent. We show that they generate the same degree function. Let $b \in A$ and $\deg_\partial(b) = 1$. By Proposition 2.1(2) it suffices to show that $\deg_\delta(b) = 1$, that is, $\alpha := \delta(b) \in A^\partial = A^\delta$. Assume that $m = \deg_\partial(\alpha) > 0$. Then there exist $a' \neq 0, a_0, a_1, \ldots, a_m \in A^\partial$ for which $a'\alpha = \sum_{j=0}^m a_j b^j$. Hence $a'\delta(\alpha) = \alpha \sum_{j=1}^m j a_j b^{j-1}$ which implies that $\deg_\delta(\delta(\alpha)) \geq \deg_\delta(\alpha)$. Contradiction. For the second statement we may put $\alpha = \delta(b)$ and $\beta = \partial(b)$. □

A mapping $D : A^j \to A$ is a multiderivation of rank j if D is antisymmetric and D is a derivation in any argument. That is, if we fix, say $a_1, \ldots, a_{j-1} \in A$, then $\partial(a) = D(a_1, \ldots, a_{j-1}, a)$ is a derivation on A . When j is the transcendence degree of A we say that D is of maximal rank. The following fact is simple.

Lemma 2.4. *Let* $a_1, \ldots, a_j \in A$ *be algebraically dependent. Then for every multiderivation* D *of rank* j *we have* $D(a_1, \ldots, a_j) = 0$ *. Furthermore, if* D *is nonzero and of maximal rank then* $D(a_1, \ldots, a_j) = 0$ *iff* $a_1, \ldots, a_j \in A$ *are algebraically dependent.*

The proof of the following fact is the exact repetition of Lemma 6 from [M-L96].

Lemma 2.5. *Let* D *be a nonzero multiderivation of maximal rank and let* $\partial(a) = D(a_1, \ldots, a_{j-1}, a)$ *for some algebraically independent* $a_1, \ldots, a_{j-1} \in A$ *(in particular,* ∂ *is nonzero). Let* $b_1, \ldots, b_{j-1} \in A^\partial$ *be algebraically independent. Put* $\delta(a) = D(b_1, \ldots, b_{j-1}, a)$ *. Then* δ *and* ∂ *are equivalent.*

Let $(x_1, \ldots, x_n)$ be a coordinate system in $\mathbf{C}^n \supset X$ and $\tilde{x}_i = x_i|_X$ for every i . Suppose that $\dim X = k$ and that X_0 is the subset of X such that $(\tilde{x}_1, \ldots, \tilde{x}_k)$ is a local holomorphic coordinate system at each point of X_0 . Without loss of generality one can assume that X_0 is a Zariski dense subset of X . Let $a_1, \ldots, a_k$ be elements of A . Consider the matrix $\{\partial a_j / \partial \tilde{x}_i \mid i, j = 1, \ldots, k;\}$ of regular functions on X_0 . Denote the determinant of this matrix by $J^X(a_1, \ldots, a_k)$. Let $J(p_1, \ldots, p_n)$ be the Jacobian of n polynomials $p_i \in \mathbf{C}^{[n]}$ with respect to $(x_1, \ldots, x_n)$ and $\pi : \mathbf{C}^{[n]} \to A = \mathbf{C}^{[n]}/I$ be the natural projection. Set $J_X(\mathcal{P})$ equal to $\pi(J(\mathcal{P}, x_1, \ldots, x_k))$ where $\mathcal{P} = (P_1, \ldots, P_{n-k})$ is an ordered sequence of polynomials on $\mathbf{C}^n$.

Lemma 2.6. *Let* $\mathcal{P}$ *and* a_i *be as before and* $P_i \in I$ *for every* i *. Put* $a_i = \pi(p_i)$ *. Then* $\pi(J(P_1, \ldots, P_{n-k}, p_1, \ldots, p_k))$ *coincides up to a sign with* $J_X(\mathcal{P}) J^X(a_1, \ldots, a_k)$ *. In particular, the last product is a regular function on* X *and, therefore,* $D(a_1, \ldots, a_k) := J_X(\mathcal{P}) J^X(a_1, \ldots, a_k)$ *is a multiderivation on* A *of maximal rank.*

Proof. If the gradients of $P_1, \ldots, P_{n-k}$ are linearly dependent at general points of X then one can see that both $\pi(J(P_1, \ldots, P_{n-k}, p_1, \ldots, p_k))$ and $J_X(\mathcal{P})$ are zeros. Thus we assume that the gradients are linearly independent at a general point z of X_0. Let $U \subset \mathbf{C}^n$ be a neighborhood of z where $(\mathcal{P}, x_1, \ldots, x_k)$ is a local holomorphic coordinate system. Consider the Jacobian of $(\mathcal{P}, p_1, \ldots, p_k)$ with respect to this system. The restriction of this Jacobian to $X \cap U$ is $J^X(a_1, \ldots, a_k)$. Since $\pi(J(\mathcal{P}, x_1, \ldots, x_k))$ coincides with $J_X(\mathcal{P})$ the Chain rule implies the desired conclusion. $\square$

Definition 2.4. Let D be as in Lemma 2.6. The derivation $\partial(a) = D(a_1, \ldots, a_{k-1}, a)$ (with fixed $a_1, \ldots, a_{k-1}$) will be called a derivation of Jacobian type. The set of all locally nilpotent derivations of A of Jacobian type will be denoted by $\mathrm{JLND}(A)$.

Proposition 2.2. [M-L04] *For every $\partial \in \mathrm{LND}(A)$ there exists an equivalent $\delta \in \mathrm{JLND}(A)$. In particular, $\mathrm{AK}(A) = \bigcap_{\partial \in \mathrm{JLND}(A)} A^\partial$.*

Remark 2.1. In general the choice of the sequence $\mathcal{P}$ which appears in an equivalent locally nilpotent derivation of Jacobian type depends on $\partial \in \mathrm{LND}(A)$. But if X is a strict complete intersection given by $P_1 = \ldots = P_{n-k} = 0$ then $\delta(a) = J_X(P_1, \ldots, P_{n-k}) J^X(a_1, \ldots, a_{k-1}, a)$ with $a_1, \ldots, a_{k-1} \in A^\partial$ is always locally nilpotent and it is equivalent to ∂ provided $a_1, \ldots, a_{k-1}$ are algebraically independent. This can be proven exactly in the same manner as it was done for hypersurfaces in [KaM-L97b].

3 Degree and semi-degree functions

Each nonzero polynomial p is the sum of nonzero monomials, and the set of these monomials will be denoted by $M(p)$.

Definition 3.1. A *weight degree function* on the polynomial algebra $\mathbf{C}^{[n]}$ is a real-valued degree function d such that $d(p) = \max\{d(m) \mid m \in M(p)\}$, where $p \in \mathbf{C}^{[n]}$ is a non-zero polynomial. Clearly, d is uniquely determined by the *weights* $d_i := d(x_i) \in \mathbf{R}$, $i = 1, \ldots, n$. A weight degree function d defines a grading $\mathbf{C}^{[n]} = \oplus_{t \in \mathbf{R}} \mathbf{C}^{[n]}_{d,t}$, where $\mathbf{C}^{[n]}_{d,t} \setminus \{0\}$ consists of all the $d-$ homogeneous polynomials of $d-$ degree t. Accordingly, for any $p \in \mathbf{C}^{[n]} \setminus \{0\}$ we have a decomposition $p = p_{t_1} + \ldots + p_{t_l}$ into a sum of $d-$ homogeneous components p_{t_i} of degree t_i where $t_1 < t_2 < \ldots t_l = d(p)$. We call $\bar{p} := p_{d(p)}$ the *principal component* of p. It is clear that $\overline{pq} = \bar{p}\bar{q}$.

Let $\widehat{I}$ be the graded ideal in $\mathbf{C}^{[n]}$ generated by the principal components $\bar{p}$, where p runs over I.

Lemma 3.1. *There exists $c_0 > 0$ such that for every d-homogeneous $\bar{q} \in \widehat{I}$ there exists $q \in I$ whose principal component is $\bar{q}$ and such that $d(q) - d(q - \bar{q}) \geq c_0$.*

Proof. Let $q_1, ..., q_l$ be elements of I such that their principal components $\bar{q}_1, ..., \bar{q}_l$ are generators of $\widehat{I}$. Let $c_0 = \min_{1 \leq i \leq l}(d(q_i) - d(q_i - \bar{q}_i))$. If $r_1, ..., r_l$ are $d-$ homogeneous polynomials (i.e. $\bar{r}_i = r_i$) such that polynomial $\bar{q} = \sum_{i=1}^l r_i \bar{q}_i$ then $d(r_i q_i) - d(r_i q_i - \bar{r}_i \bar{q}_i) \geq c_0$ and $d(r_i q_i) = d(q)$ for every i. Hence $d(q) - d(q - \bar{q}) \geq c_0$ for $q = \sum_{i=1}^l r_i q_i$. $\square$

We say that a function $s : A \longrightarrow \mathbf{R} \bigcup \{-\infty\}$ is a *semi-degree* function if in the standard definition of a degree function on A (e.g., see [Za97]) the equality $s(ab) = s(a) + s(b)$ is replaced by the inequality $s(ab) \leq s(a) + s(b)$ for all $a, b \in A$.

Lemma 3.2. *Let d be a weight degree function on $\mathbf{C}^{[n]}$ and $\pi : \mathbf{C}^{[n]} \to A = \mathbf{C}^{[n]}/I = \mathbf{C}[X]$ be the natural projection and let X contain the origin $\bar{0}$ of $\mathbf{C}^n$. For $a \in A \setminus \{0\}$ set $d_A(a) = \inf_{p \in \pi^{-1}(a)} d(p)$. For every nonzero $a \in A$ we have*

(1) *there exists a polynomial $p \in \pi^{-1}(a)$ such that $\bar{p} \notin \widehat{I}$;*

(2) *$d_A(a) = d(q)$ for a polynomial $q \in \pi^{-1}(a)$ iff $\bar{q} \notin \widehat{I}$. In particular, $d_A(a) = \min_{q \in \pi^{-1}(a)}\{d(q)\}$;*

(3) *d_A is a semi-degree function and, moreover, $d(ab) < d(a) + d(b)$ for $a, b \in A$ only in the case when there exist $p \in \pi^{-1}(a)$ and $q \in \pi^{-1}(b)$ such that $\bar{p}, \bar{q} \notin \widehat{I}$ and $\bar{p}\bar{q} \in \widehat{I}$. In particular, if $\widehat{I}$ is prime then d_A is a degree function on A.*

Proof. It is clear that $d_A(a) = d(q)$ for $q \in \pi^{-1}(a)$ when $\bar{q} \notin \widehat{I}$, and $d_A(a) < d(q)$ when $\bar{q} \in \widehat{I}$. Thus (2) follows from (1).

In order to prove (1) we show first that for nonzero $a \in A$ we have $d_A(a) \neq -\infty$. Assume the contrary. That is, there exists a sequence $p_j \in \mathbf{C}^{[n]}$, $j = 1, \ldots,$ such that $p_j \in \pi^{-1}(a)$ and $\lim\limits_{j \to \infty} d(p_j) = -\infty$. Note that $I \subset \alpha$ where $\alpha \subset \mathbf{C}^{[n]}$ is the maximal ideal that vanishes at the origin $\bar{0} \in \mathbf{C}^n$. For $p \in \mathbf{C}^{[n]}$ set $\mu(p) = \min_{m \in M(p)}\{\deg m\}$, where $\deg$ is the standard degree. Then $p \in \alpha^{\mu(p)}$. Note that $\mu(p_j) \to \infty$. Denote by $\tilde{p}$ the regular function $\pi(p) \in A$ and set the ideal $\tilde{\alpha} := (\tilde{x}_1, \ldots, \tilde{x}_n) \subset A$. Note that $a = \tilde{p}_j \in \tilde{\alpha}^{\mu(p_j)}$, $j = 1, \ldots$. Thus, $a \in \bigcap_{l \in \mathbf{N}} \tilde{\alpha}^l = \{0\} \subset A$ by the Krull Theorem (e.g., see [Hu74], Theorem 4.4), and so, $a = 0$. Hence, $d_A(a) > -\infty$ for any $a \in A \setminus \{0\}$.

Let c_0 be as in Lemma 3.1. Choose $p \in \pi^{-1}(a)$ so that $d(p) < d_A(a) + c_0$. Assume that its principal component $\bar{p}$ belongs to $\widehat{I}$. Then there exists $q \in I$ as in Lemma 3.1 such that $\bar{p} = \bar{q}$. Hence for $s = p - q$ we have $s \in \pi^{-1}(a)$ and $d(s) < d(p)$. Furthermore, since $d(q) - d(q - \bar{q}) \geq c_0$ all monomials in s whose d-degree is at least $d_A(a)$ are the same as the corresponding monomials in p. If $\bar{s} \notin \widehat{I}$ then we are done. Otherwise we can subtract from s a polynomial of the same type as q. Since the number of monomials in p whose d-degree is at least $d_A(a)$ is finite after a finite number of subtractions we obtain a polynomial $r \in \pi^{-1}(a)$ such that either $d(r) < d_A(a)$ (which is impossible) or $\bar{r} \notin \widehat{I}$. Hence we have (1).

The fact that d_A is a semi-degree function which satisfies (3) follows from the construction of d_A. □

4 Filtrations and associated graded algebras

A degree function d_A on A determines the ascending filtration $\mathcal{F} = \{F^t A\}$ on A, where $F^t A := \{a \in A \mid d_A(a) \leq t\}$ and $t \in \mathbf{R}$. Set $F_0^t A = \{a \in A \mid d_A(a) < t\}$. We want to emphasize the following two properties of this filtration (see [Za97] for the rest)

(f3) $(F^{t_1} A \setminus F_0^{t_1} A)(F^{t_2} A \setminus F_0^{t_2} A) \subset (F^{t_1+t_2} A \setminus F_0^{t_1+t_2} A)$.

(f4) If $\{t_i\} \subset \mathbf{R}$ is a decreasing sequence which is convergent to t and $a \in A$ belongs to each $F^{t_i} A$ then $a \in F^t A$.

In the case when d_A is a semi-degree function condition (f3) is replaced by

(f3′) $(F^{t_1} A \setminus F_0^{t_1} A)(F^{t_2} A \setminus F_0^{t_2} A) \subset F^{t_1+t_2} A$.

Definition 4.1. Consider the linear space $\mathrm{Gr}\ A = \oplus_{t\in\mathbf{R}}\mathrm{Gr}\ {}^t A$ where $\widehat{A}^t = \mathrm{Gr}\ {}^t A := F^t A/F_0^t A$, and introduce the following multiplication on $\mathrm{Gr}\ A$. Suppose that $f_1 \in F^{t_1}A/F_0^{t_1}A$ and $f_2 \in F^{t_2}A/F_0^{t_2}A$. Put $(f_1 + F_0^{t_1}A)(f_2 + F_0^{t_2}A)$ equal to $f_1 f_2 + F_0^{t_1+t_2}A$ if $f_1 f_2 \in F^{t_1+t_2}A \setminus F_0^{t_1+t_2}A$ and 0 otherwise (of course, the last possibility does not hold in the case when the filtration is generated by a degree function). Extend this multiplication using the distributive law. Then we call $\widehat{A} = \mathrm{Gr}\ A$ the *associated graded algebra* of the filtered algebra $(A, \mathcal{F})$. An element $\widehat{a} \in \widehat{A}$ is called d_A-homogeneous iff $\widehat{a} \in \widehat{A}^t$ for some $t \in \mathbf{R}$.

Define the mapping $\mathrm{gr}\ : A \longrightarrow \mathrm{Gr}\ A$ by $\mathrm{gr}\ f = \widehat{f} = f + F_0^t A$ when $f \in F^t A \setminus F_0^t A$. If d_A is a degree function, by property (f3) this mapping gr is a homomorphism of multiplicative semigroups.

Proposition 4.1. *Let $\widehat{A}$ be the associated graded algebra of a filtered algebra $(A, \mathcal{F})$ where the filtration $\mathcal{F}$ is generated by d_A from Lemma 3.2 (in particular $\bar{0} \in X$). Then*

$$\widehat{A} \simeq \mathbf{C}^{[n]}/\widehat{I} = \mathbf{C}[\widehat{X}]\,,$$

where $\widehat{X}$ is the affine variety in $\mathbf{C}^n$ defined by the ideal $\widehat{I}$.

Proof. Let $B = \mathbf{C}^{[n]}/\widehat{I}$. Note that $\widehat{I}$ is d-homogeneous, i.e. $\widehat{I} = \oplus_{t\in\mathbf{R}}\widehat{I}^t$ where $\widehat{I}^t = I \cap \mathbf{C}^{[n]}_{d,t}$. Put $B^t = \mathbf{C}^{[n]}_{d,t}/\widehat{I}^t$. Then B may be viewed as the graded algebra $\oplus_{t\in\mathbf{R}}B^t$ with the natural multiplication. Construct a linear mapping $\varphi_t : B^t \to \widehat{A}^t$ as follows. Let $b \in B^t$ and let $q \in \mathbf{C}^{[n]}_{d,t}$ be a representative of b, i.e. $b = q + \widehat{I}$. Suppose that b is nonzero and thus $q \notin \widehat{I}$. For every $p \in \mathbf{C}^{[n]}$ we denote by $\tilde{p}$ the regular function $\pi(p)$ on X. Put $\varphi_t(b) = \mathrm{gr}\ (\tilde{q})$. Since $q = \bar{q} \notin \widehat{I}$ Lemma 3.2 implies that $d_A(\tilde{q}) = t$ and, therefore, $\varphi_t(b)$ is a nonzero element of $\widehat{A}^t$.

We want to show that this mapping is well-defined, injective, and surjective. The injectivity follows from the fact that $\varphi_t(b)$ is nonzero for nonzero b. Let q_1 be another representative of b, i.e. $q_2 := q - q_1 \in \widehat{I}^t$. By Lemma 3.2 $d_A(\tilde{q}) = d_A(\tilde{q}_1) = t$ but $d_A(\tilde{q}_2) < t$. Hence $\mathrm{gr}\ (\tilde{q}) = \mathrm{gr}\ (\tilde{q}_1 + \tilde{q}_2) = \mathrm{gr}\ (\tilde{q}_1)$ which implies that φ_t is well-defined.

Let $a \in A$ and $d_A(a) = t$. By Lemma 3.2 there exists $p \in \pi^{-1}(a)$ such that $\bar{p} \notin \widehat{I}$ and $d(p) = d(\bar{p}) = t$. Put $a_1 = \bar{p}|_X$ and $a_2 = a - a_1 = (p - \bar{p})|_X$. By Lemma 3.2 $d_A(a_1) = t$ but $d_A(a_2) < t$ since $d(p - \bar{p}) < t$. Hence $\mathrm{gr}\ (a) = \mathrm{gr}\ (a_1 + a_2) = \mathrm{gr}\ (a_1)$. Since $\mathrm{gr}\ (a_1)$ belongs to the image of φ_t we see that φ_t is surjective.

Thus we obtained an isomorphism of linear spaces $\varphi := \oplus_{t\in\mathbf{R}} : B \to \widehat{A}$. Let $b' = q' + \widehat{I} \in B^{t'}$. If $qq' \notin \widehat{I}$ then qq' is a representative of bb'. By Lemma 3.2 $d_A(\widetilde{qq'}) = t + t'$ and, therefore, $\mathrm{gr}\ (\tilde{q})\mathrm{gr}\ (\tilde{q}') = \mathrm{gr}\ (\widetilde{qq'})$. If $qq' \in \widehat{I}$ then $bb' = 0$ and $d_A(\widetilde{qq'}) < t + t'$ by Lemma 3.2. Hence the definition of multiplication in $\widehat{A}$ implies that $\mathrm{gr}\ (\tilde{q})\mathrm{gr}\ (\tilde{q}') = 0$. Thus φ transforms the multiplication in B into the multiplication in $\widehat{A}$. That is, φ is an isomorphism of algebras which is the desired conclusion. $\square$

Remark 4.1. The proof of Proposition gives the following description of the mapping gr. Let $p \in \pi^{-1}(a)$ where $a \in A$ and let $d(p) = d_A(a)$. Then $\mathrm{gr}\ (a) = \widehat{\pi}(\bar{p})$ where $\widehat{\pi} : \mathbf{C}^{[n]} \to \widehat{A},\ q \to q|_{\widehat{X}}$ is the natural projection.

5 Associated locally nilpotent derivations

In this section $\widehat{A}$ and d_A are the same as in Proposition 4.1.

Lemma 5.1. *For every derivation ∂ on A there exists t_0 such that $\partial(F^tA) \subset F^{t+t_0}A$ for every $t \in \mathbf{R}$.* [1]

Proof. For every $q \in \mathbf{C}^{[n]}$ put $\tilde{q} = \pi(q)$ where π is as in Lemma 3.2. Let $t_0 = \max_i(d_A(\partial(\tilde{x}_i)) - d_A(\tilde{x}_i))$ where $(x_1, \ldots, x_n)$ is a coordinate system on $\mathbf{C}^n$. For a nonzero $a \in A$ choose $p \in \pi^{-1}(a)$ so that $d_A(a) = d(p)$. Let p_i be the partial derivative of p with respect to x_i . Clearly, $\tilde{p}_i = p_i(\tilde{x}_1, \ldots, \tilde{x}_n)$ and $a = p(\tilde{x}_1, \ldots, \tilde{x}_n)$. Hence $\partial(a) = \sum \tilde{p}_i \partial(\tilde{x}_i)$ and

$$d_A(\partial(a)) \leq \max_i d_A(\tilde{p}_i \partial(\tilde{x}_i)) \leq \max_i (d_A(\tilde{p}_i) + d_A(\tilde{x}_i) + t_0) \leq d_A(a) + t_0$$

(the last inequality holds since $d_A(\tilde{p}_i) + d_A(\tilde{x}_i) \leq d(p_i) + d(x_i) \leq d(p) = d_A(a)$). □

Definition 5.1. The smallest t_0 as above will be called the defect def_∂ of ∂ (the smallest t_0 exists by (f4), Section 4). For every $a \in A$ put $\hat{a} = \mathrm{gr}\,(a)$ and for every nonzero $\partial \in \mathrm{LND}(A)$ define a nonzero $\widehat{\partial} \in \mathrm{LND}(A)$ as follows: $\widehat{\partial}(\hat{a}) = \widehat{\partial(a)}$ if $d_A(\partial(a)) - d_A(a) = \mathrm{def}_\partial$, and $\widehat{\partial}(\hat{a}) = 0$ otherwise. We call $\widehat{\partial}$ the associated locally nilpotent derivation for ∂ . Clearly, $\deg_{\,\partial}(a) \geq \deg_{\,\widehat{\partial}}(\hat{a})$ for every $a \in A$. We also want to note that the associated derivation is a homogeneous derivation on graded algebra $\widehat{A}$, i.e. any element of $\widehat{A}^{\widehat{\partial}}$ coincides with a linear combination of d_A -homogeneous elements of $\widehat{A}^{\widehat{\partial}}$.

Definition 5.2. Let $\partial(a) = J_X(\mathcal{P})J^X(a_1, \ldots, a_{k-1}, a)$ be a Jacobian locally nilpotent derivation. We say that ∂ is perfect if $\hat{a}_1, \ldots, \hat{a}_{k-1}$ are algebraically independent. Denote the set of perfect derivations on A by $\mathrm{Per}(A)$.

The advantage of perfect derivations is that in the case, when $\widehat{A}$ is a domain, the associated derivations can be given by $\mathrm{gr}\,(J_X(\mathcal{P}))J^{\widehat{X}}(\hat{a}_1, \ldots, \hat{a}_{k-1}, \hat{a})$.

We shall show later (Corollary 7.1) that if $\widehat{A}$ is a domain then for a subalgebra A_0 of A , which contains m algebraically independent elements, $\mathrm{gr}\,(A_0)$ contains m algebraically independent elements as well. Applying this fact to $A_0 = A^\delta$ where $\delta \in \mathrm{LND}(A)$, by Lemma 2.5 and Proposition 2.2 we have

Corollary 5.1. *Let $\widehat{A}$ be a domain and $\delta \in \mathrm{LND}(A)$ be nonzero. Then there exists $\partial \in \mathrm{Per}(A)$ which is equivalent to δ . In particular,* $\mathrm{AK}(A) = \bigcap_{\partial \in \mathrm{Per}(A)} A^\partial$.

6 Geometrical construction of associated objects

Let Γ be a germ of a smooth curve at $o \in \Gamma$, $\Gamma^* = \Gamma \setminus o$, $\mathcal{X}$ be an affine variety over Γ (i.e. there exist a morphism $\rho : \mathcal{X} \to \Gamma$), $\mathcal{X}_c = \rho^{-1}(c), c \in \Gamma$, $\mathcal{X}^* = \mathcal{X} \setminus \mathcal{X}_o$. We suppose that there exists an isomorphism $\varphi : \mathcal{X}^* \to X \times \Gamma^*$ such that $\rho \circ \varphi^{-1}$ is the projection to the second factor. Consider the algebras of regular functions $\mathcal{A} = \mathbf{C}[\mathcal{X}] \subset \mathcal{A}^* = \mathbf{C}[\mathcal{X}^*]$,

[1] This fact was proven independently by the second author and by M. Masuda.

and $B = \mathbf{C}[\Gamma]$ which we treat as a subalgebra of $\mathcal{A}$. For every nonzero $\partial \in \mathrm{LND}(A)$ isomorphism φ generates a nonzero $\partial^* \in \mathrm{LND}(\mathcal{A}^*)$ so that $B \subset \mathrm{Ker}\, \partial^*$. Let $b \in B$ vanish at o. Since $\tilde{\partial} := b^m \partial^* \in \mathrm{LND}(\mathcal{A}^*)$ and $\mathcal{A}^*$ is finitely generated, for sufficiently large m we have

Lemma 6.1. *Every nonzero $\partial \in \mathrm{LND}(A)$ generates a nonzero $\tilde{\partial} \in \mathrm{LND}(\mathcal{A})$ so that $B \subset \mathcal{A}^{\tilde{\partial}}$ and the restriction of $\tilde{\partial}$ to each fiber $\mathcal{X}_c \simeq X, c \neq o$ is proportional to ∂.*

Put $\partial_o = \tilde{\partial}|_{\mathcal{X}_o}$ and consider $b \in B$ with a simple root at o. Suppose that the defining ideal of the divisor $\rho^*(o)$ is the principal ideal of $\mathcal{A}$ generated by b (this is the case when $\mathcal{X}$ is normal and $\rho^*(o)$ is reduced). Choosing the smallest possible m in the definition of $\tilde{\partial}$ we obtain a nonzero locally nilpotent derivation ∂_o on $\mathcal{A}_o := \mathbf{C}[\mathcal{X}_o]$. Furthermore, using the projection of $\mathcal{X}^* \simeq X \times \Gamma^*$ to the first factor, for each nonzero $a \in A$ we assign $a^* \in \mathcal{A}^*$ which in general is not regular on $\mathcal{X}$. Note that $b^n a^* \in \mathcal{A}$ for sufficiently large n. Choosing n the smallest possible we see that $a_o := b^n a^*|_{\mathcal{X}_o}$ is a nonzero element of $\mathcal{A}_o$. Define $\mathrm{gr}_o : A \to \mathcal{A}_o$ by $\mathrm{gr}\,(a) = a_o$. This construction implies

Proposition 6.1. *Let the defining ideal of $\rho^*(o)$ in $\mathcal{A}$ be the principal ideal generated by $b \in B$ which has a simple root at o, $\mathcal{A}_o, \partial_o$, and gr_o be as before. Then $\partial_o \in \mathrm{LND}(\mathcal{A}_o)$ and $\deg_{\partial}(a) \geq \deg_{\partial_o}(\mathrm{gr}_o(a))$ for every $a \in A$.*

Remark 6.1. (1) The derivation ∂_o defines a non-trivial associated $\mathbf{C}_+$-action on $\mathcal{X}_o$ which maps $\mathrm{sing}\, \mathcal{X}_o$ onto itself. Furthermore, since this action is the restriction of the associated $\mathbf{C}_+$-action of $\tilde{\partial}$ it maps $(\mathrm{sing}\, \mathcal{X}) \cap \mathcal{X}_o$ onto itself.

(2) By the Chevalley semi-continuity theorem about the dimension of the fibers of algebraic morphisms we see that the dimension of every irreducible component of $\mathcal{X}_o$ is at least $\dim X$.

Example 6.1. Let d be a weight degree function on $\mathbf{C}^{[n]}$ with integer values, i.e. $d_i = d(x_i) \in \mathbf{Z}$. Then it generates a $\mathbf{C}^*$-action G on $\mathbf{C}^{[n]}$ with $G_c(x_i) = c^{-d_i} x_i$ where $c \in \mathbf{C}^*$. Let $\Gamma = \mathbf{C}$ and $\Gamma^* = \mathbf{C}^*$, i. e. $o = 0$. Put $\mathcal{X}_c = G_c(X)$ for $c \in \mathbf{C}^*$. This defines $\mathcal{X}^*$. Set $\mathcal{X}$ equal to the closure of $\mathcal{X}^*$ in $\mathbf{C} \times \mathbf{C}^n$. In particular, $\mathcal{X}_o$ is a closed subvariety of $\mathbf{C}^n \simeq \mathbf{C}^n \times o$.

Lemma 6.2. *Let the assumption of Example 6.1 hold, and $\widehat{I}, \widehat{A}$ be as in Proposition 4.1.*

(1) *Let $p \in \mathbf{C}^{[n]}$ and let $p_c = c^l p \circ G_c$ where $d(p) = l$ and $c \in \mathbf{C}^*$. Then $p_c \to \bar{p}$ as $c \to 0$ where $\bar{p}$ is the d-principal component of p.*

(2) *The defining ideal I_o of $\mathcal{X}_o$ in $\mathbf{C}^n$ contains $\widehat{I}$ and $\dim \widehat{X} = \dim X = \dim \mathcal{X}_o$. In particular, if $\widehat{I}$ is prime (resp. primary) then $\mathcal{X}_o$ coincides with $\widehat{X}$ (resp. the reduction of $\widehat{X}$).*

Proof. Let $p = p^1 + \ldots + p^s + \bar{p}$ be the decomposition of p into d-homogeneous polynomials, i. e. p^i is homogeneous and $l_i = d(p^i) < l = d(\bar{p})$. Clearly $p_c = \bar{p} + \sum_{i=1}^{s} c^{l-l_i} p^i$ which yields (1). If $p \in I$ then p_c belongs to the defining ideal of $\mathcal{X}_c$ in $\mathbf{C}^n$. Since every point of $\mathcal{X}_o$ is a limit of points from $\mathcal{X}_c$ when $c \to 0$ we see that (1) implies the first statement of (2). Therefore, $\dim \mathcal{X}_o \leq \dim \widehat{X}$.

If $\dim \widehat{X} = m$ then there are m algebraically independent regular functions $\widehat{\pi}(p_1)$, $\ldots, \widehat{\pi}(p_m)$ on $\widehat{X}$ where $p_1, \ldots, p_m$ are polynomials and $\widehat{\pi} : \mathbf{C}^{[n]} \to \widehat{A} = \mathbf{C}[\widehat{X}]$ is the natural projection. Since $\widehat{A}$ is generated by homogeneous elements we can suppose that each p_i is d-homogeneous of weight r_i. Consider a nonzero polynomial Q in m variables such that Q is homogeneous with respect to the weights $r_1, \dots r_m$ for these variables. The fact that $\widehat{\pi}(p_1), \dots, \widehat{\pi}(p_m)$ are algebraically independent is equivalent to the fact that for every such Q the polynomial $Q(p_1, \dots, p_m)$ does not belong to $\widehat{I}$. Moreover, for every nonzero polynomial P in m variables one can find such a Q for which $Q(p_1, \dots, p_m)$ is the principal d-component of $P(p_1, \dots, p_m)$. Thus $P(p_1, \dots, p_m)$ cannot belong to I which implies that $\pi(p_1), \dots, \pi(p_m)$ are algebraically independent (recall that $\pi(p) = p|_X$) and, therefore, $\dim X \geq \dim \widehat{X}$. On the other hand by Remark 6.1 (2) $\dim \widehat{X} \geq \dim \mathcal{X}_0 \geq \dim X$. $\square$

7 The dimension of $\widehat{X}$

Let X, A, I be as in Preliminaries. Suppose that ∂ is a locally nilpotent derivation on A. For every weight degree function d on $\mathbf{C}^{[n]}$ we constructed the associated objects which were denoted by $\widehat{X}, \widehat{A}, \widehat{I}, \mathrm{gr}, \widehat{\partial}$. Since we are going to consider now different weight degree functions we shall use index d. That is, the associated objects now are $\widehat{X}_d, \widehat{A}_d, \widehat{I}_d, \mathrm{gr}_d, \widehat{\partial}_d$. The following fact is obvious.

Lemma 7.1. (1) *If d_1 and d_2 are two nonzero weight degree functions which are proportional then their associated objects coincide. In particular, $\dim \widehat{X}_d = \dim X$ for every weight degree function d with rational values.*

(2) *Let d be a weight degree function, $M(p)$ be the set of monomials that are summands in a nonzero polynomial p, and $N(p)$ be the set of Laurent monomials that are of the form $\nu = \mu_1/\mu_2$ where $\mu_1, \mu_2 \in M(p)$. Treat each ν as the vector in $\mathbf{Z}^n \subset \mathbf{R}^n$ with the coordinates that are the powers of ν. Then p is d-homogeneous iff d (as a vector in $\mathbf{R}^n$) is orthogonal to each $\nu \in N(p)$.*

Lemma 7.2. *Let d be a weight degree function with real values. Then there exists a weight degree function d_1 with rational values such that $\widehat{I}_d \subset \widehat{I}_{d_1}$, i. e. $\widehat{X}_{d_1}$ is a subvariety of $\widehat{X}_d$.*

Proof. Consider a set of d-homogeneous generators T of $\widehat{I}_d$ and $N(T) = \bigcup_{q \in T} N(q)$. Then the space $V(T)$ of vectors, that are orthogonal to $N(T)$, is nonzero (since it contains d) and for every $d_1 \in V(T)$ each d-homogeneous polynomial is d_1-homogeneous by Lemma 7.1. Since $V(T)$ is determined in $\mathbf{R}^n$ by a system of linear equations with integer coefficients, the subset of $V(T)$, that consists of points with rational coordinates, is dense in $V(T)$. Choose d_1 from this subset. Then each $q \in T$ is d_1-homogeneous. Let $p \in I$ be such that its principal d-component $\bar{p}_d = q \in T$. If d_1 is near d then the principal d_1-component of p is again q. Thus we can choose d_1 so that $\widehat{I}_{d_1}$ contains the generators of $\widehat{I}_d$ which implies the desired conclusion. $\square$

Note that the argument in the proof of Lemma 6.2 that $\dim X \geq \dim \widehat{X}_d$ works for not only integer-valued but real-valued weight degree functions d. On the other hand since $\widehat{X}_{d_1}$ is a subvariety of $\widehat{X}_d$ we have that $\dim \widehat{X}_d \geq \dim \widehat{X}_{d_1} = \dim X$ where the last equality is a consequence of Lemmas 6.2 and 7.1 (1). Thus we have

Theorem 7.1. *For every real-valued degree function* d *the dimensions of* X *and* $\widehat{X}_d$ *coincide.*

Corollary 7.1. *Let* $\widehat{A}$ *be a domain and* A_0 *be a subring of* A *which contains* m *algebraically independent elements. Then* $\mathrm{gr}\,(A_0)$ *contains* m *algebraically independent elements.*

Proof. Let m be the transcendence degree of A_0 and $l \leq m$ be the transcendence degree of $\mathrm{gr}\,(A_0)$. Since by Theorem 7.1 the transcendence degree of $\widehat{A}$ is $k = \dim X$ there exist homogeneous elements $\hat{a}_1, \ldots, \hat{a}_{k-l} \in \widehat{A}$ that are algebraically independent over $\mathrm{gr}\,(A_0)$. Let $\hat{a}_i = \mathrm{gr}\,(a_i)$ where $a_i \in A$. Assume that $l < m$. Then there exists a nonzero polynomial P in $k-l$ variables with coefficients in A_0 such that $\alpha := P(a_1, \ldots, a_{k-l}) = 0$. Consider the set $M(P)$ of monomials from P. Choose monomials from $M(P)$ whose d_A-degree is the largest and consider their sum (recall that d_A is a degree function on A by Lemma 3.2). This is another nonzero polynomial $Q(a_1, \ldots, a_{k-l})$ with coefficients in A_0. For every $\mu \in M(P)$ its image $\mathrm{gr}\,(\mu)$ in $\widehat{A}$ is a monomial in $\hat{a}_1, \ldots, \hat{a}_{k-l}$ with a nonzero coefficient from $\mathrm{gr}\,(A_0)$. Thus $\widehat{Q}(\hat{a}_1, \ldots, \hat{a}_{k-l})$ is a nonzero polynomial in $\hat{a}_1, \ldots, \hat{a}_{k-l}$ with coefficients in $\mathrm{gr}\,(A)$. This polynomial must be zero since otherwise $\mathrm{gr}\,(\alpha) \neq 0$. This contradicts our assumption that $\hat{a}_1, \ldots, \hat{a}_{k-l}$ must be algebraically independent over $\mathrm{gr}\,(A_0)$. $\square$

8 How to choose degree functions

Let $d, d_A, \widehat{A}, \widehat{I}$ and $\widehat{X} \subset \mathbf{C}^n$ be as in Lemma 3.2 and Proposition 4.1. In order to describe associated derivations on $\widehat{A}$ we need to find out which subalgebras of $\widehat{A}$ may serve as kernels of nonzero homogenous locally nilpotent derivations on $\widehat{A}$. By Theorem 7.1, $\dim \widehat{X} = k$. Hence, if $\widehat{A}$ is a domain, Proposition 2.1 (1) implies that such a subalgebra is determined by any $(k-1)$-tuple of its algebraically independent elements. We can suppose that each of these elements is d_A-homogeneous and, furthermore, irreducible by Proposition 2.1 (3). To make our search smaller we need to make the set of all irreducible d_A-homogeneous elements of $\widehat{A}$ as small as possible. Every d_A-homogeneous $\hat{a} \in \widehat{A}$ is the restriction to $\widehat{X}$ of a d-homogeneous polynomial p. Assume that p is not a monomial, i.e. we have monomials $\mu_1 \neq \mu_2 \in M(p)$. We treat d and $\nu = \mu_1/\mu_2 \in N(p)$ as elements of $\mathbf{R}^n$. By Lemma 7.1 p is d-homogeneous iff $N(p) \subset L_d := \mathbf{Z}^n \cap K_d$ where K_d is the hyperplane orthogonal to d. Thus in order to reduce the set of d-homogeneous polynomials we have to choose d so that the set L_d is the smallest possible. On the other hand we cannot make it too small if we want to keep $\widehat{I}_d$ the same (i.e. $\widehat{I}_d = \widehat{I}$). Indeed, if $\bar{P}_1, \ldots, \bar{P}_l$ are generators of $\widehat{I}$ then $\bigcup_{i=1}^{l} N(\bar{P}_i) \subset L_d$. It turns out that this is the only restriction on L_d.

Lemma 8.1. *Let* $\bar{P}_1, \ldots, \bar{P}_l$ *be as above,* $W \subset \mathbf{R}^n$ *be the subspace spanned by* $\bigcup_{i=1}^{l} N(\bar{P}_i)$*, and* $L = W \cap \mathbf{Z}^n$*. Then* d *can be chosen so that* $L_d = L$*. Furthermore, the set of such weight degree functions is dense in the subspace* $V \subset \mathbf{R}^n$ *of vectors orthogonal to* L*.*

Proof. Every $z \in \mathbf{R}^n \setminus W$ is not orthogonal to any d in an open dense subset V_z of V. The set $\bigcap_z V_z$ with z running over $\mathbf{Z}^n \setminus L$ is dense in V by the Baire category theorem. $\square$

Now every d_A -homogeneous element of $\widehat{A}$ may be viewed as the restriction to $\widehat{X}$ of a d -homogeneous polynomial p so that $M(p) \subset (\mu + L) \cap (\mathbf{Z}_{\geq 0})^n$ for any $\mu \in M(p)$. Furthermore, suppose that $P_1, \ldots, P_l$ is a Gröbner basis of I with respect to a monomial order and μ_i is the initial term of P_i . Any element of A is the restriction of a unique polynomial p such that $M(p)$ is contained in the set $S_A = (\mathbf{Z}_{\geq 0})^n \setminus \bigcup_{i=1}^l (\mu_i + (\mathbf{Z}_{\geq 0})^n)$ which we call a canonical monomial set of A . We denote also by $\mathcal{P}$ the free $\mathbf{Z}$ -submodule of L generated by the Laurent monomials $\nu \in L$ of the form $\nu = \mu_1/\mu_2$ where $\mu_1, \mu_2 \in S_A$. Remark 4.1 implies

Lemma 8.2. *In the above notation every d_A - homogeneous $\widehat{a} \in \widehat{A}$ is the restriction to $\widehat{X}$ of a d -homogeneous polynomial p so that $M(p) \subset (\mu + L) \cap S_A$ for any $\mu \in M(p)$.*

Proposition 8.1. *Let $\widehat{a}, p, \mu$ be as in Lemma 8.2. Suppose that $\widehat{a}$ is irreducible and $\widehat{u}\widehat{a}$ is not the restriction to $\widehat{X}$ of a variable for any unit $\widehat{u} \in \widehat{A}$.*

(1) *Then, replacing $\widehat{a}$ by $\widehat{u}\widehat{a}$, if necessary, one can suppose that p and the Laurent polynomial $\mu^{-1}p$ are irreducible. In particular, if $\{\nu_1, \ldots \nu_k\}$ is a $\mathbf{Z}$ -basis of $\mathcal{P}$ then $\mu^{-1}p = q(\nu_1, \ldots, \nu_k)$ where q is an irreducible polynomial.*

(2) *Let each ν_i in (1) be of the form $\nu_i = \mu_i/\mu_0$ where $\mu_0, \ldots, \mu_k$ are standard relatively prime monomials. Then there exists a standard homogeneous polynomial Q in $k+1$ variables such that $p = Q(\mu_0, \ldots, \mu_k)$.*

Proof. Since $\widehat{a}$ is irreducible, the restriction to $\widehat{X}$ of any divisor of p but one is a unit in $\widehat{A}$. Thus replacing $\widehat{a}$ by $\widehat{u}\widehat{a}$, if necessary, one can suppose that p is irreducible. Let $\mu^{-1}p = r_1 r_2$ where r_1 and r_2 are non-invertible Laurent polynomials and let μ_1 and μ_2 be (standard) monomials such that $q_1 = \mu_1 r_1$ and $q_2 = \mu_2 r_2$ are polynomials that are not divisible by variables. Then $q_1 q_2$ is a polynomial that is not divisible by any variable. Since the same is true for p we have $p = q_1 q_2$ which contradicts irreducibility of p . Thus we have (1). If s is the degree of q then $\mu_0^s q(\nu_1, \ldots, \nu_k) = Q(\mu_0, \ldots, \mu_k)$ is a standard irreducible polynomial which implies (2). $\square$

9 Non-trivial $\mathrm{AK}(A)$

If $\mathrm{AK}(A) = \mathbf{C}$ we say that A (or X) has a trivial AK -invariant. This is so, for instance, when A is a polynomial ring. Thus X with a nontrivial AK -invariant cannot be isomorphic to $\mathbf{C}^n$. Since a nonzero locally nilpotent derivation generates a nonzero associated locally nilpotent derivation we have the following.

Proposition 9.1. *Suppose that there are no nonzero locally nilpotent derivations on the associated algebra $\widehat{A}$ of A . Then there are no nonzero locally nilpotent derivations on A and, in particular, $\mathrm{AK}(A) = A$.*

If there are nonzero derivations on $\widehat{A}$ but $\mathrm{AK}(\widehat{A}) \neq \mathbf{C}$ the situation is much more complicated. We know some cases in which a non-triviality of $\mathrm{AK}(A)$ can be established. One of them is discussed below.

Let $\widehat{I}$ be the associated ideal of I for some weight degree function. We say that a weight degree function d is strongly compatible with pair $\{I, \widehat{I}\}$ if $\widehat{I}_d = \widehat{I}$ and $L_d = L$ in notation of Lemma 8.1.

Theorem 9.1. *Let $(\bar{x}, \bar{y}) = (x_1, \ldots, x_k, y_1, \ldots, y_m)$ be a coordinate system on $\mathbf{C}^n$ (where $n = m + k$) such that the restrictions of $x_1, \ldots, x_k$ to a closed subvariety X of $\mathbf{C}^n$ are algebraically independent. Suppose that the defining ideal I of X contains $P = q(\bar{x})y_1 - p(\bar{x})$ where $q(\bar{x})$ and $p(\bar{x})$ are relatively prime nonconstant polynomials. Let $\widehat{I}$ be the associated graded ideal of I for some weight degree function, L and S_A be as in Lemma 8.2. Suppose that*

(i) *for every monomials $\mu_1, \mu_2 \in S_A$ we have $\mu_1 \mu_2^{-1} \in L$ if and only if $\mu_i = \mu \mu_i'$, $i = 1, 2$ where $\mu_i' \in \mathbf{C}[\bar{x}]$;*

(ii) *there is a sequence of weight degree functions $\{d_j\}$ strongly compatible with $\{I, \widehat{I}\}$ such that the set $\{d_j(x_i)\}$ is bounded from above, and for every $\mu \in S_A \setminus \mathbf{C}[\bar{x}]$ we have $d_j(\mu) \to \infty$ as $j \to \infty$;*

(iii) *for every $i = 1, \ldots, m$, a nonzero $\partial \in \mathrm{LND}(A)$, and sufficiently large j we have $\deg_{\widehat{\partial}_{d_j}}(y_i|_{\widehat{X}}) \geq 2$.*

Then $q(\bar{x})|_X \in A^\partial$. In particular, $\mathrm{AK}(A)$ is non-trivial.

Proof. Every $a \in A$ is a restriction of a unique polynomial r such that $M(r) \subset S_A$. Let us show that if $\deg_\partial(a) \leq 1$ then $M(r) \subset S_A \cap \mathbf{C}[\bar{x}]$. Let $M(r) = M_1 \cup M_2$ where $M_1 \subset S_A \cap \mathbf{C}[\bar{x}]$ and $M_2 \subset S_A \setminus \mathbf{C}[\bar{x}]$. Assume that M_2 is not empty. Then we can suppose by (ii) that $d_j(\mu) > d_j(\kappa)$ for every $\mu \in M_2$ and $\kappa \in M_1$. Hence $M(\bar{r}) \subset M_2$ for the d_j -principal component $\bar{r}$ of r . Then (i) implies that $\bar{r}$ is divisible by some y_i and, therefore, $\deg_{\widehat{\partial}_{d_j}}(\bar{r}) \geq 2$ by (iii). By Remark 4.1 $\bar{r}|_{\widehat{X}} = \mathrm{gr}_{d_j}(a)$. Hence $\deg_\partial(a) \geq 2$ by the inequality on degree in Definition 5.2. Contradiction.

Let $b \in A$ and $\deg_\partial(b) = 1$. By Proposition 2.1 (2) every $a \in A$ is of the form $a = (\sum_{i=0}^k a_i b^i)/a'$ where $a', a_0, \ldots, a_k \in A^\partial$. Hence a is the restriction of the rational function $r_1(\bar{x})/r_2(\bar{x})$ where $r_2|_X = a' \in A^\partial$. Note that $y_1 = p(\bar{x})/q(\bar{x})$ is a nonconstant rational function on $\mathbf{C}^k_{\bar{x}}$. Hence, putting $a = y_1$, we see that r_2 is divisible by q . The desired conclusion follows from the fact that A^∂ is factorially closed. $\square$

Remark 9.1. Derksen [De97] suggested to consider instead of $\mathrm{AK}(A)$ the subring $\mathrm{DK}(A)$ of A generated by the kernels on nonzero locally nilpotent derivations, while keeping the scheme of computation the same. We say that $\mathrm{DK}(A)$ is trivial if $\mathrm{DK}(A) = A$. This is so, for instance, when $A = \mathbf{C}^{[n]}$. It follows from the proof that under the assumption of Theorem 9.1 $y_1, \ldots, y_m \notin \mathrm{DK}(A)$ and, therefore, $\mathrm{DK}(A)$ is not trivial. Furthermore, for non-triviality of $\mathrm{DK}(A)$ it suffices to require that $\deg_{\widehat{\partial}_{d_j}}(\widehat{y}_i) \geq 1$ instead of assumption (iii). In general these two invariants $\mathrm{AK}(A)$ and $\mathrm{DK}(A)$ are not equivalent, i.e. non-triviality of one of them does not imply non-triviality of the other [CrMa03].

Example 9.1. Let us show how this technique works in the case of the Russell cubic which is the hypersurface $P = x + x^2y + z^2 + t^3 = 0$ in $\mathbf{C}^4$ (the argument below is also valid for a hypersurface $x + x^m y + z^k + t^l = \mathrm{const}$ where $k, l \geq 2$ are relatively prime and $m \geq 2$), i.e. I is the principal ideal generated by P . Choose a weight degree function d so that the associated ideal $\widehat{I}$ is generated by $\bar{P} = x^2y + z^2 + t^3$. For $\widehat{X}$ given by $\bar{P} = 0$

the associated algebra $\widehat{A} = \mathbf{C}[\widehat{X}]$ by Proposition 4.1. In particular, it is a domain. Note that L from Lemma 8.1 is generated by the Laurent monomials x^2yz^{-2} and t^3z^{-2}. Choosing an appropriate monomial order we can suppose that S_A from Lemma 8.2 does not contain monomials divisible by x^2y. Hence $\mathcal{P}$ from Proposition 8.1 is generated by vector t^3z^{-2}. The same Proposition implies that every irreducible d-homogeneous element $\widehat{a}$ of $\widehat{A}$ is a restriction of either a variable or a polynomial of the form $c_1z^2+c_2t^3$ with $c_1, c_2 \in \mathbf{C}^*$. If in the last case $\widehat{a} \in \widehat{A}^{\widehat{\partial}}$ for a nonzero $\widehat{\partial} \in \widehat{A}$ then $z, t \in \widehat{A}^{\widehat{\partial}}$ by Corollary 2.1. Thus $\widehat{A}^{\widehat{\partial}}$ contains always a pair of irreducible algebraically independent d-homogeneous elements which are the restrictions of variables. Denote these variables by ξ and ζ. Note that $(x, y) \neq (\xi, \zeta)$ since otherwise the specialization of x and y (as in Definition 2.2) leads to a non-trivial $\mathbf{C}_+$-action on the curve $z^2 + t^3 = c$ (where $c \in \mathbf{C}$) which is absurd. Similarly (ξ, ζ) cannot coincide with (z, t), (y, z), or (y, t). This leaves two cases: either $(\xi, \zeta) = (x, z)$ or $(\xi, \zeta) = (x, t)$. In the first case the specialization leads to the curve $y = t^3 + c$ where $c \in \mathbf{C}$. Clearly $\deg_{\partial}(y) \geq 3$ for every nonzero locally nilpotent derivation on this curve. Similarly, in the second case $\deg_{\partial}(y) \geq 2$ which yields assumption (iii) of Theorem 9.1. Assumption (i) in the case of the Russell cubic means that for every monomials $\mu_1, \mu_2 \in S_A$ we have $\mu_1\mu_2^{-1} \in L$ if and only if $\mu_i = \mu\mu_i',\ i = 1, 2$ where $\mu_i' \in \mathbf{C}[x, z, t]$. This follows readily from the description of S_A and L since $d(y)$ is not a $\mathbf{Q}$-linear combination of $d(z)$ and $d(t)$. Note also that one can change d so that $d(z)$ and $d(t)$ remain fixed and $d(y) \to \infty$ while $d(y)+2d(x) = 2d(z) = 3d(t)$. Thus assumption (ii) of Theorem 9.1 is also satisfied and the AK-invariant of the Russell cubic is non-trivial.

10 Threefolds ramified over $\mathbf{C}^3$.

It was shown in [KaM-L97a] that every smooth contractible surface of Kodaira logarithmic dimension 1 is isomorphic to a hypersurface $Y_{m,n,k}$ in $\mathbf{C}^3_{x,y,z}$ given by $P_{m,n,k}(x, y, z) = 0$ where $P_{m,n,k}(x, y, z) = z^{-m}[(z^my + g(z))^k - (z^mx + f(z))^n + z]$; $m \geq 1$, $n > k \geq 2$, $(k, n) = 1$; $f(0) = g(0) = 1$, and f and g are polynomials of degree at most $m - 1$ which are chosen so that $P_{m,n,k}$ is a polynomial. Consider the hypersurface $X_{m,n,k,l} \subset \mathbf{C}^4_{x,y,z,t}$ given by $R_{m,n,k,l}(x, y, z, t) = t^l - P_{m,n,k}(x, y, z) = 0$ where $l \geq 2$. Then aim of this section is to show that whenever $X_{m,n,k,l}$ is contractible its AK-invariant is nontrivial, i.e. it is an exotic algebraic structure on $\mathbf{C}^3$. This will be done in Proposition 10.1 and Lemma 10.1 below.

It is easy to check that the zero fiber of $P_{1,n,k}$ (resp. $R_{1,n,k,l}$) is a general fiber. By the Némethi-Sebastiani-Thom theorem [Ne91] $X_{1,n,k,l}$ is homotopy equivalent to the joint of $Y_{1,n,k}$ and the general fiber of t^l (which consists of l points). Thus $X_{1,n,k,l}$ is contractible as $Y_{1,n,k}$ is contractible, and furthermore, $X_{1,n,k,l}$ is diffeomorphic to $\mathbf{R}^6$ by [ChDi].

Proposition 10.1. *Let $X := X_{m,n,k,l}$. Then $\mathrm{AK}(A) = A$ unless $k = l = 2$ and m is even. In particular, $X_{1,n,k,l}$ is an exotic structure on $\mathbf{C}^3$.*

Proof. We choose a weight degree function d so that the principal d-component of $R_{m,n,k,l}$ is $Q(x, y, z, t) = t^l + z^{m(n-1)}x^n - z^{m(k-1)}y^k$, i.e. $\widehat{X}$ is a hypersurface given by $Q(x,y,z,t)=0$. By Proposition 9.1 it suffices to show that any associated locally nilpotent derivation $\widehat{\partial}$ on $\widehat{A}$ is zero. Assume the contrary and by abusing notation denote the restrictions of variables to $\widehat{X}$ by the same letters.

Case 1: $k>2$. Since $n>k$ one can see that, when $m(k-1)>1$ (which holds for $k>2$) the singular locus of $\widehat{X}$ is the union of the surface given by $z=t=0$ and the line given by $x=y=0$. The singular surface coincides with the reduction of zeros of z and a general orbit O of the associated $\mathbf{C}_+$ -action does not meet it by Remark 6.1 (1). Hence the image of the projection of O to the z -axis does not contain 0 which implies that z is constant on O . Hence $z\in\widehat{A}^{\widehat{\partial}}$. The specialization of z produces a non-trivial $\mathbf{C}_+$ -action on the Pham-Brieskorn surface $S_{n,k,l}=\{t^l-x^n+y^k=0\}\subset\mathbf{C}^3_{x,y,t}$ in contradiction with [KaZa00, Lemma 4]. Thus we can suppose that $k=2$.

Case 2: $k=2,m=1$, i.e. $\widehat{X}$ is given by

$$t^l+z^{(n-1)}x^n-zy^2=0. \tag{1}$$

We can suppose that the lattice L from Lemma 8.1 is generated by the Laurent monomials $z^{(n-2)}x^ny^{-2}$ and zy^2t^{-l} , and $\mathcal{P}$ from Proposition 8.1 is generated by $z^{(n-2)}x^ny^{-2}$. The same Proposition implies that up to a unit factor any irreducible d_A -homogeneous element of $\widehat{A}$ is a restriction of a polynomial which is either a variable or of the form

$$c_1z^{(n-2)}x^n-c_2y^2 \tag{2}$$

with $c_1,c_2\in\mathbf{C}^*$. If algebraically independent irreducible d_A -homogeneous elements $\widehat{a}_1,\widehat{a}_2\in\widehat{A}^{\widehat{\partial}}$ are the restrictions of two polynomials as in (2) then one can see that the restrictions of $z^{(n-2)}x^n$ and y^2 to $\widehat{X}$ are in $\widehat{A}^{\widehat{\partial}}$. Since $\widehat{A}^{\widehat{\partial}}$ is factorially closed x,y,z , and, therefore, t are in $\widehat{A}^{\widehat{\partial}}$. Hence $\widehat{\partial}$ is the zero derivation. If only $\widehat{a}_1$ is of the form $c_1z^{(n-2)}x^n-c_2y^2$ then we have two possibilities : $c_1/c_2=1$ and $c_1/c_2\neq1$. In the latter case the specialization of $\widehat{a}_1$ (as in Definition 2.2) and (1) gives two equations $t^l=z(c'y^2+c_0')$ and $z^{n-2}x^n=cy^2+c_0$ with $c,c'\neq0$. One can see that $\widehat{a}_2$ cannot be the restriction of a variable, since the specialization of any variable in combination with the two equations leads to a curve without non-trivial $\mathbf{C}_+$ -actions in contradiction with Lemma 2.1. When $c_1/c_2=1$ equation (1) implies that t is in $\widehat{A}^{\widehat{\partial}}$ and the specialization of t leads to a surface $z^{n-2}x^n-y^2=c$. Specializing a variable on this surface we get the same contradiction. The similar argument works also in the case when $\widehat{a}_1$ and $\widehat{a}_2$ are the restrictions of variables.

Case 3: $k=2,m>1$. As we mentioned in Case 1 $z\in\widehat{A}^{\widehat{\partial}}$. Thus after the specialization of $\widehat{a}_1=z$ we have again a non-trivial $\mathbf{C}_+$ -action on $S_{n,2,l}$. As $n\geq3$ and the action can be non-trivial only in the case of dihedral surface [KaZa00] we have $l=2$. If m is odd Proposition 8.1 implies (in the same manner as in Case 2) that up to a unit factor every irreducible d_A -homogeneous element of $\widehat{A}$ is the restriction of a polynomial which is either a variable or a polynomial of the form $c_1z^{(n-2)m}x^n-c_2y^2$. One can check now that the specialization of algebraically independent irreducible d_A -homogeneous $\widehat{a}_1$ and $\widehat{a}_2$ from $\widehat{A}^{\widehat{\partial}}$ produces a curve without a non-trivial $\mathbf{C}_+$ -action. □

If m is even and $k=l=2$ then $\mathrm{AK}(\widehat{A})$ is different from $\widehat{A}$. However, this case is not a trouble since $X_{m,n,2,2}$ is not homeomorphic to a Euclidean space.

Lemma 10.1. *If m is even then for $i \neq 0,2$ the integer homology $H_i(X_{m,n,2,2})$ are trivial, and $H_2(X_{m,n,2,2}) = \mathbf{Z}_n$.*

Proof. Let $m = 2j$. Replace t by $t = u - z^j y$. Then $R_{m,n,2,2}$ becomes $u^2 - 2y(uz^j + g(z)) + z^{-m}[(z^m x + f(z))^n - g^2(z) - z]$. Hence $M' := X_{m,n,2,2}$ is the affine modification of $M = \mathbf{C}^3_{x,z,u}$ along divisor $D = \{uz^j + g(z) = 0\} \subset M$ with center $C = \{u^2 + z^{-m}[(z^m x + f(z))^n - g^2(z) - z] = uz^j + g(z) = 0\} \subset D$ (see [KaZa99] for definitions). Note that $D \simeq G \times \mathbf{C}_x$ where G is the hyperbola in $\mathbf{C}^2_{z,u}$ given by $uz^j + g(z) = 0$. As $u^2 + z^{-m}[(z^m x + f(z))^n - g^2(z) - z] = z^{-m}[(z^m x + f(z))^n - z]$ on C , the natural projection $C \to G$ is an n -sheeted unramified covering. Hence it induces the homomorphism $H_1(C) = \mathbf{Z} \to \mathbf{Z} = H_1(G)$ whose image is $n\mathbf{Z}$. The exceptional divisor $E \subset M'$ is naturally isomorphic to $C \times \mathbf{C}_y$. Thus if $\sigma : M' \to M$ is the natural projection $\sigma|_E : E \to C \subset D$ induces the homomorphism $\tau_1 : H_1(E) = \mathbf{Z} \to \mathbf{Z} = H_1(D)$ whose image is $n\mathbf{Z}$. As $\sigma^*(D) = E$ this affine modification satisfies the assumption of [KaZa99, Th. 3.1] which says that the modification induces an isomorphism between $H_*(M')$ and $H_*(M)$ iff it induces an isomorphism between $H_*(E)$ and $H_*(D)$. Thus M' is not contractible. In order to be more specific let us consider the commutative diagram from [KaZa99, Th. 3.1]

$$\begin{array}{ccccccccccc}
\ldots \longrightarrow & H_{j-1}(E) & \longrightarrow & H_j(\breve{M}') & \longrightarrow & H_j(M') & \longrightarrow & H_{j-2}(E) & \longrightarrow & H_{j-1}(\breve{M}') & \longrightarrow \ldots \\
& \downarrow \tau_* & & \simeq \downarrow \breve{\sigma}_* & & \downarrow \sigma_* & & \downarrow \tau_* & & \simeq \downarrow \breve{\sigma}_* & \\
\ldots \longrightarrow & H_{j-1}(D) & \longrightarrow & H_j(\breve{M}) & \longrightarrow & H_j(M) & \longrightarrow & H_{j-2}(D) & \longrightarrow & H_{j-1}(\breve{M}) & \longrightarrow \ldots
\end{array}$$

where $\breve{M} = M \setminus D, \breve{M}' = M' \setminus E$ and $\breve{\sigma}_* : H_*(\breve{M}') \to H_*(\breve{M})$ is induced by the natural isomorphism $\breve{M}' \to \breve{M}$. As $G \subset \mathbf{C}^2$ is a general fiber of $uz^j + g(z)$, the fundamental group and, therefore, the first homology of $\mathbf{C}^2 \setminus G$ are isomorphic to $\mathbf{Z}$ [Ka93]. By the additivity of Euler characteristics [Du87] we have $\chi(\mathbf{C}^2 \setminus G) = 1$. Hence $H_2(\mathbf{C}^2 \setminus G) \simeq \mathbf{Z}$ since $\mathbf{C}^2 \setminus G$ is homotopy equivalent to a 2-dimensional CW-complex and $H_2(\mathbf{C}^2 \setminus G)$ has no torsion [Mi63]. As $\breve{M} \simeq \breve{M}' \simeq (\mathbf{C}^2 \setminus G) \times \mathbf{C}$ we have $H_3(\breve{M}') = H_3(\breve{M}) = 0$ and $H_2(\breve{M}') = H_2(\breve{M}) = \mathbf{Z}$. Therefore, we have the commutative diagram

$$\begin{array}{ccccccccccc}
\ldots \longrightarrow & 0 & \longrightarrow & H_3(M') & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z} & \longrightarrow & H_2(M') & \longrightarrow 0 \\
& & & \downarrow & & \downarrow \tau_1 & & \downarrow \simeq & & & \\
\ldots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 & \longrightarrow 0
\end{array}$$

This implies that $H_2(M') = \mathbf{Z}_n$ and $H_3(M') = 0$. As $\pi_1(M') = \pi_1(M) = 0$ [KaZa99, Prop. 3.1] we have $H_1(M') = 0$. □

This type of examples makes us to repeat the question which appeared first in [OP].

Conjecture. *Let X be a smooth contractible affine algebraic variety of dimension 3 such that* $\mathrm{AK}(A) = \mathbf{C}$ *(resp.* $\mathrm{DK}(A) = A$ *). Then X is isomorphic to* $\mathbf{C}^3$.

11 On Masuda's question

In all previous computations of AK -invariant we dealt essentially with the case when the free $\mathbf{Z}$-module $\mathcal{P}$ from Proposition 8.1 was one-dimensional. If we study the invariant on the hypersurface X of $\mathbf{C}^5$ given by $x + x^2y + z^2 + t^3 + u^5 = \text{const}$ we encounter the situation when $\mathcal{P}$ is of dimension 2 which is more complicated.

Lemma 11.1. *Let q_1 and q_2 be irreducible algebraically independent homogenous (in a standard sense) polynomials on $\mathbf{C}^3$ such that each component of general fibers of morphism $f = (q_1, q_2) : \mathbf{C}^3 \to \mathbf{C}^2$ is a (straight) line in $\mathbf{C}^3$. Then q_1 and q_2 are linear.*

Proof. Consider the fiber $f^{-1}(0, c_2)$ where $c_2 \in \mathbf{C}^*$. It contains a line ℓ (which is the limit of lines contained in general fibers). This line does not contain the origin as the origin belongs to $f^{-1}(0,0)$. The plane that contains the origin and ℓ is a component of $q_1^{-1}(0)$ because $f^{-1}(0, c_2) \subset q_1^{-1}(0)$ and q_1 is homogeneous. Since q_1 is irreducible it is linear. □

Lemma 11.2. *Let X be an affine algebraic threefold with a dominant morphism $\tau : X \to \mathbf{C}^3_{z,t,u}$. Suppose that $\partial \in \mathrm{LND}(A)$ is such that A^∂ contains two algebraically independent elements a_1 and a_2 of the form*

$$a_i = q_i(z^2, t^3, u^5), \ i = 1, 2 \tag{3}$$

where each q_i is homogeneous. Then two of three variables z, t, and u are contained in A^∂.

Proof. Assume the contrary. Consider the composition $\rho : X \to \mathbf{C}^3_{\xi,\zeta,\eta}$ of τ with map $\mathbf{C}^3_{z,t,u} \to \mathbf{C}^3_{\xi,\zeta,\eta}$ given by $(\xi, \zeta, \eta) = (z^2, t^3, u^5)$. Let $g = (a_1, a_2) : X \to \mathbf{C}^2$, and $f : \mathbf{C}^3 \to \mathbf{C}^2$ be given by $f(\xi, \zeta, \eta) = (q_1(\xi, \zeta, \eta), q_2(\xi, \zeta, \eta))$, i.e. $g = f \circ \rho$.

Consider first the case when $z, t, u \notin A^\partial$. Note that any coordinate plane $H \subset \mathbf{C}^3_{\xi,\zeta,\eta}$ is not the preimage of a curve in $\mathbf{C}^2$ under f. Indeed, otherwise there exists a polynomial r on $\mathbf{C}^2$ such that $r \circ f$ vanishes on H, i.e. $r \circ f$ is divisible by a variable. Hence $r \circ g = r \circ f \circ \rho$ is divisible by one of elements z, t, u. This contradicts our assumption that $z, t, u \notin A^\partial$ because A^∂ is factorially closed and $r \circ g \in A^\partial$. Hence $f(H)$ is dense in $\mathbf{C}^2$ and, therefore, for general $c \in \mathbf{C}^2$ the curve $\Gamma_c = f^{-1}(c)$ meets any coordinate hyperplane H of $\mathbf{C}^3$ transversally.

Since for general c each component C' of the curve $G_c = g^{-1}(c)$ is an orbit of a $\mathbf{C}_+$-action, each component C of $\Gamma_c = \rho(G_c)$ is a polynomial curve because it is the image of morphism $\rho|_{C'} = (h_1, h_2, h_3) : C' \simeq \mathbf{C} \to C \subset \mathbf{C}^3$. Condition that Γ_c meets each coordinate plane transversally means that each h_i has simple roots only. Furthermore, $\rho|_{C'} : C' \simeq \mathbf{C} \to C$ factors through a Galois covering $\tau(C') \to C$ ramified over points $\rho(h_1^{-1}(0)), \rho(h_2^{-1}(0)), \rho(h_3^{-1}(0))$ with orders 2,3, and 5 respectively. The Riemann-Hurwitz formula implies that each h_i must have at most one root as otherwise C' cannot be isomorphic to $\mathbf{C}$. Hence C is a straight line in $\mathbf{C}^3_{\xi,\zeta,\eta}$. By Lemma 11.1 both q_1 and q_2 are linear. Replacing them with their linear combinations we can suppose that q_1 is, say, of the form $c_1\xi + c_2\zeta$ which implies that $c_1 z^2 + c_2 t^3$ is invariant by the $\mathbf{C}_+$-action on X. By Corollary 2.1 z and t are in A^∂ contrary to our assumption.

Now suppose that one of variables, say u, is in A^{∂}. Then specializing u we can treat a_i as a polynomial in z^2 and t^3. Hence Corollary 2.1 implies that both z, t are contained in A^{∂} contrary to our assumption. □

Proposition 11.1. *Let* X *be the hypersurface of* $\mathbf{C}^5$ *given by* $x+x^2y+z^2+t^3+u^5 = \mathrm{const}$. *Then* $x|_X \in \mathrm{AK}(A)$.

Proof. With an appropriate choice of a weight degree function d we can suppose that $\widehat{X}$ is the hypersurface $x^2y + z^2 + t^3 + u^5 = 0$ in $\mathbf{C}^5$, that L from Lemma 8.1 is generated by $x^2yz^{-2}, t^3z^{-2}, u^5z^{-2}$, and, therefore, $\mathcal{P}$ from Proposition 8.1 is generated by t^3z^{-2} and u^5z^{-2} (because one can choose S_A without monomials divisible by x^2y). As in Example 9.1 we see that the assumptions (i) and (ii) of Theorem 9.1 hold (where $\mu_1, \mu_2 \in \mathbf{C}[x, z, t, u]$ for assumption (i)). Thus it suffices to check assumption (iii) of that Theorem. By Proposition 8.1 every d_A-homogeneous irreducible element of $\widehat{A}$ is the restriction to $\widehat{X} = X$ of either a variable or a polynomial $q(z^2, t^3, u^5)$ where q is a (standard) homogeneous polynomial. By abusing notation we denote the restriction of x, y, z, t, u to $\widehat{X}$ by the same letters. Let $\widehat{\partial} \in \mathrm{LND}(\widehat{A})$ be homogeneous. If $\widehat{a} \in \widehat{A}^{\widehat{\partial}}$ is one of z, t, u then specializing $\widehat{a}$ we reduce the dimension of the problem and, therefore, we can see as in Example 9.1 that $\deg_{\widehat{\partial}}(\widehat{y}) \geq 2$.

Let us show that such an element $\widehat{a}$ exists by assuming the contrary. As $\dim X = 4$ by Proposition 2.1 one can find irreducible d_A-homogeneous algebraically independent elements $\widehat{a}_1, \widehat{a}_2, \widehat{a}_3 \in \widehat{A}^{\widehat{\partial}}$. Note that x and y cannot belong to $\widehat{A}^{\widehat{\partial}}$ simultaneously since otherwise their specialization leads to a nontrivial $\mathbf{C}_+$-action on the surface S given by $z^2 + t^3 + u^5 = \mathrm{const}$ in $\mathbf{C}^3$. But it has no such actions by Proposition 9.1 since (with an appropriate choice of a weight degree function) the associated surface $\widehat{S}$ is the Pham-Brieskorn surface $z^2 + t^3 + u^5 = 0$ which has no nontrivial $\mathbf{C}_+$-action [KaZa00, Lemma 4]. Thus $\widehat{a}_1, \widehat{a}_2 \in \mathbf{C}[z, t, u]$ and by Lemma 11.2 we can suppose that they coincide with two of the variables z, t, u. □

Remark 11.1. (1) The same application of the Smith theory, which shows that the Russell cubic is contractible [KoRu97], implies that the hypersurface $x+x^2y+z^2+t^3+u^5 = 0$ is also contractible and, therefore, diffeomorphic to $\mathbf{R}^8$ [ChoDi94]. Since its AK-invariant is nontrivial it is an exotic algebraic structure on $\mathbf{C}^4$. (2) Note also that the argument of Proposition 11.1 is applicable for hypersurfaces $x+x^my+z^k+t^l+u^s = \mathrm{const}$ where $k, l, s \geq 2$ are pairwise prime and $m \geq 2$.

References

[ChoDi94] A.D.R. Choudary, A. Dimca, *Complex hypersurfaces diffeomorphic to affine spaces*, Kodai Math. J. **17** (1994), 171–178.

[CrMa03] A. Crachiola, S. Maubach, *The Derksen invariant vs. the Makar-Limanov invariant*, Proc. AMS **131** (2003), 3365-3369.

[Da97] D. Daigle, *On some properties of locally nilpotent derivations*, J. Pure Appl. Algebra **114** (1997), 221-230.

[De97] H. Derksen, *Constructive invariant theory and the linearization problem*, Ph. D. thesis, Basel, 1997.

[Du87] A. Durfee, *Algebraic varieties which are a disjoint union of subvarieties,* In: Geometry and Topology. Proc. Conf. Athens/Ga, 1985. Lect. Notes Pure Appl. Math. **105** (1987), 99-102.

[FLN92] M. Ferrero, Y. Lequain, A. Nowicki, *A note on locally nilpotent derivations*, J. of Pure and Appl. Algebra, **79**(1992), 45–50.

[Hu74] T. W. Hungerford, *Algebra*, Springer-Verlag New York Inc., New York, 1974, p. 512.

[Ka93] Sh. Kaliman, *Smooth contractible hypersurfaces in* $\mathbf{C}^n$ *and exotic algebraic structures on* $\mathbf{C}^3$, Math. Zeitschrift **214** (1993), 499–510.

[KaM-L97a] Sh. Kaliman, L. Makar-Limanov, *Affine algebraic manifolds without dominant morphisms from Euclidean spaces*, Rocky Mount. J. Math., **27**(1997), 601–609.

[KaM-L97b] Sh. Kaliman, L. Makar-Limanov, *On Russell–Koras contractible threefolds*, J. of Alg. Geom. **6**(1997), 247–268.

[KaKoM-LRu97] Sh. Kaliman, M. Koras, L. Makar-Limanov, P. Russell, $\mathbf{C}^*$ *-actions on* $\mathbf{C}^3$ *are linearizable*, Electronic Research Announcement of the AMS, **3**(1997), 63–71.

[KaZa99] Sh. Kaliman, M. Zaidenberg, *Affine modifications and affine hypersurfaces with a very transitive automorphism group*, Trans. Groups, **4**, (1999), 53–95.

[KaZa00] Sh. Kaliman, M. Zaidenberg, *Miyanishi's characterization of the affine 3-space does not hold in higher dimensions*, Annales de l'Institut Fourier **50:6** (2000), 1649-1669.

[KaZa01] Sh. Kaliman, M. Zaidenberg, *Families of affine planes: The existence of a cylinder*, Michigan Math. J. **49** (2001), 353–367.

[Ka02] Sh. Kaliman *Polynomials with generic* $\mathbf{C}^2$ *-fibers are equivalent to a linear polynomial*, Pacific J. Math. **203:1** (2002), 161–189.

[KoRu97] M. Koras, P. Russell, *Contractible threefolds and* $\mathbf{C}^*$ *-actions on* $\mathbf{C}^3$, J. Algebraic Geom., **6**(1997), no.4, 671–695.

[M-L96] L. Makar-Limanov, *On the hypersurface* $x + x^2y + z^2 + t^3 = 0$ *in* $\mathbf{C}^4$ *or a* $\mathbf{C}^3$ *-like threefold which is not* $\mathbf{C}^3$, Israel Math. J. **96**(1996), 419–429.

[M-L04] L. Makar-Limanov, *Locally nilpotent derivations of affine domains*, Max-Planck-Institut preprint, MPIM2004-92.

[Mi63] J. Milnor, *Morse Theory*, Princeton Univ. Press, Princeton, NJ, 1963.

[Ne91] A. Némethi, *Global Sebastiani–Thom theorem for polynomial maps*, J. Math. Soc. Japan, **43** (1991), 213–218.

[OP] *Open Problems*, Workshop on Group Actions on Rational Varieties, edited by G. Freudenburg and P. Russell, CICMA Report 2002-04 (to appear).

[Po86] V. L. Popov, *Contractions of actions of reductive algebraic groups*, (Russian) Mat. Sb. (N.S.) **130(172)** (1986), no. 3, 310–334, 431.

[Re68] R. Rentschler, *Opérations du groupe additif sur le plane affine*, C.R. Acad. Sci. Paris, **267** (1968), 384–387.

[Za97] M. Zaidenberg, *Lectures on exotic algebraic structures on affine spaces*, Schriftenreihe des Graduiertenkollegs Geometrie und Mathematische Physik, Heft 24, Institut Für Mathematik, Ruhr-Universität Bochum, Juli 1997, 56p.

Department of Mathematics, University of Miami, Coral Gables, FL 33124, U.S.A.
E-mail address: kaliman@math.miami.edu

Department of Mathematics, Wayne State University, Detroit, MI 48202, U.S.A.
E-mail address: lml@math.wayne.edu