# SMOOTH AFFINE SURFACES WITH NON-UNIQUE $\mathbb{C}^*$-ACTIONS

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Abstract. In this paper we complete the classification of effective $\mathbb{C}^*$-actions on smooth affine surfaces up to conjugation in the full automorphism group and up to inversion $\lambda \mapsto \lambda^{-1}$ of $\mathbb{C}^*$. If a smooth affine surface $V$ admits more than one $\mathbb{C}^*$-action then it is known to be Gizatullin i.e., it can be completed by a linear chain of smooth rational curves. In [$\mathrm{FKZ}_3$] we gave a sufficient condition, in terms of the Dolgachev-Pinkham-Demazure (or DPD) presentation, for the uniqueness of a $\mathbb{C}^*$-action on a Gizatullin surface. In the present paper we show that this condition is also necessary, at least in the smooth case. In fact, if the uniqueness fails for a smooth Gizatullin surface $V$ which is neither toric nor Danilov-Gizatullin, then $V$ admits a continuous family of pairwise non-conjugated $\mathbb{C}^*$-actions depending on one or two parameters. We give an explicit description of all such surfaces and their $\mathbb{C}^*$-actions in terms of DPD presentations. We also show that for every $k > 0$ one can find a Danilov-Gizatullin surface $V(n)$ of index $n = n(k)$ with a family of pairwise non-conjugate $\mathbb{C}_+$-actions depending on $k$ parameters.

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## 1. Introduction

The classification of $\mathbb{C}^*$-actions on normal affine surfaces up to equivariant isomorphism is a widely studied subject and by now well understood, see e.g., [FlZa$_1$]. However from this classification it is not clear which of these surfaces are *abstractly* isomorphic. This leads to the question of classifying all equivalence classes of $\mathbb{C}^*$-actions on a given surface $V$ under the equivalence relation generated by conjugation in $\mathrm{Aut}(V)$ and inversion $t \mapsto t^{-1}$ of $\mathbb{C}^*$. In this paper we give a complete solution to this problem for smooth affine surfaces. In particular we obtain the following result.

**Theorem 1.0.1.** *Let $V$ be a smooth affine $\mathbb{C}^*$-surface. Then its $\mathbb{C}^*$-action is unique up to equivalence if and only if $V$ does not belong to one of the following classes.*
*(1) $V$ is a toric surface;*
*(2) $V = V(n)$ is a Danilov-Gizatullin surface of index $n \geq 4$ (see below);*
*(3) $V$ is a special smooth Gizatullin surface of type I or II (see Definition 1.0.4 below).*
*Furthermore, $V$ admits at most two conjugacy classes of $\mathbb{A}^1$-fibrations $V \to \mathbb{A}^1$ if and only if $V$ is not one of the surfaces in (2) or (3).*

**1.0.2.** Here two $\mathbb{A}^1$-fibrations $\varphi_1, \varphi_2 : V \to \mathbb{A}^1$ are called conjugated if $\varphi_2 = \beta \circ \varphi_1 \circ \alpha$ for some $\alpha \in \mathrm{Aut}(V)$ and $\beta \in \mathrm{Aut}(\mathbb{A}^1)$. Let us describe in more detail the exceptions (1)-(3) in Theorem 1.0.1.

Obviously uniqueness of $\mathbb{C}^*$-actions fails for affine toric surfaces. Restricting the torus action to one-dimensional subtori yields an infinite number of equivalence classes of $\mathbb{C}^*$-action on such a surface.

A surface $V = V(n)$ as in (2) is by definition the complement to an ample section $C$ of a Hirzebruch surface $\Sigma_k \to \mathbb{P}^1$ with [1] $C^2 = n$. By a remarkable theorem of Danilov and Gizatullin [DaGi, Theorem II.5.8.1] [2] this surface only depends on $n$ and neither on $k$ nor on the choice of the section. As was observed by Peter Russell, for $n \geq 4$ there are non-equivalent $\mathbb{C}^*$-actions on $V(n)$; see [FKZ$_2$, 5.3] for a full classification of them. It is also shown in *loc.cit.* that there are at least $n-1$ non-conjugated $\mathbb{A}^1$-fibrations $V(n) \to \mathbb{A}^1$. In Section 6.3 we give a complete classification of all $\mathbb{A}^1$-fibrations on $V(n)$. It turns out that there are even families of pairwise non-conjugated

[1]Our enumeration of the Danilov-Gizatullin surfaces differs from that in [FKZ$_2$]. In this enumeration e.g., $V_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1 \backslash \Delta$, where $\Delta$ is the diagonal.

[2]See Corollary 4.8 in [CNR], [FKZ$_4$], or Corollary 6.2.4 below for alternative proofs.

$\mathbb{A}^1$-fibrations depending on an arbitrary number of parameters, if $n$ is sufficiently large; see Corollary 6.3.20.

**1.0.3.** To describe the special surfaces as in (3) we recall[3] that a normal affine surface is said to be *Gizatullin* if it can be completed by a zigzag that is, by a linear chain of smooth rational curves. Any normal non-Gizatullin surface admits at most one $\mathbb{C}^*$-action up to equivalence (see [Be] for the smooth and [FlZa$_2$] for the general case). Thus, if for a normal affine $\mathbb{C}^*$-surface uniqueness of $\mathbb{C}^*$-action fails, it must be Gizatullin.

By a result of Gizatullin [Gi$_1$] (see also [Du$_1$, FKZ$_2$]) any non-toric Gizatullin surface has a completion $(\bar{V}, D)$ with a boundary $D = \bar{V} \backslash V$ which is a *standard* zigzag. This means that $D$ is a zigzag with dual graph

$$(1) \qquad \Gamma_D : \quad \begin{array}{ccccccc} 0 & & 0 & & w_2 & & & & w_n \\ \circ & — & \circ & — & \circ & — & \cdots & — & \circ \\ C_0 & & C_1 & & C_2 & & & & C_n \end{array} ,$$

where $C_0^2 = C_1^2 = 0$ and $w_i = C_i^2 \le -2,\ i = 2, \ldots, n$.

Such a completion $\bar{V}$ can be constructed starting from the quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and the curves

$$C_0 = \{\infty\} \times \mathbb{P}^1, \quad C_1 = \mathbb{P}^1 \times \{\infty\}, \quad C_2 = \{0\} \times \mathbb{P}^1$$

by a sequence of blowups on $C_2 \backslash C_1$ and infinitesimally near points, see 3.2.1. An exceptional curve not belonging to the zigzag is called a *feather* [FKZ$_2$]. If in this process a feather $F$ is created by a blowup on component $C_\mu$ of the zigzag then we call $C_\mu$ the *mother component* of $F$. This component can be different from the component of the zigzag where the feather is attached. Furthermore, the feathers depend on the completion $(\bar{V}, D)$ chosen. When a completion is deformed then not only the feathers are deformed, but also their components of attachment can change. If this happens then we say that a feather jumps.

**Definition 1.0.4.** A smooth Gizatullin surface $V$ is said to be *special* with data $(n, r, t)$ if it admits a standard completion $(\bar{V}, D)$ such that

(a) every component $C_{i+1}$, $i \ge 2$, of the zigzag $D$ is created by a blowup on $C_i \backslash C_{i-1}$, and

(b) $n \ge 3$ and the divisor formed by the feathers can be written as

$$F_2 + F_{t1} + \cdots + F_{tr} + F_n, \quad r \ge 0,$$

where $F_2$, $F_{t\rho}$, $1 \le \rho \le r$, and $F_n$ have mother component $C_2$, $C_t$ and $C_n$, respectively. Such a surface $V$ is called a special surface of

- type I if either $r = 1$ or $r \ge 2$ and $t \in \{2, n\}$;
- type II if $r \ge 2$ and $2 < t < n$.

The remaining special surfaces with $r = 0$ are just the Danilov-Gizatullin surfaces $V_n$ with $n \ge 3$. They are neither of type I nor II.

In [FlZa$_3$, FKZ$_3$] we have shown that a $\mathbb{C}^*$-action on a smooth affine surface $V$ is unique up to equivalence unless $V$ belongs to one of the classes (1)-(3) in Theorem 1.0.1 (see also 3.5.2 and 3.5.3 below). A similar uniqueness theorem holds for $\mathbb{A}^1$-fibrations

[3] See [Gi$_1$] or [FKZ$_3$].

$V \to \mathbb{A}^1$, see [FKZ$_3$, 5.13] and Proposition 6.4.1 in Sect. 6.4. Thus Theorem 1.0.1 is a consequence of the following result.

**Theorem 1.0.5.** *Let $V$ be a special smooth Gizatullin surface. If $V$ is of type I then the equivalence classes of $\mathbb{C}^*$-actions on $V$ form in a natural way a 1-parameter family, while in case of type II they form a 2-parameter family. Similarly, conjugacy classes of $\mathbb{A}^1$-fibrations $V \to \mathbb{A}^1$ contain, in the case of type I special surfaces, a one-parameter family, while in case of type II they contain two-parameter families.*

In particular any special Gizatullin surface admits a $\mathbb{C}^*$-action. To construct a one- or two-parameter family of such $\mathbb{C}^*$-actions let us recall [FlZa$_2$] that on a non-toric Gizatullin surface there can exist only hyperbolic $\mathbb{C}^*$-actions. These actions can be described via the following DPD-presentation [FlZa$_1$].

**1.0.6.** Any hyperbolic $\mathbb{C}^*$-surface can be presented as

$$V = \operatorname{Spec} A, \qquad \text{where} \quad A = A_0[D_+, D_-] = A_0[D_+] \oplus_{A_0} A_0[D_-]$$

for a pair of $\mathbb{Q}$-divisors $(D_+, D_-)$ on a smooth affine curve $C = \operatorname{Spec} A_0$ satisfying the condition $D_+ + D_- \le 0$. Here

$$A_0[D_\pm] = \bigoplus_{k\ge 0} H^0(C, \mathcal{O}_C(\lfloor kD_\pm \rfloor))u^{\pm k} \subseteq \operatorname{Frac}(A_0)[u, u^{-1}]\,,$$

where $\lfloor D \rfloor$ stands for the integral part of a divisor $D$ and $u$ is an independent variable. Two pairs $(D_+, D_-)$ and $(D'_+, D'_-)$ are said to be *equivalent* if $D'_\pm = D_\pm \pm \operatorname{div} \varphi$ for a rational function $\varphi$ on $C$.

In terms of these DPD-presentations we can reformulate Theorem 1.0.5 in the following more precise form.

**Theorem 1.0.7.** *(a) A smooth Gizatullin surface $V$ equipped with a hyperbolic $\mathbb{C}^*$-action is special if and only if $V$ admits a DPD-presentation $V = \operatorname{Spec} \mathbb{C}[z][D_+, D_-]$ with*

$$(D_+, D_-) = \left( -\frac{1}{t-1}[p_+]\ , -\frac{1}{n-t+1}[p_-] - D_0 \right)\,, \tag{2}$$

*where $p_+ \ne p_-$, $2 \le t \le n$, $n \ge 3$ and $D_0 = \sum_{i=1}^r [p_i]$ is a reduced divisor on $C \cong \mathbb{A}^1$ such that $r \ge 0$* [4] *and all points $p_i$ are different from $p_\pm$.*

*(b)* [FlZa$_1$, Theorem 4.3(b)] *Two such $\mathbb{C}^*$-surfaces $V$, $V'$ given by pairs of divisors $(D_+, D_-)$ and $(D'_+, D'_-)$ as in (a) are equivariantly isomorphic if and only if $(D_+, D_-)$ and $(D'_+, D'_-)$ are equivalent up to interchanging $D_+$ and $D_-$, if necessary, and up to an automorphism of the underlying curve $C = \mathbb{A}^1$.*

*(c) Two surfaces $V$, $V'$ as in (b) are (abstractly) isomorphic if and only if the unordered pairs* [5] $(\deg\{D_+\}, \deg\{D_-\})$ *and* $(\deg\{D'_+\}, \deg\{D'_-\})$ *coincide and the integral part $\lfloor -D_+ - D_- \rfloor$ is equivalent to $\lfloor -D'_+ - D'_- \rfloor$ up to an automorphism of the underlying curve $C = \mathbb{A}^1$.*

[4] For $r = 0$ we obtain the Danilov-Gizatullin surfaces.

[5] $\{D\}$ denotes the fractional part of the $\mathbb{Q}$-divisor $D$.

In case $r = 0$, (c) is just the theorem of Danilov and Gizatullin cited above. The generalized Isomorphism Theorem of (c) is our principal new result, which occupies the major part of the paper.

The standard boundary zigzag of a special smooth Gizatullin surface $V$ as in (a) is

$$[[0, 0, (-2)_{t-2}, -2 - r, (-2)_{n-t}]] , \tag{3}$$

where $[[(-2)_a]]$ represents a chain of $(-2)$-curves of length $a$ (cf. 2.1.1 below). Here the numbers $n, r, t$ have the same meaning as in Definition 1.0.4. In particular,

- $V$ is of type I if and only if either $r = 1$ or $r \geq 2$ and one of the coefficients of $p_\pm$ in (2) equals $-1$,
- $V$ is of type II if and only if $r \geq 2$ and both coefficients of $p_\pm$ are in the interval $]-1, 0[$.

Comparing part (b) and (c) of the theorem, the position of $p_+$ is essential for the equivariant isomorphism type of $V$ while it does not affect the abstract isomorphism type of $V$ unless the coefficient of $p_+$ in $D_+$ is integral. Dually the same holds for $p_-$. Thus Theorem 1.0.7 implies Theorem 1.0.5, since fixing $D_0$ we can vary $p_+$ and $p_-$ for surfaces of type II and one of these points for surfaces of type I. This preserves the isomorphism type of $V$, but changes the equivalence class of the $\mathbb{C}^*$-action and of the $\mathbb{A}^1$-fibration.

**1.0.8.** To make the result above more concrete, consider for instance the normalization $V$ of the singular surface in $\mathbb{A}^3$ given by the equation

$$x^{n-1}y = (z - p_-)(z-1)^{n-1}q^{n-1}(z), \quad \text{where} \quad q(z) = \prod_{i=1}^{r}(z - p_i), \quad r \geq 0,\ n \geq 3,$$

and $1, p_-, p_1 \ldots, p_r \in \mathbb{C}$ are pairwise different. Such a surface carries a hyperbolic $\mathbb{C}^*$-action

$$\lambda.(x, y, z) = (\lambda x, \lambda^{1-n} y, z) ,$$

which amounts to a DPD-presentation (2) with $t = 2$ and $p_+ := 1$; see Example 4.10 in [$\text{FlZa}_1$]. For $r \geq 1$ these are special smooth Gizatullin surfaces of type I whereas for $r = 0$, $V = V(n)$ is the Danilov-Gizatullin surface of index $n$.

According to Theorem 1.0.7(b),(c) fixing a polynomial $q \in \mathbb{C}[z]$ of degree $r \geq 2$ and varying $p_- \in \mathbb{A}^1 \setminus \{p_1, \ldots, p_{r+1}\}$ the resulting surfaces $V$ are all abstractly but not equivariantly isomorphic, in general.

Let us give a brief overview of the contents of the various sections. After recalling in Section 2 some necessary preliminaries, we prove in Section 3 that the configuration of points in $\lfloor -D_+ - D_- \rfloor$ represents an invariant of the (abstract) isomorphism type of a Gizatullin $\mathbb{C}^*$-surface $V$, see Corollary 3.5.5 and Remark 3.5.6. This remains valid more generally without assuming the existence of a $\mathbb{C}^*$-action, see Theorem 3.4.1.

In Section 6 we establish that for special $\mathbb{C}^*$-surfaces this configuration of points together with the numbers $\deg\{D_\pm\}$ is the only invariant of the isomorphism type of $V$, see Theorem 6.2.1. This yields Theorem 1.0.7. The proof proceeds in two steps. In Section 4 we deal with special Gizatullin surfaces of $(-1)$-type, which are characterized by the property, that all feathers are $(-1)$-curves. The hard part in the proof of the general case is to transform any standard completion of a special surface into one of

(−1)-type. This problem is solved in Sections 5 and 6.1 using an explicit coordinate description of special surfaces.

Sections 6.2-6.4 contain applications of the main results. For instance, we show in Section 6.3 how to classify $\mathbb{C}_+$-actions or, equivalently, $\mathbb{A}^1$-fibrations on Gizatullin surfaces, see Theorem 6.3.18. In Section 6.4 we strengthen our previous uniqueness result [FlZa$_3$] for $\mathbb{A}^1$-fibrations on (singular, in general) Gizatullin surfaces.

The authors are grateful to Peter Russell for inspiring discussions concerning the results of Section 6.3.

## 2. Preliminaries

In this section we recall some necessary notions and facts from [DaGi] and [FKZ$_3$].

### 2.1. Standard zigzags and reversions.

**2.1.1.** Let $X$ be a complete normal algebraic surface, and let $D$ be an SNC (i.e. a simple normal crossing) divisor $D$ with rational components contained in the smooth part $X_{\text{reg}}$ of $X$. We say that $D$ is a *zigzag* if the dual graph $\Gamma_D$ of $D$ is linear i.e.,

$$\Gamma_D : \quad \underset{C_0}{\overset{w_0}{\circ}} \!\!-\!\!-\!\!-\!\! \underset{C_1}{\overset{w_1}{\circ}} \!\!-\!\!-\!\!-\; \cdots \;-\!\!-\!\!-\!\! \underset{C_n}{\overset{w_n}{\circ}} \;, \tag{4}$$

where $w_i = C_i^2$, $i = 0, \ldots, n$, are the weights of $\Gamma_D$. We abbreviate this chain by $[[w_0, \ldots, w_n]]$. We also write $[[\ldots, (w)_k, \ldots]]$ if a weight $w$ occurs at $k$ consecutive places.

**2.1.2.** A zigzag $D$ is called *standard* if its dual graph $\Gamma_D$ is either $[[w_0, \ldots, w_n]]$ with all $w_i \le -2$, or $\Gamma_D$ is one of the chains[6]

$$[[0]]\,, \quad [[0,0]]\,, \quad [[0,0,0]] \text{ or } [[0,0,w_2,\ldots,w_n]], \text{ where } n \ge 2 \text{ and } w_j \le -2 \;\; \forall j. \tag{5}$$

A linear chain $\Gamma$ is said to be *semi-standard* if it is either standard or one of

$$[[0, w_1, w_2, \ldots, w_n]], \quad [[0, w_1, 0]]\,, \text{ where } n \ge 1,\; w_1 \in \mathbb{Z}, \text{ and } w_j \le -2 \;\; \forall j \ge 2\,. \tag{6}$$

**2.1.3.** Every Gizatullin surface $V$ admits a *standard completion* $(\bar V, D)$ i.e., a completion by a standard zigzag $D$, see [DaGi, Du$_1$] or Theorem 2.15 in [FKZ$_1$]. Similarly we call $(\bar V, D)$ a *semi-standard completion* if its boundary zigzag is semi-standard.

The standard boundary zigzag is unique up to reversion

$$D = [[0,0,w_2,\ldots,w_n]] \rightsquigarrow [[0,0,w_n,\ldots,w_2]] =: D^\vee\,. \tag{7}$$

We say that $D$ is *symmetric* if $D = D^\vee$.

The reversion of a zigzag, regarded as a birational transformation of the weighted dual graph, admits a factorization into a sequence of inner elementary transformations (see 1.4 in [FKZ$_3$] and [FKZ$_1$]). By an *inner elementary transformation* of a weighted graph we mean blowing up at an edge incident to a 0-vertex of degree 2 and blowing

[6] The case of the zigzag $[[w_0, \ldots, w_n]]$ with all $w_i \le -2$ was unfortunately forgotten in [FKZ$_1$, Lemma 2.17], in [FKZ$_2$, 2.8] and in [FKZ$_3$, 1.2]. However, such a chain and also $[[0]]$ cannot appear as boundaries of affine surfaces. Hence this omission does not affect any of the results of these papers. By abuse of notation, we often denote an SNC divisor and its dual graph by the same letter.

down the image of this vertex. Given $[[0,0,w_2,\dots,w_n]]$ we can successively move the pair of zeros to the right

$$[[0,0,w_2,\dots,w_n]] \rightsquigarrow [[w_2,0,0,w_3,\dots,w_n]] \rightsquigarrow \dots \rightsquigarrow [[w_2,\dots,w_n,0,0]]$$

by inner elementary transformations, which gives the reversion. An *outer elementary transformation* consists in blowing up at a 0-vertex $v$ of degree $\le 1$ and blowing down the image of this vertex. We note that such elementary transformations are defined on the level of graphs as well as on the level of divisors $D$ on $\bar V$. An inner elementary transformation of $D$ on $\bar V$ is uniquely determined by the corresponding elementary transformation of its dual graph while in the case of outer elementary transformations it depends on a parameter, namely the center of blowup on the curve corresponding to $v$.

If $(\bar V, D)$ is a standard completion of a Gizatullin surface $V$, then reversing the zigzag $D$ by a sequence of inner elementary transformations we obtain from $(\bar V, D)$ a new completion $(\bar V^\vee, D^\vee)$, which we call the *reverse standard completion*. It is uniquely determined by $(\bar V, D)$.

2.2. **Symmetric reconstructions.** Given a Gizatullin surface, any two SNC completions are related via a birational transformation which is called a *reconstruction*, see [FKZ$_3$, Definition 4.1]. For further use we recall the necessary notions and facts.

**2.2.1.** Given weighted graphs $\Gamma$ and $\Gamma'$, a *(combinatorial) reconstruction* $\gamma$ of $\Gamma$ into $\Gamma'$ consists in a sequence

$$\gamma:\quad \Gamma=\Gamma_0 \overset{\gamma_1}{\dashrightarrow} \Gamma_1 \overset{\gamma_2}{\dashrightarrow} \cdots \overset{\gamma_n}{\dashrightarrow} \Gamma_n=\Gamma',$$

where each arrow $\gamma_i$ is either a blowup or a blowdown. The graph $\Gamma'$ is called the *end graph* of $\gamma$. The inverse sequence $\gamma^{-1}=(\gamma_n^{-1},\dots,\gamma_1^{-1})$ yields a reconstruction of $\Gamma'$ with end graph $\Gamma$.

Such a reconstruction is said to be

- *admissible* if it involves only blowdowns of at most linear vertices, inner blowups or outer blowups at end vertices;
- *symmetric* if it can be written in the form $(\gamma,\gamma^{-1})$. Clearly in this case the end graph is again $\Gamma$.

Similarly, given two SNC completions $(X,D)$ and $(Y,E)$ of a normal surface $V$, a reconstruction of $(X,D)$ into $(Y,E)$ consists in a sequence of blowups and blowdowns

$$\tilde\gamma:\quad X=X_0 \overset{\tilde\gamma_1}{\dashrightarrow} X_1 \overset{\tilde\gamma_2}{\dashrightarrow} \cdots \overset{\tilde\gamma_n}{\dashrightarrow} X_n=Y \quad ,$$

performed on $D$ and on its subsequent total transforms. Such a reconstruction induces a combinatorial reconstruction of $\Gamma_D$ into $\Gamma_E$. We say that $\tilde\gamma$ is admissible or symmetric if $\gamma$ is so.

For symmetric reconstructions the following hold (see Proposition 4.6 in [FKZ$_3$]).

**Lemma 2.2.2.** *Given two standard completions $(X,D)$ and $(Y,E)$ of a normal Gizatullin surface $V\not\cong \mathbb{A}^1\times\mathbb{A}^1_*$,* [7] *after replacing, if necessary, $(X,D)$ by its reversion $(X^\vee,D^\vee)$, there exists a symmetric reconstruction of $(X,D)$ into $(Y,E)$.*

[7] Here $\mathbb{A}^1_* := \mathbb{A}^1\setminus\{0\}$.

### 2.3. Generalized reversions.

**Definition 2.3.1.** We say that two standard completions $(\bar V, D)$, $(\bar V', D')$ of a Gizatullin surface $V$ are *evenly linked* if there is a symmetric reconstruction of $(\bar V, D)$ into $(\bar V', D')$. In particular, the dual graphs of $D$ and $D'$ are then the same. Otherwise $(\bar V, D)$ and $(\bar V', D')$ are called *oddly linked*.

By Lemma 2.2.2, $(\bar V', D')$ is always evenly linked to one of the completions $(\bar V, D)$ or $(\bar V^\vee, D^\vee)$.

**Definition 2.3.2.** We let $(\bar V, D)$ be a semi-standard completion a Gizatullin surface $V$ with boundary zigzag $[[0, -m, w_2, \ldots, w_n]]$, where $m \geq 0$. Moving the zero vertex to the right by elementary transformations we can transform this to a semi-standard zigzag $[[w_2, \ldots, w_m, -k, 0]]$ for every $k \geq 0$. We call the resulting semi-standard completion $(\bar V', D')$ a *generalized reversion* of $(\bar V, D)$.

Transforming $[[0, -m, w_2, \ldots, w_n]]$ into the standard zigzag $[[0, 0, w_2, \ldots, w_n]]$ requires outer elementary transformations; see 2.1.3. To transform further the latter zigzag into $[[w_2, \ldots, w_m, -k, 0]]$ only inner elementary transformations are needed. Thus the resulting semi-standard completion $(\bar V', D')$ depends on parameters, namely on the choice of the centers of outer blowups when passing from $[[0, -m, w_2, \ldots, w_n]]$ to $[[0, 0, w_2, \ldots, w_n]]$.

The following proposition follows from the connectedness part of Theorem I.1.2 in [DaGi]. We provide an independent proof relying on [$\mathrm{FKZ}_1$].

**Proposition 2.3.3.** *For any two semi-standard completions $(\bar V, D)$, $(\bar V', D')$ of a Gizatullin surface $V$, $(\bar V', D')$ can be obtained from $(\bar V, D)$ by a sequence of generalized reversions*

$$(\bar V, D) = (\bar V_0, D_0) \quad \rightsquigarrow \quad (\bar V_1, D_1) \quad \rightsquigarrow \quad \ldots \quad \rightsquigarrow \quad (\bar V_l, D_l) = (\bar V', D')\,.$$

*Proof.* By Lemma 3.29 in [$\mathrm{FKZ}_1$] we can find a sequence of semi-standard completions

$$(\bar V, D) = (\bar V_0, D_0), \quad (\bar V_1, D_1), \quad \ldots, \quad (\bar V_l, D_l) = (\bar V', D')$$

such that every step $(\bar V_i, D_i) \rightsquigarrow (\bar V_{i+1}, D_{i+1})$ is dominated by a completion $(W_i, E_i)$ of $V$ with a linear zigzag $E_i$. Thus it is sufficient to show the assertion in case where $l = 1$ i.e., $(\bar V, D)$ and $(\bar V', D')$ are dominated by a completion $(W, E)$ with a linear zigzag $E$. We can perform elementary transformations of $(\bar V, D)$ and $(\bar V', D')$ to obtain standard completions $(\bar V_0, D_0)$ and $(\bar V_0', D_0')$, respectively, such that all these surfaces are dominated by a suitable admissible blowup of $(W, E)$. Replacing $(\bar V, D)$ and $(\bar V', D')$ by $(\bar V_0, D_0)$ and $(\bar V_0', D_0')$, respectively, we are reduced to the case where both $(\bar V, D)$ and $(\bar V', D')$ are standard completions of $V$. The result follows now from Proposition 3.4 in [$\mathrm{FKZ}_1$], which says that a birational transformation between standard graphs $\Gamma_D \dashrightarrow \Gamma_{D'}$ dominated by a linear graph is either the identity or the reversion. □

## 3. The principle of matching feathers

Consider a Gizatullin surface $V$. By Gizatullin's Theorem [$\mathrm{Gi}_1$] (see also [$\mathrm{FKZ}_1$]), the sequence of weights $[[w_2, \ldots, w_n]]$ (up to reversion) of a standard boundary zigzag $D$ of $V$ is a discrete invariant of the abstract isomorphism type of $V$. In this section we introduce a more subtle continuous invariant of $V$ called the *configuration invariant*. This is a point in the product of certain configuration spaces, up to reversing the order in the product. Although it is defined using a standard completion of $V$, in Corollary 3.4.3 below we establish that this point is an invariant of the open surface $V$.

3.1. **Configuration spaces.** The configuration invariant takes values in configurations of points on $\mathbb{A}^1$ and $\mathbb{A}^1_* = \mathbb{A}^1 \setminus \{0\}$. We recall shortly the necessary notions.

**3.1.1.** We let $\mathcal{M}_s^+$ denote the configuration space of all $s$-points subsets $\{\lambda_1, \ldots, \lambda_s\}$ of the affine line $\mathbb{A}^1$. This is a Zariski open subset of the Hilbert scheme of $\mathbb{A}^1$. By the main theorem on symmetric functions $\mathcal{M}_s^+$ can be identified with the set of all monic polynomials $P = X^s + \sum_{j=1}^s a_j X^{s-j}$ of degree $s$, whose discriminant is nonzero. This identification

$$\{\lambda_1, \ldots, \lambda_s\} \mapsto P = \prod_{j=1}^s (X - \lambda_j)$$

sends $\mathcal{M}_s^+$ onto the principal Zariski open subset $D(\mathrm{discr}) := \mathbb{A}^s \backslash \{\mathrm{discr}(P) = 0\}$ of $\mathbb{A}^s$.

The affine group $\mathrm{Aut}(\mathbb{A}^1)$ acts on $\mathcal{M}_s^+$ in a natural way. By restriction we obtain an action on $\mathcal{M}_s^+$ of the normal subgroup $\mathbb{G}_a$ of translations. The quotient $\mathcal{M}_s^+/\mathbb{G}_a$ can be identified with the space, say, $U_0$ of all monic polynomials $P = X^s + \sum_{j=2}^s a_j X^{s-j}$ with $a_1 = 0$ and with nonzero discriminant. The residual action of the multiplicative group $\mathbb{C}^* \simeq \mathrm{Aut}(\mathbb{A}^1)/\mathbb{G}_a$ on $U_0$ is given by

$$P = X^s + \sum_{j=2}^s a_j X^{s-j} \mapsto t.P := X^s + \sum_{j=2}^s t^j a_j X^{s-j}\,, \qquad t \in \mathbb{C}^*\,.$$

Consequently, the quotient

$$\mathfrak{M}_s^+ = \mathcal{M}_s^+/\mathrm{Aut}(\mathbb{A}^1)$$

exists and is an affine variety of dimension $s - 2$. More precisely, we can identify $\mathfrak{M}_s^+$ with the principal Zariski open subset of the weighted projective space $\mathbb{P}(2, 3, \ldots, s)$ given by the discriminant i.e., $\mathfrak{M}_s^+ = D_+(\mathrm{discr})$, see e.g., [Li] or [ZaLi, Ch. 1, Example 2].

**3.1.2.** Let $\mathcal{M}_s^*$ be the part of $\mathcal{M}_s^+$ consisting of all subsets of $\mathbb{A}^1_*$. Similarly as before this can be identified with the space of all monic polynomials $P$ of degree $s$ with $P(0) \neq 0$. The group $\mathbb{C}^*$ acts on $\mathcal{M}_s^*$, and the quotient $\mathfrak{M}_s^*$ embeds as a principal Zariski open subset into the weighted projective space $\mathbb{P}(1, \ldots, s)$.

3.2. **The configuration invariant.** We consider a smooth Gizatullin surface $V = \bar{V} \setminus D$ completed by a semi-standard zigzag

$$(8)\quad D: \quad \underset{C_0}{\overset{0}{\circ}} \text{—} \underset{C_1}{\overset{-m}{\circ}} \text{—} \underset{C_2}{\overset{w_2}{\circ}} \text{—} \cdots \text{—} \underset{C_i}{\overset{w_i}{\circ}} \text{—} \cdots \text{—} \underset{C_n}{\overset{w_n}{\circ}}\,, \qquad w_i \leq -2 \ \ \forall i \geq 2.$$

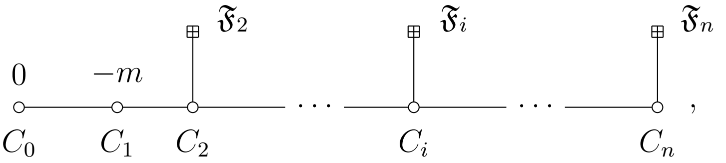
We associate to $(\bar{V}, D)$ a point in the product of configuration spaces

$$\mathfrak{M} := \mathfrak{M}_{s_2}^{\tau_2} \times \ldots \times \mathfrak{M}_{s_n}^{\tau_n}$$

for suitable numbers $s_i$, $2 \leq i \leq n$, where $\tau_i \in \{+, *\}$ depends on the component $C_i$ as described below. This point occurs to be an invariant of $V$, i.e. it depends only on the isomorphism class of $V$ and not on the choice of a semi-standard completion $(\bar{V}, D)$.

**3.2.1.** To define this invariant we need to recall the notion of extended divisor. Let $(\bar{V}, D)$ be a semi-standard completion of a Gizatullin surface $V$. Then the linear system $|C_0|$ on $\bar{V}$ defines a morphism $\Phi_0 : \bar{V} \to \mathbb{P}^1$ with at most one degenerate fiber, say, $\Phi_0^{-1}(0)$ while the fiber $C_0 = \Phi^{-1}(\infty)$ is non-degenerate. The reduced SNC divisor $D_{\text{ext}} = D \cup \Phi_0^{-1}(0)$ is called the *extended divisor* of the completion $(\bar{V}, D)$. By Proposition 1.11 of [FKZ$_3$], this divisor has dual graph

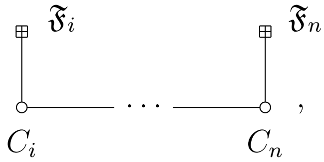
(9) $D_{\text{ext}}$ :



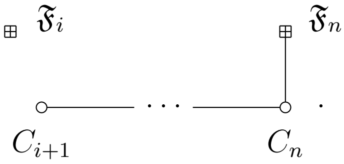

where $\mathfrak{F}_i = \{F_{i\rho}\}_{1\le\rho\le r_i}$ $(2 \le i \le n)$ is a collection of pairwise disjoint *feathers* attached to the component $C_i$, $i \ge 2$. A feather is a linear chain of smooth rational curves on $\bar{V}$. In our particular case, where $V$ is assumed to be smooth and affine, each of these feathers consists of just one smooth rational curve $F_{ij}$ with self-intersection $w_{ij} := F_{ij}^2 \le -1$. Given $i$ we have $w_{ij} \le -2$ for at most one of the feathers $F_{ij}$, see Proposition 2.6 in [FKZ$_3$].

As in [FKZ$_3$] we let $D_{\text{ext}}^{\ge i}$ denote the branch of $D_{\text{ext}}$ at the vertex $C_{i-1}$ containing $C_i$, while $D_{\text{ext}}^{>i}$ stands for $D_{\text{ext}}^{\ge i} - C_i$.

$D_{\text{ext}}^{\ge i}$ :




$D_{\text{ext}}^{>i}$ :




Likewise, we let $D^{\ge i} = D \cap D_{\text{ext}}^{\ge i}$ and $D^{>i} = D \cap D_{\text{ext}}^{>i}$.

Assume now that $(\bar{V}, D)$ is a standard completion so that $m = 0$ in (9). The linear systems $|C_0|, |C_1|$ on $\bar{V}$ define a morphism

$$\Phi = \Phi_0 \times \Phi_1 : \bar{V} \to \mathbb{P}^1 \times \mathbb{P}^1,$$

called the *standard morphism*, which is birational according to [FKZ$_2$], Lemma 2.19.

Decomposing $\Phi$ into a sequence of blowups we can grow $\bar{V}$ starting with the quadric $\mathbb{P}^1 \times \mathbb{P}^1$, see [FKZ$_3$]. For a feather $F_{i\rho}$ we let $C_{\mu_{i\rho}}$ denote its *mother component*. The latter means that $F_{i\rho}$ was born by a blowup with center on $C_{\mu_{i\rho}}$ under this decomposition of $\Phi$. Since the zigzag $D$ is connected this defines $C_{\mu_{i\rho}}$ in a unique way. By Proposition 2.6 in [FKZ$_3$], $\mu_{i\rho} = i$ (i.e., $C_i$ is the mother component for $F_{i\rho}$) if and only if $F_{i\rho}^2 = -1$, otherwise $\mu_{i\rho} < i$. In the latter case we say that the feather $F_{i\rho}$ jumped.

**Definition 3.2.2.** (1) We let

$$s_\mu := \#\{(i,\rho) : 2 \le i \le n,\ 1 \le \rho \le r_i,\ \text{and}\ \mu_{i\rho} = \mu\}$$

be the number of feathers $F_{i\rho}$ whose mother component is $C_\mu$.

(2) We say that $C_\mu$ $(2 \le \mu \le n)$ is a component *of type* $*$, or a $*$-*component* for short, if

(i) $D_{\text{ext}}^{\ge\mu+1}$ is not contractible[8], and

[8]We say that a divisor $G$ on a smooth projective surface is contractible if it can be contracted into a smooth point. In this case the dual graph $\Gamma_G$ is also said to be contractible.

(ii) $D_{\text{ext}}^{\geq\mu+1} - F_{ij}$ is not contractible for every feather $F_{ij}$ of $D_{\text{ext}}^{\geq\mu+1}$ with mother component $C_\tau$, where $\tau < \mu$.

Otherwise $C_\mu$ is called a component of type $+$, or simply a $+$-*component*. For instance, $C_2$ and $C_n$ are always components of type $+$. We let $\tau_\mu = *$ in the first case and $\tau_\mu = +$ in the second one.

**Remark 3.2.3.** It is easily seen that, in the process of blowing up starting from the quadric, every $*$-component $C_\mu$ with $\mu \geq 3$ appears as a result of an inner blowup of the previous zigzag, while an outer blowup of a zigzag creates a $+$-component (cf. 2.1.3 and 1.0.3 in the Introduction).

Given a component $C_\mu$ we denote by $p_{\mu\rho}$, $1 \leq \rho \leq s_\mu$, the following collection of points on $C_\mu \backslash C_{\mu-1} \cong \mathbb{A}^1$. For every feather $F_{\mu\rho}$ of self-intersection $-1$ we let $p_{\mu\rho}$ be its intersection point with $C_\mu$. This gives $r_\mu$ or $r_\mu - 1$ points on $C_\mu \backslash C_{\mu-1}$ depending on whether the feathers $F_{\mu\rho}$ attached to $C_\mu$ are all $(-1)$-curves or not. If there is a feather $F_{ij}$ with mother component $C_\mu$ and with $i > \mu$ then we also add the intersection point $c_{\mu+1}$ of $C_\mu$ and $C_{\mu+1}$ to our collection. Thus $s_\mu$ is one of the numbers $r_\mu - 1$, $r_\mu$ or $r_\mu + 1$, and the points

$$p_{\mu\sigma} \in C_\mu, \ 1 \leq \sigma \leq s_\mu$$

are just the locations on $C_\mu$ in which the feathers with mother component $C_\mu$ are born by a blowup. We call them the *base points* of the associated feathers.

The collection $(p_{\mu\sigma})_{1\leq\sigma\leq s_\mu}$ defines a point $Q_\mu$ in the configuration space $\mathfrak{M}^+_{s_\mu}$.

Suppose further that $C_\mu$ is a component of type $*$. We consider then $Q_\mu$ as a collection of points in $C_\mu \backslash (C_{\mu-1} \cup C_{\mu+1})$. Note that the intersection point $c_{\mu+1}$ of $C_\mu$ and $C_{\mu+1}$ cannot be one of the points $p_{\mu\sigma}$ because of (ii) in Definition 3.2.2(2). Identifying $C_\mu \backslash (C_{\mu-1} \cup C_{\mu+1})$ with $\mathbb{C}^*$ in such a way that $c_{\mu+1}$ corresponds to 0 and $c_\mu$ to $\infty$, we obtain a point in the configuration space $\mathfrak{M}^*_{s_\mu}$. Thus in total we obtain a point

$$Q(\bar{V}, D) := (Q_2, \ldots Q_n) \in \mathfrak{M} = \mathfrak{M}^{\tau_2}_{s_2} \times \ldots \times \mathfrak{M}^{\tau_n}_{s_n}$$

called the *configuration invariant* of $(\bar{V}, D)$.

Performing in $(V, D)$ elementary transformations with centers at the component $C_0$ in (8) does neither change $\Phi_0$ nor the extended divisor (except for the self-intersection $C_1^2$) and thus leaves $s_i$ and $Q(\bar{V}, D)$ invariant. Hence we can define these invariants for any semi-standard completion $(\bar{V}, D)$ of $V$ by sending it via elementary transformations with centers on $C_0$ into a standard completion.

3.3. **Matching feathers.** In the following proposition we show that reversion of the boundary zigzag of length $n$ leads to the same configuration invariant. To formulate this result, it is convenient to use systematically the notation

$$t^\vee = n - t + 2 \qquad (10)$$

for an integer $t \in \mathbb{Z}$.

**Proposition 3.3.1.** (Matching Principle) *Let $V = \bar{V} \setminus D$ be a smooth Gizatullin surface completed by a standard zigzag $D$. Consider the reversed completion $(\bar{V}^\vee, D^\vee)$ with boundary zigzag $D^\vee = C_0^\vee \cup \ldots \cup C_n^\vee$, associated numbers $s'_2, \ldots, s'_n$ and types*

*$\tau'_2, \ldots, \tau'_n$. Then $s_i = s'_{i^\vee}$ and $\tau_i = \tau'_{i^\vee}$ for $i = 2, \ldots, n$. Furthermore, the associated points $Q(\bar V, D)$ and $Q(\bar V^\vee, D^\vee)$ in $\mathfrak{M}$ coincide under the natural identification*

$$\mathfrak{M} = \mathfrak{M}_{s_2}^{\tau_2} \times \ldots \times \mathfrak{M}_{s_n}^{\tau_n} \cong \mathfrak{M}_{s'_2}^{\tau'_2} \times \ldots \times \mathfrak{M}_{s'_n}^{\tau'_n} .$$

The proof is given in 3.3.2-3.3.11 below. It uses the following construction.

**3.3.2. Correspondence fibration.** Let us consider a standard completion $(\bar V, D)$ of $V$ and the reversed completion $(\bar V^\vee, D^\vee)$. Thus $D = C_0 \cup \ldots \cup C_n$ is a standard zigzag $[[0, 0, w_2, \ldots, w_n]]$ as in (8) and $D^\vee = C_0^\vee \cup \ldots \cup C_n^\vee$ is the standard zigzag $[[0, 0, w_n, \ldots, w_2]]$. Let $D_{\mathrm{ext}}$ and $D^\vee_{\mathrm{ext}}$ denote the corresponding extended divisors and let $F_{i\rho}$ and $F^\vee_{j\rho}$ be the feathers attached to $C_i$, $C^\vee_j$, respectively.

Using inner elementary transformations we can move the pair of zeros in the zigzag $[[0, 0, w_2, \ldots, w_n]]$ several places to the right. In this way we obtain a new completion, say, $(W, E)$ of $V$ with boundary zigzag $E = [[w_2, \ldots, w_{t-1}, 0, 0, w_t, \ldots, w_n]]$ for some $t \in \{2, \ldots, n+1\}$. For $t = 2$, $E = D = [[0, 0, w_2, \ldots, w_n]]$ is the original zigzag, while for $t = n + 1$, $E = D^\vee = [[w_2, \ldots, w_n, 0, 0]]$ is the reversed one. The transformed components of $E$ are

$$E = C_n^\vee \cup \ldots \cup C^\vee_{t^\vee} \cup C_{t-1} \cup C_t \cup \ldots \cup C_n ,$$

where we identify $C_i \subseteq \bar V$ and $C^\vee_j \subseteq \bar V^\vee$ with their proper transforms in $W$ ($t-1 \le i \le n$, $t^\vee \le j \le n$). In particular $E = D^{\ge t-1} \cup D^{\vee \ge t^\vee}$ with new weights $C_{t-1}^2 = C_{t^\vee}^{\vee 2} = 0$. There are natural isomorphisms

$$\begin{aligned} W\backslash D^{\vee\ge t^\vee} &= W\backslash (C_n^\vee \cup \ldots \cup C^\vee_{t^\vee}) \cong \bar V\backslash (C_0 \cup \ldots \cup C_{t-2}) \qquad \text{and} \\ W\backslash D^{\ge t-1} &= W\backslash (C_{t-1} \cup \ldots \cup C_n) \cong \bar V^\vee\backslash (C_0^\vee \cup \ldots \cup C^\vee_{t^\vee-1}) . \end{aligned} \tag{11}$$

In the proof of the Matching Principle 3.3.1 we use the following fibration.

**Definition 3.3.3.** The map

$$\psi : W \to \mathbb{P}^1$$

defined by the linear system $|C_{t-1}|$ on $W$ will be called the *correspondence fibration* for the pair of curves $(C_t, C^\vee_{t^\vee})$.

The components $C_t$ and $C^\vee_{t^\vee}$ represent sections of $\psi$. Since the feathers of $D_{\mathrm{ext}}$ and $D^\vee_{\mathrm{ext}}$ are not contained in the boundary zigzags they are not contracted in $W$. We denote their proper transforms in $W$ by the same letters. It will be clear from the context where they are considered.

We use below the following technical facts.

**Lemma 3.3.4.** *(a) The divisor $D^{\ge t+1}_{\mathrm{ext}}$ is contained in some fiber $\psi^{-1}(q)$, $q \in \mathbb{P}^1$. Similarly, $D^{\vee\ge t^\vee+1}_{\mathrm{ext}}$ is contained in some fiber $\psi^{-1}(q^\vee)$. The points $q$ and $q^\vee$ are uniquely determined unless $D^{\ge t+1}_{\mathrm{ext}}$ and $D^{\vee\ge t^\vee+1}_{\mathrm{ext}}$ are empty, respectively.*
*(b) A fiber $\psi^{-1}(p)$ can have at most one component $C$ not belonging to $D^{\ge t}_{\mathrm{ext}} \cup D^{\vee \ge t^\vee}_{\mathrm{ext}}$. Such a component $C$ meets both $D^{\ge t}$ and $D^{\vee\ge t^\vee}$.*

*Proof.* (a) follows immediately for the divisor $D^{\ge t+1}_{\mathrm{ext}}$ since it is connected and disjoint from the full fiber $C_{t-1}$ of $\psi$. By symmetry the assertion holds also for the divisor $D^{\vee\ge t^\vee+1}_{\mathrm{ext}}$.

(b) Let $C$ be a component of the fiber $\psi^{-1}(p)$ belonging neither to $D_{\rm ext}^{>t}$ nor to $D_{\rm ext}^{\vee>t^\vee}$. We claim that it meets both $D^{\geq t}$ and $D^{\vee\geq t^\vee}$. Assume on the contrary that it does not meet e.g., $D^{\geq t}$. Since the affine surface $V$ does not contain complete curves and $V = W\backslash E$, we have $C\cdot E = C\cdot D^{\vee\geq t^\vee} \neq 0$. Thus the proper transform $C'$ of $C$ on $\bar V^\vee$ must be a feather of $D_{\rm ext}^\vee$. Indeed, $C'\cdot C_0^\vee = 0$, see (11). Hence $C'$ is a component of the only degenerate fiber $(\Phi_0^\vee)^{-1}(0)$ of $\Phi_0^\vee : \bar V^\vee \to \mathbb{P}^1$ and does not belong to $D^\vee$. Since $C'\cdot D^{\vee\geq t^\vee} \neq 0$, we must have $C' \subseteq D_{\rm ext}^{\vee>t^\vee}$ on $\bar V^\vee$. This contradicts our assumption that $C$ does not belong to $D_{\rm ext}^{\vee>t^\vee}$ on $W$, and so the claim follows.

Finally, there can be at most one such fiber component $C$ since the fiber $\psi^{-1}(p)$ does not contain cycles and meets only once each of the sections $C_t$ and $C_{t^\vee}^\vee$ of $\psi$. □

**Corollary 3.3.5.** *If, in the notation as in Lemma 3.3.4(a), $q \neq q^\vee$ then each of the divisors $D_{\rm ext}^{\geq t+1}$ and $D_{\rm ext}^{\vee\geq t^\vee+1}$ is either empty or contractible.*

*Proof.* We suppose that $q\neq q^\vee$ and $D_{\rm ext}^{\geq t+1} \neq \emptyset$. By Lemma 3.3.4(a) the latter divisor is contained in the fiber $\psi^{-1}(q)$. This fiber contains also a component $C$ meeting the section $C_{t^\vee}^\vee$. Clearly such a curve $C$ is neither a component of the zigzag nor a feather of $D_{\rm ext}^{\geq t+1}$ and so not a component of $D_{\rm ext}^{\geq t+1}$. Since $q \neq q^\vee$ the fiber over $q$ cannot contain any component of $D_{\rm ext}^{\vee\geq t^\vee+1}$. Thus by Lemma 3.3.4(b) $\psi^{-1}(q) = C \cup D_{\rm ext}^{\geq t+1}$. Since the multiplicity of $C$ in the fiber is 1, the remaining part $D_{\rm ext}^{\geq t+1}$ of the fiber can be blown down. Symmetrically, the same holds for $D_{\rm ext}^{\vee\geq t^\vee+1}$. □

The following lemma is crucial in the proof of Proposition 3.3.1, see 3.3.11 below.

**Lemma 3.3.6.** *Let $F_{i\rho}$ be a feather of the extended divisor $D_{\rm ext}$ attached to $C_i$ and with mother component $C_\tau$. For an index $t$ with $\tau \leq t \leq i$, we consider the correspondence fibration $\psi : W \to \mathbb{P}^1$ as defined in 3.3.3 above. Then the following hold.*

*(a) $F_{i\rho}$ is contained in a fiber $\psi^{-1}(q_{i\rho})$ on $W$ for some point $q_{i\rho} \in \mathbb{P}^1$.*
*(b) The fiber $\psi^{-1}(q_{i\rho})$ contains as well a feather $F_{j\sigma}^\vee$ of $D_{\rm ext}^{\vee>t^\vee}$ meeting $F_{i\rho}$. This feather $F_{j\sigma}^\vee$ has mother component $C_{\tau^\vee}^\vee$.*
*(c) The feather $F_{j\sigma}^\vee$ in (b) is uniquely determined by $F_{i\rho}$, and the points $q_{i\rho}$ in (a) are all different.*

*Proof.* Let $F_{i\rho}$ be a feather with mother component $C_\tau$, where $\tau\leq t\leq i$. Since $F_{i\rho}$ does not meet $C_{t-1}$ it is vertical with respect to $\psi$ and so contained in a fiber over some point $q_{i\rho}\in\mathbb{P}^1$, proving (a). We note that by the same reasoning, any feather $F_{j\sigma}^\vee$ of $D_{\rm ext}^\vee$ with $j \geq t^\vee$ is a fiber component of $\psi$ on $W$.

To deduce (b), let us start with the case $i = t = \tau$ so that $F_{t\rho}$ is a $(-1)$-feather. In this case the fiber $\psi^{-1}(q_{i\rho})$ cannot be irreducible and so $F_{t\rho}$ meets some other component, say, $C$ of $\psi^{-1}(q_{i\rho})$. Clearly, $q_{t\rho} \neq q$ (see Lemma 3.3.4(a)) and so $C\cdot D_{\rm ext}^{\geq t} = 0$. By Lemma 3.3.4 (b) $C$ belongs either to $D_{\rm ext}^{\vee>t^\vee}$ or to $D_{\rm ext}^{>t}$. Since $F_{t\rho}$ cannot meet any other feather of $D_{\rm ext}$ and cannot meet the boundary zigzag twice, $C$ must be one of the feathers, say, $F_{j\sigma}^\vee$ of $D_{\rm ext}^{\vee>t^\vee}$.

Let us show that $F_{j\sigma}^\vee$ has mother component $C_{t^\vee}^\vee$. Since $F_{t\rho}\cdot C_t = 1$, the feather $F_{t\rho}$ has multiplicity 1 in the fiber $\psi^{-1}(q_{t\rho})$. Thus the remaining part $\psi^{-1}(q_{t\rho}) - F_{t\rho}$ can be blown down to a $(-1)$-curve. After this contraction we must still have $F_{t\rho}^2 = -1$. Hence this remaining $(-1)$-curve must be the image of $C = F_{j\sigma}^\vee$. Moreover, after this contraction $F^\vee_{j\sigma} \cdot C^\vee_{t^\vee} = 1$, hence $F^\vee_{j\sigma}$ appears under a blowup with center on $C^\vee_{t^\vee}$. Thus the mother component of $F^\vee_{j\sigma}$ is indeed $C^\vee_{t^\vee}$, as stated.

Consider further the case where $i > t = \tau$ so that $F_{i\rho}$ is contained in the fiber $\psi^{-1}(q)$, see Lemma 3.3.4(a). According to Proposition 2.6(b) in [FKZ$_3$] the divisor $A := D^{\geq t+1}_{\mathrm{ext}} - F_{i\rho}$ is contractible to a point on $C_t$. Since $C_t$ is the mother component of $F_{i\rho}$ in $D_{\mathrm{ext}}$, after this contraction $F_{i\rho}$ becomes a $(-1)$-curve with $F_{i\rho} \cdot E = F_{i\rho} \cdot C_t = 1$. Replacing $W$ by the contracted surface $W/A$ and arguing as before the result follows as well in this case.

If $i, t, \tau$ with $\tau \leq t \leq i$ are arbitrary, then we pass to the correspondence fibration $\psi' : W' \to \mathbb{P}^1$ for the pair $(C_\tau, C^\vee_{\tau^\vee})$, see Definition 3.3.3. By what was shown already there is a feather $F^\vee_{j\sigma}$ of $D^{\vee > \tau^\vee}_{\mathrm{ext}}$ with mother component $C^\vee_{\tau^\vee}$ meeting $F_{i\rho}$ on $W'$. Since $D^{\vee > \tau^\vee}_{\mathrm{ext}} \subseteq D^{\vee > t^\vee}_{\mathrm{ext}}$ these feathers $F_{i\rho}$ and $F^\vee_{j\sigma}$ also meet on the surface $W$. Being both fiber components of $\psi$ (see the proof of (a) above), they meet within the same fiber. This completes the proof of (b).

Finally, (c) is a simple consequence of the fact that the fibers of $\psi$ cannot contain cycles and intersect with index 1 each of the sections $C_t$, $C^\vee_{t^\vee}$ of $\psi$. □

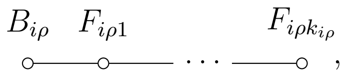
Lemma 3.3.6 motivates the following definition.

**Definition 3.3.7.** Consider a pair $(F_{i\rho},\, F^\vee_{j\sigma})$, where $F_{i\rho}$ is a feather of $D_{\mathrm{ext}}$ attached to component $C_i$ and $F^\vee_{j\sigma}$ is a feather of $D^\vee_{\mathrm{ext}}$ attached to component $C^\vee_j$. This pair is called a *pair of matching feathers*, or simply a *matching pair*, if $i + j \geq n + 2$ and $F_{i\rho}$ and $F^\vee_{j\sigma}$ meet on $V$.

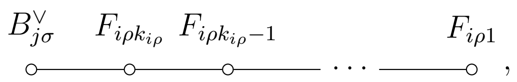
**Remark 3.3.8.** Thus if two feathers $F_{i\rho}$ and $F^\vee_{j\sigma}$ with $i + j \geq n + 2$ meet on $V$, then $(F_{i\rho},\, F^\vee_{j\sigma})$ is a matching pair. By Lemma 3.3.6 every feather $F_{i\rho}$ of $D_{\mathrm{ext}}$ has a unique matching feather $F^\vee_{j\sigma}$ of $D^\vee_{\mathrm{ext}}$, and vice versa. Moreover, if $F_{i\rho}$ has mother component $C_\tau$ then its matching feather $F^\vee_{j\sigma}$ has mother component $C^\vee_{\tau^\vee}$.

The condition $i + j \geq 2$ here is essential. Indeed, every feather $F_{t-1,\rho}$ represents a section of $\psi$, hence it meets every fiber of $\psi$. Since it cannot meet $D^{\geq t}_{\mathrm{ext}}$, it meets every feather $F^\vee_{t^\vee,\sigma}$ with $F^{\vee 2}_{t^\vee,\sigma} = -1$ on the affine surface $V$.

**Remark 3.3.9.** One can treat in a similar way feathers of arbitrary length instead of length one. Such feathers appear under the minimal resolution of a singular Gizatullin surface. These are linear chains

$F_{i\rho}$ :



where the chain $F_{i\rho} - B_{i\rho}$ (if non-empty) contracts to a singular point on $V$ and $B_{i\rho}$ is attached to component $C_i$. The curve $B_{i\rho}$ is also called the *bridge curve* of the feather $F_{i\rho}$. For instance, an $A_k$-singularity on $V$ leads to an $A_k$-feather, where $k_{i\rho} = k$ and $F_{i\rho} - B_{i\rho}$ is a chain of $(-2)$-curves of length $k$. Again, a matching principle provides a one-to-one correspondence between feathers $F_{i\rho}$ and $F^\vee_{j\sigma}$ such that the mother component of the bridge curve $B_{i\rho}$ of $F_{i\rho}$ is equal to the mother component of the tip of $F^\vee_{j\sigma}$. Moreover $F^\vee_{j\sigma}$ has dual graph




so that the tip of $F^\vee_{j\sigma}$ is just $F_{i\rho 1}$.

In the next lemma we show that the reversion respects the type of components of the zigzag.

**Lemma 3.3.10.** *$C_t$ is a $*$-component if and only if $C^\vee_{t^\vee}$ is. Furthermore in this case the points $q$ and $q^\vee$ in Lemma 3.3.4(a) are equal.*

*Proof.* Let $C_t$ be a $*$-component of the zigzag $D$; see Definition 3.2.2.(2). Let us first deduce the second assertion. If on the contrary $q \neq q^\vee$ then by Corollary 3.3.5 above, $D^{\geq t+1}_{\text{ext}}$ is contractible contradicting (i) in Definition 3.2.2. Thus $q = q^\vee$.

It remains to check that also $C^\vee_{t^\vee}$ is a $*$-component i.e., conditions (i) and (ii) in Definition 3.2.2 are satisfied.

To check (i), we assume on the contrary that $D^{\vee\geq t^\vee+1}_{\text{ext}}$ is contractible. After contracting this divisor in the fiber of $\psi$ over $q = q^\vee$ there is a component $F$ that meets the section $C^\vee_{t^\vee}$ with multiplicity 1. The rest, say, $R$ of the remaining fiber is as well contractible. Clearly $F$ cannot be a component of the zigzag $D^{\geq t+1}$. If $F$ is a feather of $D^{\geq t+1}_{\text{ext}}$ then $R = D^{\geq t+1}_{\text{ext}} - F$ is contractible, which is only possible if $F$ has mother component $C_\tau$, $\tau \leq t$. The latter contradicts condition (ii) in Definition 3.2.2. Otherwise $F = C$ is an extra component of the fiber $\psi^{-1}(q)$ (see Lemma 3.3.4(b)), and the argument as in the proof of Corollary 3.3.5 shows that $R = D^{\geq t+1}_{\text{ext}}$. Since $R$ is contractible, again we get a contradiction, this time to (i) of Definition 3.2.2.

Let us finally check that (ii) in Definition 3.2.2 holds. We need to show that for every feather $F^\vee$ of $D^{\vee\geq t^\vee+1}_{\text{ext}}$ with mother component $C^\vee_\tau$, where $\tau \leq t^\vee$, the divisor $D^{\vee\geq t^\vee+1}_{\text{ext}} - F^\vee$ cannot be contracted. Indeed, otherwise after contracting this divisor, $F^\vee$ meets the section $C^\vee_{t^\vee}$. The remaining fiber is $D^{\geq t+1}_{\text{ext}} + F^\vee$, since $F^\vee$ meets a matching feather $F$ in $D^{\geq t+1}_{\text{ext}}$. Hence $D^{\geq t+1}_{\text{ext}}$ is contractible, and again we arrive at a contradiction. □

Now we are ready to deduce Proposition 3.3.1.

**3.3.11.** *Proof of Proposition 3.3.1.* By Lemma 3.3.6(b) the map $\psi$ provides a one-to-one correspondence between the feathers of $D_{\text{ext}}$ with mother component $C_t$ and the feathers of $D^\vee_{\text{ext}}$ with mother component $C^\vee_{t^\vee}$. Moreover using Lemma 3.3.6(c) it provides a one-to-one correspondence between the base points $p_{t\rho}$ and $p^\vee_{t^\vee,\rho}$ of the feathers as considered in 3.2.1. By virtue of Lemma 3.3.10 $C_t$ is a $*$-component if and only if $C^\vee_{t^\vee}$ is, proving the assertion. □

3.4. **Invariance of the configuration invariant.** Theorem 3.4.1 and its Corollary 3.4.3 below are the main results of Section 3.

**Theorem 3.4.1.** *Given two semi-standard completions $(\bar V, D)$, $(\bar V', D')$ of a Gizatullin surface $V$, for the corresponding configuration invariants $s_i$, $s'_i$ and $Q(\bar V, D) \in \mathfrak{M}$, $Q(\bar V', D') \in \mathfrak{M}'$ as introduced in 3.2.1 the following hold.*

*(1) If $(\bar V, D)$ and $(\bar V', D')$ are evenly linked then $s_i = s'_i$ for $i = 2, \ldots, n$ and the points $Q(\bar V, D)$ and $Q(\bar V', D')$ of $\mathfrak{M} = \mathfrak{M}'$ coincide.*

*(2) If $(\bar V, D)$ and $(\bar V', D')$ are oddly linked then $s_i = s'_{i^\vee}$ for $i = 2, \ldots, n$ and the points $Q(\bar V, D)$ and $Q(\bar V', D')$ of $\mathfrak{M}$ and $\mathfrak{M}'$ coincide under the identification*

$$\mathfrak{M} = \mathfrak{M}^{\tau_2}_{s_2} \times \ldots \times \mathfrak{M}^{\tau_n}_{s_n} \cong \mathfrak{M}^{\tau'_n}_{s'_n} \times \ldots \times \mathfrak{M}^{\tau'_2}_{s'_2} = \mathfrak{M}' .$$

*Proof.* (1) By Proposition 2.3.3 $(\bar{V}', D')$ can be obtained by a sequence of generalized reversions

$$(\bar{V}, D) = (\bar{V}_0, D_0) \quad \rightsquigarrow \quad (\bar{V}_1, D_1) \quad \rightsquigarrow \quad \ldots \quad \rightsquigarrow \quad (\bar{V}_l, D_l) = (\bar{V}', D')\,,$$

where $(\bar{V}_i, D_i)$, $0 \le i \le l$, are semi-standard completions of $V$. Moreover $l$ is even if $(\bar{V}, D)$ and $(\bar{V}, D)$ are evenly linked, and odd otherwise. Hence it suffices to show the theorem for a generalized reversion of semi-standard completions. Since elementary transformations on $(\bar{V}, D)$ with centers on the component $C_0$ in (8) do not change the extended divisor (except for the weight $C_1^2$) and leave $s_i$ and $Q(V, D)$ invariant, we can reduce the statement to the case where $(V, D)$ and, symmetrically, $(V', D')$ are standard. The assertion now follows from Proposition 3.3.1. □

**Definition 3.4.2.** Given a configuration space $\mathfrak{M} = \mathfrak{M}_{s_2}^{\tau_2} \times \ldots \times \mathfrak{M}_{s_n}^{\tau_n}$ we consider the reversed product

$$\mathfrak{M}^\vee = \mathfrak{M}_{s_n}^{\tau_n} \times \ldots \times \mathfrak{M}_{s_2}^{\tau_2}\,.$$

By the *symmetric configuration invariant* of a completion $(\bar{V}, D)$ of a Gizatullin surface $V$ we mean the unordered pair

$$\tilde{Q}(\bar{V}, D) = \left\{Q(\bar{V}, D), Q(\bar{V}^\vee, D^\vee)\right\}, \quad \text{where } Q(\bar{V}, D) \in \mathfrak{M} \text{ and } Q(\bar{V}^\vee, D^\vee) \in \mathfrak{M}^\vee.$$

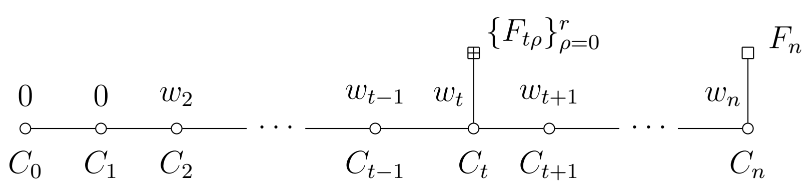
Theorem 3.4.1 leads immediately to the following result.

**Corollary 3.4.3.** *The sequence $(s_i)_{2\le i\le n}$ (up to reversion) and the pair $\tilde{Q}(V) := \tilde{Q}(\bar{V}, D)$ are invariants of the isomorphism type of $V$.*

3.5. **The configuration invariant for $\mathbb{C}^*$-surfaces.** According to [FlZa$_1$] a normal non-toric $\mathbb{C}^*$-surface admits a hyperbolic DPD-presentation $V = \operatorname{Spec} \mathbb{C}[u][D_+, D_-]$. If, moreover, $V$ is Gizatullin, then there are (not necessarily different) points $p_\pm$ with $\operatorname{supp}\{D_\pm\} \subseteq \{p_\pm\}$, see [FlZa$_2$, Section 4]. By [FKZ$_2$] $V$ admits an equivariant standard completion $(\bar{V}, D)$, which is unique up to reversion. Concerning the structure of this completion we can summarize the main results from [FKZ$_2$] and Section 3 in [FKZ$_3$] for smooth $\mathbb{C}^*$-surfaces as follows. We recall that the *parabolic* component is the unique component $C_t$ of the zigzag with $t \ge 2$ consisting of fixed points of the $\mathbb{C}^*$-action.

**Proposition 3.5.1.** *If $V$ is non-toric and smooth then it admits a unique equivariant standard completion $(\tilde{V}, D)$ with extended divisor*

(12) $D_{\text{ext}}$ :




*and with boundary zigzag $D$ represented by the bottom line in (12) such that $C_t$ is an attractive parabolic component. Here $w_t = \deg(\lfloor D_+\rfloor + \lfloor D_-\rfloor) \le -2$, $F_n$ is a single feather (possibly empty) and $\{F_{t\rho}\}_{\rho=0}^r$ is a non-empty collection of feathers with all $F_{t\rho}$, $\rho \ge 1$, being $(-1)$-curves. Furthermore the following hold:*

*(a) Suppose that $p_+ \ne p_-$ or one of the fractional parts $\{D_\pm\}$ of the divisors $D_\pm$ is zero. Then $F_n$ is a $(-1)$-curve*[9]*, $F_{t0}$ with $F_{t0}^2 = 1-t$ has mother component $C_2$ and*

[9]Hence it is non-empty.

*$w_i = -2$ for $i \neq 0, 1, t$. Up to equivalence the pair $(D_+, D_-)$ is*

$$D_+ = -\frac{1}{t-1}[p_+], \qquad D_- = -\frac{1}{n-t+1}[p_-] - \sum_{i=1}^{r}[p_i] \tag{13}$$

*with pairwise different points $p_+, p_-, p_1, \dots, p_r$, where $r = -2 - w_t \geq 0$. The feathers $F_{t0}, \dots, F_{tr}$ are attached to the points $p_+, p_1, \dots, p_r$ of $C_t \backslash C_{t-1} \cong \mathbb{A}^1$ whereas $p_-$ corresponds to the intersection point $C_t \cap C_{t+1}$ if $t < n$ and to $C_n \cap F_n$ if $t = n$.*

*(b) $\{D_+(p_+)\} = 0$ iff $t = 2$ and, similarly, $\{D_-(p_-)\} = 0$ iff $t = n$.*

*(c) Assume that $p_+ = p_- =: p$. Then the $F_{t\rho}$ are $(-1)$-feathers $\forall \rho \geq 0$ while $F_n = \emptyset$ if and only if $D_+(p) + D_-(p) = 0$. Moreover the feathers $F_{t0}, \dots, F_{tr}$ are attached to the points of the reduced divisor $\lfloor -D_+ - D_- \rfloor = \sum_{i=0}^{r}[p_i]$ considered as points of $C_t \backslash C_{t-1} \cong \mathbb{A}^1$ while $p_-$ corresponds to the intersection point $C_t \cap C_{t+1}$.*

For the proof we refer the reader to [$\mathrm{FKZ}_3$, §3], in particular to Proposition 3.10 and Remark 3.11(2).

**3.5.2.** We recall the following conditions $(\alpha_*)$ and $(\beta)$ of Theorem 0.2 in [$\mathrm{FKZ}_3$].

$(\alpha_*)$ supp $\{D_+\} \cup$ supp $\{D_-\}$ is either empty or consists of one point $p$, where $D_+(p) + D_-(p) \leq -1$ or both fractional parts $\{D_+(p)\}$, $\{D_-(p)\}$ are nonzero.

$(\beta)$ supp $\{D_+\} = \{p_+\}$ and supp $\{D_-\} = \{p_-\}$ for two distinct points $p_+, p_-$, where $D_+(p_+) + D_-(p_+) \leq -1$ and $D_+(p_-) + D_-(p_-) \leq -1$.

By Theorem 0.2 of [$\mathrm{FKZ}_3$] for a normal $\mathbb{C}^*$-surface satisfying one of these conditions the $\mathbb{C}^*$-action is unique up to conjugation and inversion. In the next proposition we clarify for which smooth surfaces these conditions do not hold. This yields in particular part (a) of Theorem 1.0.7 from the Introduction.

**Proposition 3.5.3.** *Let $V$ be a smooth Gizatullin $\mathbb{C}^*$-surface with DPD-presentation $V = \operatorname{Spec} \mathbb{C}[u][D_+, D_-]$. Then the following conditions are equivalent.*

*(i) Neither $(\alpha_*)$ nor $(\beta)$ is fulfilled.*

*(ii) $(D_+, D_-)$ is equivalent to a pair as in (13) with $n \geq 3$.*

*(iii) $V$ is special*[10].

*Proof.* If (i) holds then supp $\{D_+\} \cup$ supp $\{D_-\}$ is non-empty and so by Proposition 3.5.1(b) $n \geq 3$ in (12). Moreover $p_+ \neq p_-$ or one of the fractional parts $\{D_\pm\}$ is zero. Hence (ii) follows from 3.5.1(a). The implication (ii)⇒(i) is easy and left to the reader.

If (ii) holds then inspecting (12) $V$ is special with $n \geq 3$ and so (iii) holds. To prove the converse, assume that $V$ is special with $n \geq 3$. If $p_+ \neq p_-$ or one of the divisors $\{D_\pm\}$ is zero, then we can conclude by Proposition 3.5.1(a). So assume that both divisors $\{D_\pm\}$ are non-zero and $p_+ = p_-$. In particular by 3.5.1(b) $3 \leq t \leq n-1$.

According to Definition 1.0.4, the extended divisor has a feather $F_2$ with mother component $C_2$ and another one $F_n$ with mother component $C_n$. Comparing with (12) $F_2$ must be attached to $C_t$ with $F_2^2 \leq -2$. This contradicts Proposition 3.5.1(c). □

**Remark 3.5.4.** Let as before $V = \operatorname{Spec} \mathbb{C}[u][D_+, D_-]$ be a $\mathbb{C}^*$-surface with $(D_+, D_-)$ as in (13) of Proposition 3.5.1. If $r = 0$ then $V$ is a Danilov-Gizatullin surface, see [$\mathrm{FKZ}_2$, 5.2] or 6.2.3 below. Furthermore, if $n \geq 3$ then $V$ is special of type I if either $r = 1$ or $r \geq 2$ and one of the fractional parts $\{D_\pm\}$ vanishes (i.e., $t = 2$ or $t = n$). Otherwise it is of type II.

[10] In particular $n \geq 3$, see Definition 1.0.4(b).

We note the following important consequence.

**Corollary 3.5.5.** *Assume that* $V = \operatorname{Spec}\mathbb{C}[u][D_+, D_-]$ *is a special smooth* $\mathbb{C}^*$*-surface with*
$$\lfloor -D_+ - D_- \rfloor = q_1 + \ldots + q_s\,.$$
*Then the following hold.*

*(1) The configuration invariant of* $V$ *is given by the point* $(q_1, \ldots, q_s) \in \mathfrak{M}_s^+$*.*

*(2) The numbers* $\deg\{D_+\}$*,* $\deg\{D_-\}$ *and* $s$ *uniquely determine the zigzag* $D$ *of a standard completion of* $V$ *up to reversion.*

*Proof.* According to Proposition 3.5.3 $(D_+, D_-)$ is up to equivalence a pair as in (13). Using the description of the zigzag in Proposition 3.5.1, (2) follows.

To deduce (1), we assume first that $2 < t < n$. With the notations as in Proposition 3.5.1(a), the feather $F_{t0}$ has mother component $C_2$ while the feathers $F_{t1}, \ldots, F_{tr}$ are $(-1)$-feathers attached to $C_t \backslash C_{t-1} \cong \mathbb{A}^1$ in the points $p_1, \ldots, p_r$. Since in this case $\lfloor D_+ + D_- \rfloor = p_1 + \ldots + p_r$, the assertion follows.

In the case $t = 2 < n$ we have $\lfloor D_+ + D_- \rfloor = p_+ + p_1 + \ldots + p_r$ while $F_0$ is an additional $(-1)$-feather attached to $C_2 = C_t$ in $p_+$. Hence we can conclude as before. Replacing $F_0$ by $F_n$ and $p_+$ by $p_-$ the same argument works also in the case $2 < t = n$. □

**Remark 3.5.6.** Let $V = \operatorname{Spec}\mathbb{C}[u][D_+, D_-]$ be a non-special smooth $\mathbb{C}^*$-surface such that the divisors $\{D_+\}$ and $\{D_-\}$ are both nonzero and supported on the same point $p = p_+ = p_-$. In this case the parabolic component $C_t$ in (12) is of $*$-type. Indeed, condition (ii) in Definition 3.2.2 is empty. Condition (i) follows in the case $D_+(p) + D_-(p) \neq 0$ by Lemma 3.21 in [FKZ$_3$] and in the case $D_+(p) + D_-(p) = 0$ from the fact that $F_n = \emptyset$, see 3.5.1(c).

Now again the conclusion of Corollary 3.5.5 holds. Indeed, part (1) with $\mathfrak{M}_s^+$ replaced by $\mathfrak{M}_s^*$ is a consequence of Proposition 3.5.1(c), while part (2) follows from the fact that the weights $w_2, \ldots, w_{t-1}$ and $w_{t+1}, \ldots, w_n$ of the boundary zigzag in (12) are uniquely determined by $\deg\{D_+\}$ and $\deg\{D_-\}$, respectively (see [FKZ$_3$, Proposition 3.10]).

The preceding remark can be used to deduce uniqueness of $\mathbb{C}^*$-actions for all non-special smooth Gizatullin surfaces; cf. the more general Theorem 0.2 in [FKZ$_3$], which covers as well the singular case.

**Corollary 3.5.7.** *If* $V$ *is a non-special smooth* $\mathbb{C}^*$*-surface then its* $\mathbb{C}^*$*-action is unique up to equivalence.*

*Proof.* Assume that $V = \operatorname{Spec}\mathbb{C}[t][D_+, D_-]$ is non-special. Then either both $D_\pm$ are integral, or both fractional parts $\{D_\pm\}$ are non-zero and supported by the same point $p = p_+ = p_-$.

Suppose first that both $D_\pm$ are integral. Comparing with Proposition 3.5.1(a) $D$ is then a zigzag with $n = t = 2$, and the configuration invariant of $V$ is given by $(p_-, p_+, p_1, \ldots, p_r)$. Hence the pair $(D_+, D_-)$ is uniquely determined up to equivalence.

Suppose now that both fractional parts $\{D_\pm\}$ are non-zero and supported by the same point $p$. Let $\lfloor -D_+ - D_- \rfloor = \sum_{i=0}^r p_i$. By Remark 3.5.6 and Theorem 3.5.1(c) the values $\{D_+(p)\}$ and $\{D_-(p)\}$ are, up to interchanging, uniquely determined by the boundary zigzag and so by the abstract isomorphism type of $V$. Since $V$ is smooth, using Theorem 4.15 in [FlZa$_1$] we have $-1 < D_+(p) + D_-(p) \le 0$. Hence if we require that $-1 < D_+(p) < 0$ then $D_\pm(p)$ are uniquely determined.

Applying again Remark 3.5.6, the parabolic component $C_t$ is of $*$-type. By Theorem 3.5.1(c) under a suitable isomorphism $C_t \backslash C_{t-1} \cong \mathbb{A}^1$ the configuration invariant of $V$ is the point in $\mathfrak{M}^*_{r+1}$ given by the subset $\{p_0, \ldots, p_r\}$ of $C_t \backslash (C_{t-1} \cup C_{t+1})$. Since by *loc.cit.* $p$ corresponds to the intersection point $C_t \cap C_{t+1}$ the abstract isomorphism type of $V$ determines the pair $(D_+, D_-)$ up to equivalence. □

## 4. Special Gizatullin surfaces of $(-1)$-type

The main result of this section says that the isomorphism type of a special surface (as introduced in Definition 1.0.4) is uniquely determined by its configuration invariant provided that all feathers are $(-1)$-curves, see Proposition 4.4.1. To this purpose we introduce presentations of Gizatullin surfaces. Given such a surface $V$, we define certain natural group actions on the set of all presentations of $V$. They change the completion while leaving the affine surface unchanged. The most important ones are a 2-torus action, elementary shifts, and backward elementary shifts. Given two special smooth Gizatullin surfaces $V$, $V'$of $(-1)$-type with the same zigzag and configuration invariant, we show that they admit a common presentation, hence are isomorphic (see the proof of Proposition 4.4.1). To achieve this we gradually change two given presentations of $V$ and $V'$ by means of the above actions until they become equal.

4.1. **Presentations.** Every smooth Gizatullin surface $V$ can be constructed along with a standard completion $(\bar{V}, D)$ via a sequence of blowups starting from the quadric $\mathbb{P}^1 \times \mathbb{P}^1$. If all components $C_i$ of $D$ with $i \geq 2$ are of $+$-type then the necessary sequence of blowups can be described in the following way (cf. Corollary 4.1.6).

**4.1.1.** We let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ denote the quadric, where $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$. In $Q$ we consider the curves

$$C_0 = \{\infty\} \times \mathbb{P}^1 , \quad C_1 = \mathbb{P}^1 \times \{\infty\} , \quad \text{and} \quad C_2 = \{0\} \times \mathbb{P}^1 .$$

We choose finite sets of points

$$M_2, c_3, M_3, \ldots, c_n, M_n$$

among the points of $C_2$ and infinitesimally near points as follows.

(a$_2$) $M_2 \subseteq C_2 \backslash C_1$ is a finite subset. Blowing up $X_2 = Q$ with centers at the points of $M_2$ we obtain a surface $X_3$.

(b$_2$) $c_3 \in X_3$ is a point on $C_2 \backslash C_1$, where by abuse of notation we identify the curves $C_1$, $C_2$ with their proper transforms in $X_3$. Blowing up $X_3$ with center at $c_3$ leads to a surface $\tilde{X}_3$ with an exceptional curve $C_3$ over $c_3$.

(a$_3$) $M_3 \subseteq C_3 \backslash C_2$ is a finite subset, where we identify the curves $C_2$ and $C_3$ with their proper transforms in $\tilde{X}_3$. We assume that none of the points of $M_3$ is contained in an exceptional curve over $M_2$. Blowing up $\tilde{X}_3$ with centers at the points of $M_3$ we obtain a surface $X_4$.

(b$_3$) $c_4 \in X_4$ is a point on $C_3 \backslash C_2$, where we identify again the curves $C_2$ and $C_3$ with their proper transforms in $X_4$. Blowing up $c_4$ leads to a surface $\tilde{X}_4$ with an exceptional curve $C_4$.

Iterating this procedure we finally arrive at a smooth rational projective surface

$$X_n = X(M_2, c_3, M_3, \ldots, c_n, M_n) . \tag{14}$$

We emphasize that in each step ($a_i$) we require that

(15) none of the points of $M_i$ is in an exceptional curve over $M_j$ with $j < i$.

As before we identify the curves $C_i$ with their proper transforms in $X_n$. The smooth open surface

$$V_n = V(M_2, c_3, M_3, \dots, c_n, M_n) = X_n \backslash D\,,$$

where $D = C_0 \cup \ldots \cup C_n \subseteq X_n$ stands for the boundary zigzag, is an affine Gizatullin surface, see Lemma 4.1.4.

**Definition 4.1.2.** We call $X_n$ as in (14) a *presentation* of $V = V_n$. It is called a presentation of $(-1)$-*type*, or simply a $(-1)$-*presentation*, if $c_{i+1} \notin M_i\ \forall i$. Equivalently, this means that at each step ($b_i$) the point $c_{i+1}$ is not contained in any of the exceptional curves over the points of $M_i$.

We say that a standard completion $(\bar V, D)$ of a Gizatullin surface $V$ is of $(-1)$-*type*, or simply a $(-1)$-*completion*, if all feathers of the extended divisor $D_{\text{ext}}$ are $(-1)$-feathers or, equivalently, are attached to their mother components. It is easily seen that $(X_n, D)$ as above gives a $(-1)$-completion of $V_n$ if and only if $X_n$ is a $(-1)$-presentation.

**4.1.3.** We let $\Pi(s_2, \dots, s_n)$ denote the space of all presentations $X(M_2, c_3, \dots, c_n, M_n)$ with $|M_i| = s_i$. The tower of smooth fibrations

$$\Pi(s_2, \dots, s_n) \to \Pi(s_2, \dots, s_{n-1}) \to \ldots \to \{\cdot\}$$

shows that $\Pi(s_2, \dots, s_n)$ is a smooth quasiprojective variety (non-affine, in general).

The next lemma is immediate from the construction in 4.1.2; we leave the details to the reader.

**Lemma 4.1.4.** *For a presentation as in 4.1.2 and with $s_i = |M_i|$ the following hold.*

*(a) If $n \geq 3$ then $D = C_0 \cup \ldots \cup C_n \subseteq X_n$ represents a zigzag*

(16) $$\Gamma_D = [[0, 0, -s_2 - 1, -s_3 - 2, \dots, -s_{n-1} - 2, -s_n - 1]]\,,$$

*while for $n = 2$ we have $\Gamma_D = [[0, 0, -s_2]]$. Consequently, if $s_2, s_n \geq 1$ for $n \geq 3$ and $s_2 \geq 2$ for $n = 2$, then $V_n$ is a Gizatullin surface with standard completion $(X_n, D)$.*

*(b) Letting $M_i = \{p_{ij}\}$, for a point $p_{ij} \in M_i$ we let $F_{ij}$ denote*[11] *the proper transform in $X_n$ of the exceptional curve over $p_{ij}$ that was generated in step ($a_i$). Then $F_{ij}$ is a feather of $D_{\text{ext}}$ with mother component $C_i$, and every feather of $D_{\text{ext}}$ appears in this way.*

*(c) All components $C_2, \dots, C_n$ of the zigzag $D$ are $+$-components.*[12]

*(d) The configuration invariant $Q(\bar V, D) \in \mathfrak{M} = \mathfrak{M}^+_{s_2} \times \ldots \times \mathfrak{M}^+_{s_n}$ of $V$ as in 3.2.1 is given by the sequence $(M_2, \dots, M_n)$, where $M_i$ is viewed as a point in the configuration space $\mathfrak{M}^+_{s_i}$ of $s_i$-tuples of distinct points in $C_i \backslash C_{i-1} \cong \mathbb{A}^1$.*

To show that $V_n$ in (a) is affine it suffices to observe that the zigzag $D$ supports an ample divisor $\sum_{i=0}^n m_i C_i$ with $0 < m_0 \ll m_1 \ll \ldots \ll m_n$, due to the Nakai-Moishezon criterion (cf. e.g., [$\text{Gi}_1$, $\text{Du}_1$]).

We have the following criteria for a surface to admit a presentation.

[11] Attention: now $i$ stands for the index of the mother component, whereas in Section 3 it means the index of the component of attachment.

[12] See Definition 3.2.2.

**Lemma 4.1.5.** *Let $(\bar V, D)$ be a standard completion of a smooth Gizatullin surface $V$, and let $\Phi : \bar V \to Q = \mathbb{P}^1 \times \mathbb{P}^1$ be the standard morphism so that $\bar V$ is obtained from $Q$ by a sequence of blowups. Then the following conditions are equivalent.*

*(a) Every standard completion $(\bar V, D)$ of $V$ arises from a presentation $\bar V = X_n$.*
*(b) $V$ arises from a presentation as in 4.1.2.*
*(c) Every component $C_{s+1}$, $s = 2, \dots, n-1$, of the zigzag is created by a blowup on $C_s \backslash C_{s-1}$.*
*(d) The dual graph of $D$ is as in (16), where $s_i$ is the number of feathers with mother component $C_i$.*

*Proof.* (a)⇒(b) is trivial while (b)⇒(c) follows from the definitions. To deduce (c)⇒(d) and (d)⇒(a) we proceed by induction on the length $n$ of the zigzag. In case $n = 2$ both implications are evident. Clearly every feather with mother component $C_n$ is a $(-1)$-feather. If $n \geq 3$, blowing down all $(-1)$-feathers of $C_n$ this component becomes a $(-1)$-curve and can be blown down too. This results in a zigzag of shorter length and so the induction argument works.

Since by Proposition 4.4.1 condition (d) does not depend on the choice of the completion, (d)⇒(a) follows as well. □

**Corollary 4.1.6.** *Every standard completion of a special surface (see Definition 1.0.4 (a)) admits a presentation.*

4.2. **Reversed presentation.** In this subsection we study how a presentation changes when we reverse the boundary zigzag. The matching principle 3.3.6 yields the following result.

**Corollary 4.2.1.** *Let $X_n$ as in 4.1.2 be a presentation of a special smooth Gizatullin surface $V = X_n \backslash D$, and let $(X_n^\vee, D^\vee)$ be the reversion of the completion $(X_n, D)$ with reversed zigzag $D^\vee = C_0^\vee \cup \ldots \cup C_n^\vee$. Then there is a presentation*

$$X_n^\vee \cong X(M_n, c_3^\vee, M_{n-1}, \dots, c_n^\vee, M_2)$$

*for suitable points $c_3^\vee, \dots, c_n^\vee$, where we identify $M_\mu$ with a subset of $C^\vee_{\mu^\vee} \backslash C^\vee_{\mu^\vee - 1}$ via the correspondence fibration $\psi$ for the pair $(C_\mu, C^\vee_{\mu^\vee})$ as in Definition 3.3.3.*

Next we describe the positions of matching feathers in $X_n$ and $X_n^\vee$.

**Proposition 4.2.2.** *Let $X = X_n$ be as in Definition 4.1.2, and let $F$ and $F^\vee$ be a pair of matching feathers with mother components $C_\mu$ and $C^\vee_{\mu^\vee}$, respectively, and with the same base point $p \in M_\mu$. Then the following hold.*

*(a) $F$ is attached to component $C_{k+1}$ with $\mu < k+1 < n$ if and only if*

$$(i) \quad p = c_{\mu+1}, \quad c_{i+1} = c^\vee_{i^\vee+1} \quad \text{for} \quad \mu < i \leq k, \quad \text{and} \quad (ii) \quad c_{k+2} \neq c^\vee_{k^\vee}\,.$$

*Similarly, $F$ is attached to component $C_n$ with $\mu < n$ if and only if (i) is satisfied with $k = n-1$.*

*(b) $F^\vee$ is attached to component $C^\vee_{l^\vee+1}$ with $\mu^\vee < l^\vee + 1 < n$ if and only if*

$$(i) \quad p = c^\vee_{\mu^\vee+1}, \quad c_{i+1} = c^\vee_{i^\vee+1} \quad \text{for} \quad l \leq i < \mu, \quad \text{and} \quad (ii) \quad c_l \neq c^\vee_{l^\vee+2}\,.$$

*Similarly, $F^\vee$ is attached to component $C_n^\vee$ with $\mu^\vee < n$ if and only if (i) is satisfied with $l^\vee = n-1$.*

*Proof.* To deduce (a), assume that $F$ is attached to $C_{k+1}$ with $\mu \leq k$. It is clear that then $p = c_{\mu+1}$ since otherwise in the construction the feather $F$ would be attached to $C_\mu$ as a $(-1)$-feather. Let us consider the correspondence fibration $\psi : (W, E) \to \mathbb{P}^1$ for the pair $(C_i, C_{i^\vee})$ as in Definition 3.3.3. In the notation of Lemma 3.3.4(a) we have $q = c_{i+1}$ and $q^\vee = c^\vee_{i^\vee+1}$. If $F$ is attached to $C_{k+1}$ and $\mu + 1 \leq i \leq k$ then $F$ appears in the fiber $\psi^{-1}(c_{i+1})$ (cf. Lemma 3.3.4(a)). Since $F$ meets its matching feather $F^\vee$ and the latter sits in the fiber $\psi^{-1}(c^\vee_{i^\vee+1})$, this forces $c_{i+1} = c^\vee_{i^\vee+1}$ in this range. However, if $i = k + 1$ and $k + 1 \leq n - 1$ then $F$ is not any longer contained in $\psi^{-1}(c_{k+2})$ while $F^\vee$ and then also $F$ is still in $\psi^{-1}(c^\vee_{k^\vee})$. This shows that indeed $c_{k+2} \neq c^\vee_{k^\vee}$.

Dualizing (a) also (b) follows. □

**Corollary 4.2.3.** *Suppose that $F$ is a feather of $X$ with mother component $C_\mu$ and $G^\vee$ a feather of $X^\vee$ with mother component $C^\vee_{\nu^\vee}$. If $F$ and $G^\vee$ are attached to the components $C_{k+1}$ of $X$ and $C^\vee_{l^\vee+1}$ of $X^\vee$, respectively, then the intervals of integers $[\mu, k+1]$ and $[l-1, \nu]$ have at most one point in common.*

*Proof.* If $\mu = k+1$ or $\nu^\vee = l^\vee + 1$, i.e. $\nu = l - 1$, then the assertion is trivial. So assume for the rest of the proof that $\mu < k + 1$ and $\nu > l - 1$. Let $p \in M^\vee_{\nu^\vee} \cong M_\nu$ be the base point of $G^\vee$. We claim that:

(1) $\nu \notin ]\mu, k]$ and, dually, $\mu \notin [l, \nu[$. Clearly it suffices to show the first part. If on the contrary $\nu \in ]\mu, k]$ then by Proposition 4.2.2(a) $c_{\nu+1} = c^\vee_{\nu^\vee+1}$ while by 4.2.2(b) $p = c^\vee_{\nu^\vee+1}$ hence $c_{\nu+1} \in M_\nu$. This contradicts (15) in 4.1.1.

(2) $l - 1 \notin ]\mu, k]$ and, dually, $k + 1 \notin [l, \nu[$. Again it suffices to show the first part. If on the contrary $l - 1 \in ]\mu, k]$ then by (i) in 4.2.2(a) $c_l = c^\vee_{l^\vee+2}$, contradicting (ii) in 4.2.2(b).

(3) $k + 1 \neq \nu$ and, dually, $l - 1 \neq \mu$. As before it suffices to show the first part. If on the contrary $\nu = k + 1$ then by Proposition 4.2.2(b) $p = c^\vee_{\nu^\vee+1}$. The feather $F$ is attached to $C_\nu$ and contained in the fiber over $c^\vee_{\nu^\vee+1}$, since it has to meet its dual feather. Moreover $G^\vee$ and then also its dual feather $G$ are in this fiber. Thus the two feathers $G$ and $F$ are attached to the same point of $C_\nu$, which gives a contradiction.

Obviously (1)-(3) imply our assertion. □

4.3. **Actions of elementary shifts on presentations.** Here we develop our principal tool in the proof of the main theorem.

**4.3.1.** We fix a coordinate system $(x, y)$ on the affine plane $\mathbb{A}^2$, and we let $\mathrm{Aut}_y(\mathbb{A}^2)$ denote the group of all automorphisms $h : \mathbb{A}^2 \to \mathbb{A}^2$ stabilizing the $y$-axis $\{x = 0\}$. Such an automorphism can be written as

$$h : (x, y) \longrightarrow (ax, by + P(x)), \quad \text{where } a, b \in \mathbb{C}^* \text{ and } P \in \mathbb{C}[x]. \tag{17}$$

With notations as in Definition 4.1.1, we identify $\mathbb{A}^2$ with the complement of the curve $C_0 \cup C_1$ on the quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$, so that the $y$-axis is identified with $C_2 \setminus C_1$. Clearly $h$ extends to a birational transformation, which is biregular on $Q \setminus C_0$ and stabilizes the curves $C_1$, $C_2$.

The group $\mathrm{Aut}_y(\mathbb{A}^2)$ acts on presentations $X_n := X(M_2, c_3, \ldots, c_n, M_n)$. Indeed, given $h \in \mathrm{Aut}_y(\mathbb{A}^2)$, in the inductive construction of 4.1.1 the set $M_2 \subseteq C_2$ is moved by $h$ into a new set of point, say $M'_2 \subseteq C_2$. Thus $h$ induces a morphism $X(M_2)\backslash C_0 \to X(M'_2)\backslash C_0$. Under this map $c_3$ is mapped to a point $c'_3$, yielding again a morphism

$X(M_2, c_3)\backslash C_0 \to X(M_2', c_3')\backslash C_0$. Continuing in this way we obtain finally a transformed presentation

$$h_*(X_n) := X_n' := X(M_2', c_3', \dots, c_n', M_n')$$

together with an isomorphism (also denoted by $h$)

$$h : X(M_2, c_3, \dots, c_n, M_n)\backslash C_0 \xrightarrow{\cong} X(M_2', c_3', \dots, c_n', M_n')\backslash C_0\,. \tag{18}$$

In particular the affine surfaces $V = X_n\backslash D$ and $V' := X_n'\backslash D'$ are isomorphic under $h$. Furthermore, $h$ maps $D_{\text{ext}} - C_0$ isomorphically onto $D'_{\text{ext}} - C_0$, where $D_{\text{ext}}$ and $D'_{\text{ext}}$ are the extended divisors of $X_n$ and $X_n'$, respectively.

**Remarks 4.3.2.** 1. The automorphism $h$ as in (17) extends to an automorphism of the Hirzebruch surface $\Sigma_t$, where $t = \deg(P)$. We can replace $X_n$ and $X_n'$ by the corresponding semi-standard completions of $V$ and $V'$, respectively, with boundary zigzags $[[0, -t, \dots]]$, by performing on both surfaces a sequence of inner elementary transformations with centers at $C_0 \cap C_1$. Then $h$ extends to a biregular map between these new completions, sending $D_{\text{ext}}$ isomorphically onto $D'_{\text{ext}}$.

2. It is easy to see that the assignment

$$X(M_2, c_3, \dots, c_n, M_n) \mapsto X(M_2', c_3', \dots, c_n', M_n')$$

defines a regular action of the group $\text{Aut}_y(\mathbb{A}^2)$ on the presentation space $\Pi(s_2, \dots, s_n)$ as in 4.1.3.

Let us study in detail the action of the *elementary shifts* $h = h_{a,t} \in \text{Aut}_y(\mathbb{A}^2)$, where

$$h_{a,t} : (x, y) \mapsto (x, y + ax^{t-2}) \qquad \text{with} \quad a \in \mathbb{C} \quad \text{and} \quad t \geq 2\,. \tag{19}$$

**Lemma 4.3.3.** *Let $X_n$ as in (4.1.2) be a $(-1)$-presentation. Then for every $a \in \mathbb{C}$ and $t \geq 2$ the elementary shift $h = h_{a,t}$ induces the identity on $C_2 \cup \ldots \cup C_{t-1}$ and a translation $x \mapsto x + a$ in a suitable coordinate on $C_t \setminus C_{t-1}$. In particular,*

$$X_n' = h_*(X_n) = X(M_2, c_3, \dots, M_{t-1}, c_t, a + M_t, a + c_{t+1}, M_{t+1}', \dots, c_n', M_n')$$

*for some $c_{i+1}'$ and $M_i'$ for $i > t$.*

*Proof.* For $t = 2$ the assertion is evidently true. So we assume in the sequel that $t \geq 3$. Since $X_n$ is a $(-1)$-presentation it can be obtained by first creating the zigzag $D = C_0 \cup \ldots \cup C_n$ by successive blowups with centers at $c_3, \dots, c_n$ and then blowing up the points of $M_2, \dots, M_n$. Let as before $C_2 \setminus C_1 = \{x = 0\}$ in coordinates $(x, y)$ in $\mathbb{A}^2$. After a suitable translation we may suppose that $c_3 = (0, 0)$. The blowup with center at $c_3$ can be written in coordinates as

$$(x_3, y_3) = (x, y/x)\,, \quad \text{or, equivalently,} \quad (x, y) = (x_3, x_3 y_3).$$

In these coordinates, the exceptional curve $C_3$ is given by $x_3 = 0$ and the proper transform of $C_2$ by $y_3 = \infty$. The elementary shift $h_{a,t}$ can be written as

$$h_{a,t} : (x_3, y_3) \mapsto (x_3, y_3 + ax_3^{t-3})\,. \tag{20}$$

In particular $h_{a,3}$ yields the identity on the curve $C_3\backslash C_2 \cong \mathbb{A}^1$ if $t > 3$ and the translation by $a$ if $t = 3$. The formulas (20) remain the same after replacing the coordinates $(x_3, y_3)$ by the new ones $(x_3, y_3 - \delta_4)$, where $c_4 = (0, \delta_4)$. Thus we may assume that $c_4 = (0, 0)$ in the coordinate system $(x_3, y_3)$. Now the lemma follows easily by induction. □

**4.3.4.** Consider now the action of the 2-torus $\mathbb{T} = \mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{A}^2$ given by

$$(\lambda_1, \lambda_2).(x, y) = (\lambda_1 x, \lambda_2 y), \quad (\lambda_1, \lambda_2) = \lambda \in \mathbb{T}.$$

It leaves both axes invariant and extends to a biregular action on the quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Hence in this case the map $h$ in (18) is biregular, while in general it can have points of indeterminacy. By Lemma 4.3.3 we can use the induced $\mathbb{T}$-action on presentations in the following way.

**Corollary 4.3.5.** *Let as before $X_n = X(M_2, c_3, \ldots, c_n, M_n)$ be a $(-1)$-presentation. Then it can be transformed by a suitable sequence of elementary shifts into a new one such that the points $c_3, \ldots, c_n$ and one of the points $p_{n,j} \in C_n$ to which a feather is attached, are fixed by the induced $\mathbb{T}$-action.*

*Proof.* After a suitable translation we may assume that $c_3 = (0, 0)$, so this point is fixed by the torus action. Using induction on $t$, suppose that the presentation is already transformed by a sequence of elementary shifts into a new one such that $c_3, \ldots, c_t$ are invariant under the torus action. Then for $t < n$, $\mathbb{T}$ acts on $C_t$. Applying Lemma 4.3.3, the elementary shift $h_{a,t}$ with a suitable $a \in \mathbb{C}$ moves $c_{t+1}$ into the second fixed point of the $\mathbb{T}$-action on $C_t$, as required. Similarly, if $t = n$ then we can achieve that $p_{n,j}$ is the second fixed point of $\mathbb{T}$ on $C_n$. □

**Lemma 4.3.6.** *Let*

$$X_n = X(M_2, c_3, \ldots, M_{t-1}, c_t, M_t, c_{t+1}, \ldots, c_n, M_n) \quad \textit{and}$$
$$X'_n = X(M_2, c_3, \ldots, M_{t-1}, c_t, M'_t, c'_{t+1}, \ldots, c'_n, M'_n)$$

*be presentations with reversed presentations*[13]

$$X_n^\vee = X(M_n, c_3^\vee, \ldots, c_{t^\vee}^\vee, M_t, c_{t^\vee+1}^\vee, M_{t-1}, \ldots, c_n^\vee, M_2) \qquad \textit{and}$$
$$X_n'^\vee = X(M'_n, c_3'^\vee, \ldots, c_{t^\vee}'^\vee, M'_t, c_{t^\vee+1}'^\vee, M_{t-1}, \ldots, c_n'^\vee, M_2),$$

*respectively. Then we have*

$$c_i^\vee = c_i'^\vee \qquad \forall i = t^\vee + 1, \ldots, n. \tag{21}$$

*Proof.* Starting with the completion $(X_n, D)$ of $V$ we consider the correspondence fibration $\psi : (W, E) \to \mathbb{P}^1$ for the pair of curves $(C_t, C_{t^\vee}^\vee)$ as in Definition 3.3.3. Similarly, we let $\psi' : (W', E') \to \mathbb{P}^1$ be the correspondence fibration associated to $(X'_n, D')$ for the pair of curves $(C'_t, C_{t^\vee}'^\vee)$. To obtain the part $D^{\vee \geq t^\vee} = C_n^\vee \cup \ldots \cup C_{t^\vee}^\vee$ of the reversed zigzag $D^\vee$ only inner elementary transformations with centers at the components $C_0 = C'_0, \ldots, C_{t-1} = C'_{t-1}$ are required. It follows that $C_{i^\vee}^\vee = C_{i^\vee}'^\vee$ for all $i \leq t - 1$ i.e., for all $i^\vee \geq t^\vee + 1$. In particular (21) hold. □

Our next aim is to study the behavior of $(-1)$-presentations under reversion, see Proposition 4.3.8. Let us first give an example showing that reversion does not necessarily preserve $(-1)$-type.

**Example 4.3.7.** Consider a $\mathbb{C}^*$-surface $V$ and its equivariant standard completion $(\bar{V}, D)$ as in Proposition 3.5.1. By virtue of that Proposition, if $t = 2$ in the extended divisor (12) then $V$ is special of $(-1)$-type. Passing to the inverse $\mathbb{C}^*$-action $\lambda \mapsto \lambda^{-1}$

[13] See Corollary 4.2.1.

amounts to interchanging the divisors $D_+$ and $D_-$, see [FlZa$_1$]. According to Proposition 3.5.1(a) the reverse equivariant completion $(\bar V^\vee, D^\vee)$ has an extended divisor with $t=n$ and $(F^\vee_{n,0})^2 = 1-n$. Thus for $n\ge 3$, $(\bar V^\vee, D^\vee)$ is not of $(-1)$-type, while $(\bar V, D)$ is.

**Proposition 4.3.8.** *Given a $(-1)$-presentation $X_n = X(M_2, c_3, \ldots, c_n, M_n)$, by applying a finite sequence of elementary shifts as in (19) we can transform $X_n$ into a $(-1)$-presentation*

$$X_n^o := X(M_2^o, c_3^o, \ldots, c_n^o, M_n^o)$$

*such that the reversion of $(X_n^o, D^o)$ is again of $(-1)$-type.*

*Proof.* Let

$$X_n^\vee = X(M_n, c_3^\vee, \ldots, c_{t^\vee}^\vee, M_t, c_{t^\vee+1}^\vee, M_{t-1}, \ldots, c_n^\vee, M_2)$$

be the reversion of $X_n$, see Corollary 4.2.1. With a suitable coordinate on $\mathbb{A}^1 \cong C_t\backslash C_{t-1}$, the elementary shift $h_{a,t}$ transforms $X_n$ into a $(-1)$-presentation

$$X_n' = X(M_2, c_3, \ldots, c_t, a+M_t, a+c_{t+1}, M_{t+1}', c_{t+2}', \ldots c_n', M_n') \tag{22}$$

with reversion

$$X_n'^\vee = X(M_n', c_3'^\vee, \ldots, c_{t^\vee}'^\vee, a+M_t, c_{t^\vee+1}^\vee, M_{t-1}, \ldots, c_n^\vee, M_2)\,,$$

see Lemmas 4.3.3 and 4.3.6. Choosing $a$ general we may suppose that

$$c_{t^\vee+1}^\vee \notin M_t^o := a + M_t\,. \tag{$(*)_t$}$$

Applying successively shifts $h_{a_t,t}$, $t=2,\ldots,n-1$, with general $a_3,\ldots,a_n\in\mathbb{C}$ the resulting surface $X_n^o$ satisfies $(*)_t$ for all $t=2,\ldots n-1$. Thus $X_n^{o\vee}$ is of $(-1)$-type, as required. ☐

**Definition 4.3.9.** Applying an elementary shift $h=h_{a,t}$ to the reversed presentation $X_n^\vee$ we obtain a presentation

$$h_{a,t^\vee}^\vee(X_n) := (h_{a,t}(X_n^\vee))^\vee\,,$$

which we call a *backward elementary shift.* If $X_n^\vee$ is of $(-1)$-type, then according to Lemmas 4.3.3 and 4.3.6, $h^\vee_{a,t^\vee}$ transforms a presentation

$$X_n = X(M_2, c_3, \ldots, c_{t-1},\ M_{t-1},\ c_t,\ M_t,\ c_{t+1},\ M_{t+1}, \ldots, c_n,\ M_n)$$

into

$$X_n' = X(M_2', c_3', \ldots, c_{t-1}',\ M_{t-1}',\ c_t',\ a+M_t,\ c_{t+1},\ M_{t+1}, \ldots, c_n,\ M_n)\,. \tag{23}$$

Clearly then $X_n'^\vee$ is as well of $(-1)$-type. However note that a backward shift can transform a $(-1)$-presentation $X_n$ into one not of $(-1)$-type.

In the sequel we fix, for every $t$ in the range $2\le t<n$, an isomorphism

$$\alpha_t : C_t\backslash C_{t-1} \xrightarrow{\cong} \mathbb{A}^1 \quad \text{with} \quad \alpha_t(C_t\cap C_{t+1}) = \{0\} \tag{24}$$

so that $[M_t]\in\mathfrak{M}^*_{s_t}$ (see 3.1.2).

**Lemma 4.3.10.** *Let*

$$X_n = X(M_2, c_3, M_3 \dots, c_n, M_n) \quad \textit{and} \quad X'_n = X(M'_2, c'_3, M'_3 \dots, c'_n, M'_n)$$

*be $(-1)$-presentations of (possibly different) Gizatullin surfaces $V$ and $V'$, respectively. Assume that the reversed presentations $X_n^\vee$ and $X_n'^\vee$ are also of $(-1)$-type. If the corresponding invariants*

$$(s_i)_{2\le i\le n}, \quad Q(X_n, D) \in \mathfrak{M} \quad \textit{and} \quad (s'_i)_{2\le i\le n}, \quad Q(X'_n, D') \in \mathfrak{M}$$

*are equal (cf. 3.2.2), then there exists a presentation*

$$X''_n = X(M''_2, c''_3, M''_3, \dots, c''_n, M''_n) \quad \textit{of} \quad V$$

*such that for every $t \ge 2$,*

*(a) $\alpha'_t(M'_t) = \alpha''_t(M''_t)$ when fixing suitable isomorphisms $\alpha'_t : C'_t \backslash C'_{t-1} \to \mathbb{A}^1$ and $\alpha''_t : C''_t \backslash C''_{t-1} \to \mathbb{A}^1$ as in (24);*

*(b) $X''_n$ and its reversion are both of $(-1)$-type.*

*Proof.* By assumption we have

(25) $\lambda_i\alpha'_i(M'_i) = \alpha_i(M_i) + a_i \quad$ for some $\quad \lambda_i \in \mathbb{C}^* \quad$ and $\quad a_i \in \mathbb{C}, \quad i = 2, \dots, n-1\,.$

After changing one of the isomorphisms $\alpha_i$, $\alpha'_i$ appropriately we may assume that $\lambda_i = 1$. Applying a suitable backward shift $h^\vee_{a,t^\vee}$ we can translate $M_t$ to $M''_t = M_t + a_t \cong M'_t$. Under this transformation $c_i$ and $M_i$ remain unchanged for $i > t$, see (23). Moreover the relations (25) remain valid for $i < t$ with possibly new coefficients $\lambda_i$, $a_i$. Applying decreasing induction starting with $t = n-1$ we can thus achieve that $a_t = 0$ for $t = 2, \dots, n-1$, as required. The transformed presentation $X''_n$ is then necessarily of $(-1)$-type since $M''_t \subseteq C''_t \backslash (C''_{t-1} \cup C''_{t+1})$ by construction. Moreover the reversed presentation is as well of $(-1)$-type since we only applied backwards shifts, proving also (b). □

4.4. **Isomorphisms of special surfaces of $(-1)$-type.** The following Proposition is the main result of Section 4.

**Proposition 4.4.1.** *Two special smooth Gizatullin surfaces $V$ and $V'$ with standard $(-1)$-completions $(\bar V, D)$ and $(\bar V', D')$* [14] *are isomorphic if and only if $D' \cong D$ or $D' \cong D^\vee$ and the configuration invariants*

(26) $$\tilde Q(\bar V, D) \quad \textit{and} \quad \tilde Q(\bar V', D')$$

*are equal.*

*Proof.* The 'only if' statement follows from Theorem 3.4.1. To show the converse we note first that according to Corollary 4.1.6 and Lemma 4.1.5 $\bar V$ and $\bar V'$ admit $(-1)$-presentations. Using Proposition 4.3.8, applying appropriate elementary shifts we can achieve that also the reversed completions are of $(-1)$-type.

Since $V$ and $V'$ are special, the standard zigzags of our completions have the form

$$D = [[0, 0, (-2)_{t-2}, -2-r, (-2)_{n-t}]] \quad \text{and} \quad D' = [[0, 0, (-2)_{t'-2}, -2-r', (-2)_{n'-t'}]]\,,$$

respectively, where by our assumption either $t' = t$ or $t' = t^\vee$. Replacing $(\bar V, D)$ by the reversion $(\bar V^\vee, D^\vee)$, if necessary, we may restrict to the case where $t' = t$ and the configurations invariants $Q(\bar V, D)$ and $Q(\bar V', D')$ are equal, see Theorem 3.4.1.

[14] See Definition 4.1.2.

Assume first that $2 < t = t' < n$. As we already remarked, the completions $(\bar V, D)$ and $(\bar V', D')$ arise from $(-1)$-presentations

$$\bar V = X_n = X(M_2, c_3, \ldots, M_t \ldots, c_n, M_n) \text{ and } \bar V' = X'_n = X(M'_2, c'_3, \ldots, M'_t \ldots, c'_n, M'_n),$$

respectively, where

$$s_2 = s'_2 = s_n = s'_n = 1\,, \quad s_t = s'_t = r, \quad \text{and} \quad s_i = s'_i = 0 \quad \forall i \notin \{2, t, n\}\,.$$

According to Lemma 4.3.10 we may assume that for every $j = 2, \ldots, n-1$ the configurations $M_j$, $M'_j$ coincide under appropriate isomorphisms $C_j \backslash (C_{j-1} \cup C_{j+1}) \cong \mathbb{C}^* \cong C'_j \backslash (C'_{j-1} \cup C'_{j+1})$. In particular, they are proportional whatever are these isomorphisms.

Applying now a sequence of elementary shifts, by Corollary 4.3.5 we may suppose that the points $c_3, \ldots, c_n$ and the unique point of $M_n$ are fixed under the torus action,[15] and similarly for $c'_3, \ldots, c'_n$ and $M'_n$. In particular $c_i = c'_i$ for $i = 3, \ldots, n$ and $M_n = M'_n$.

After these shifts the configurations $M_i$ and $M'_i$ are contained in the same curve $C_i = C'_i$. By our assumption they are proportional for $2 \le i < n$ and equal for $i = n$.

Using further the torus action on one of the surfaces we can move the point of $M_2$ into that of $M'_2$ so that $M_2 = M'_2$. There is a one-parameter subgroup of the torus acting trivially on $C_2$. It is easily seen that, since $t > 2$, this subgroup acts nontrivially on $C_t$. With this $\mathbb{C}^*$-action we can move $M_t$ on $C_t$ into the position of $M'_t$ (that is proportional to $M_t$), keeping $M_2$ and $M_n$ fixed. Now the presentations became equal, and so they define isomorphic Gizatullin surfaces $V$ and $V'$.

The same argument works also in the case, where $t = t' = 2 < n$ or $2 < t = t' = n$. We leave the details to the reader. □

To give right away an application let us deduce Theorem 1.0.7(c) in the Introduction in a particular case.

**Corollary 4.4.2.** *Let $V$, $V'$ be smooth Gizatullin $\mathbb{C}^*$-surfaces with DPD-presentations*

$$V = \operatorname{Spec} \mathbb{C}[u][D_+, D_-] \qquad \text{and} \qquad V' = \operatorname{Spec} \mathbb{C}[u][D'_+, D'_-]\,.$$

*Suppose that $D_+ = D'_+ = 0$. Then $V$ and $V'$ are isomorphic if and only if*

$$\deg\{D_-\} = \deg\{D'_-\} \quad \text{and} \quad \lfloor D_- \rfloor = \lfloor \beta^*(D'_-) \rfloor \tag{27}$$

*for some automorphism $\beta$ of $\mathbb{A}^1$.*

*Proof.* Since $D_+ = 0$, the pair $(D_+, D_-)$ is equivalent to a pair in (13) with $t = 2$. Hence by Proposition 3.5.3 either $n = 2$ in (13) i.e., $\{D_-\} = 0$, or $V$ is special. The same holds for $V'$.

Suppose first that (27) is fulfilled. In the case where $\{D_-\} = 0$ the pairs $(D_+, D_-)$ and $(D'_+, D'_-)$ are equivalent in the sense of 1.0.6 and so $V$ and $V'$ are equivariantly isomorphic. For the remaining part of the proof we assume that $\{D_-\} \neq 0$. Letting $\lfloor D_- \rfloor = \{p_1, \ldots, p_r\}$, by Corollary 3.5.5(1) the configuration invariants of both $V$ and $V'$ are given by $Q(\bar V, D) = (p_i)_{0 \le i \le r} \in \mathfrak{M}_{r+1}$. Since $\deg\{D_-\} = \deg\{D'_-\}$ the zigzags of the standard completions of $V$ and $V'$ are equal, see Corollary 3.5.5(2). Since $t = 2$ in (12), both $V$ and $V'$ arise from $(-1)$-presentations. Applying Proposition 4.4.1, $V$ and $V'$ are isomorphic, as required.

[15]Possibly after such transformations $X_n^\vee$ is not any longer of $(-1)$-type; however this does not matter in the rest of the proof.

Conversely, if $V \cong V'$ then their standard boundary zigzags coincide up to reversion. Inspecting Proposition 3.5.1 it follows that $\deg\{D_-\} = \deg\{D'_-\}$. Moreover, by Proposition 4.4.1 the configuration invariants of $V$ and $V'$ are the same. Since they are given by $\lfloor D_- \rfloor$ and $\lfloor D'_- \rfloor$, respectively, (27) follows.

□

**Remark 4.4.3.** The $\mathbb{C}^*$-surfaces as in Corollary 4.4.2 are just normalizations of the hypersurfaces in $\mathbb{A}^3$ as in 1.0.8 in the Introduction.

## 5. Shifting presentations and moving coordinates

In this section we provide further technical tools that will enable us in the next section to deduce Theorem 1.0.7 from the Introduction (see Theorem 6.2.1 below). For certain special Gizatullin surfaces of type I this theorem was already proved in Section 4. The strategy of proof for the remaining surfaces is similar, namely to apply Proposition 4.4.1. However, given a special Gizatullin surface $V$ of type II with standard completion $(\bar{V}, D)$ usually neither $(\bar{V}, D)$ nor its reverse completion is of $(-1)$-type, as it was the case for the special type I surfaces already treated. For instance, if in the presentation the blowup centers $c_3, \ldots, c_{k+1}$ are points of a feather generated by the blowup of $M_2$ then the elementary shifts are the identity on many of the subsequent components of the zigzag. It is not enough to shift just the first moving blowup center, either in the presentation or in its reversion, as in the proof of Proposition 4.4.1 above. We have to take into account also second order motions, which makes the proof considerably more involved.

5.1. **Coordinate description of a presentation.** As a principal technical tool we use a sequence of coordinate charts on our presentation $X_n$. They appear in the recursive construction of $X_n$ when describing the blowups in coordinates. This procedure is similar to the Hamburger-Noether algorithm for resolving a plane curve singularity, and our coordinates are analogous to those in the Hamburger-Noether tables, see e.g., [Ru]. One of the main insights is that they allow an explicit description of the correspondence fibration $\psi : W \to \mathbb{P}^1$ as in Definition 3.3.3, see Proposition 5.2.1 below.

**5.1.1.** On the quadric $X_1 = Q = \mathbb{P}^1 \times \mathbb{P}^1$ we consider the affine chart

$$U_1 = Q \backslash (C_0 \cup C_1) \cong \mathbb{A}^2$$

with affine coordinates $(x_1, y_1) = (x, y)$, where as before

$$C_0 = \{x = \infty\}, \quad C_1 = \{y = \infty\}, \quad \text{and} \quad C_2 = \{x = 0\}\,. \tag{28}$$

We let $V = V_n$ be a special smooth Gizatullin surface with a presentation $X = X(M_2, c_3, \ldots, c_n, M_n)$ obtained from the quadric $Q$ by a sequence of blowups as in Definition 4.1.1. We decompose this presentation into a sequence of single blowups

$$X = X_N \to X_{N-1} \to \ldots \to X_1 = Q\,,$$

so that first we blow up the points of $M_2$ on $C_2$ to create the corresponding feathers (in any order), then $c_3$ to create $C_3$, then the points on $M_3 \subseteq C_3$ to create feathers etc., as prescribed by Definition 4.1.2.

We say that the blowup $X_i \to X_{i-1}$ is of type (F) if it creates a feather, say, $F_i$. Otherwise $X_i \to X_{i-1}$ is called of type (C), in which case we let $F_i = \emptyset$.

Starting with the coordinate system $(x_1, y_1) = (x, y)$ on $X_1 = Q$ we construct recursively coordinate charts $U_i \cong \mathbb{A}^2$ on the intermediate surfaces $X_i$ with coordinates $(x_i, y_i)$. They satisfy the following properties.

$(1)_i$ If $C_s$ is the last curve of the zigzag constructed on $X_i$ then

$$U_i = X_i \backslash \left( C_0 \cup C_1 \cup \ldots \cup C_{s-1} \cup \bigcup_{j \le i} F_j^\vee \right) \cong \mathbb{A}^2 . \tag{29}$$

where $F_j^\vee = \emptyset$ if the blowup $X_j \to X_{j-1}$ is of type (C), and $F_j^\vee$ is the curve described explicitly below, when this blowup is of type (F). As we shall show in 5.2.2 $F_j^\vee$ is the matching feather of $F_j$.

$(2)_i$ $C_s \cap U_i = \{x_i = 0\}$ and $y_i | C_s$ yields an affine coordinate on $C_s \backslash C_{s-1} \cong \mathbb{A}^1$.

$(3)_i$ If $X_i \to X_{i-1}$ is of type (F) then $F_i \cap U_i = \{y_i = 0\}$.

Clearly these properties are satisfied for $U_1$ and $(x_1, y_1)$ when we let $F_1^\vee = \emptyset$. Assume that the coordinate chart $U_i \subseteq X_i$ with coordinates $(x_i, y_i)$ was already constructed so that $(1)_i$, $(2)_i$, $(3)_i$ are satisfied. We introduce the coordinate chart $U_{i+1}$ on $X_{i+1}$ as follows.

**Type F:** In the next blowup $X_{i+1} \to X_i$ with center at a point $p = (0, d) \in M_s \subseteq C_s$ a feather $F_{i+1}$ is created. Then we let

$$(x_{i+1}, y_{i+1}) = \left( \frac{x_i}{y_i - d},\ y_i \right) \qquad \text{so that} \quad (x_i, y_i) = ((y_{i+1} - d)x_{i+1}, y_{i+1}) \ . \tag{30}$$

We let $F_{i+1}^\vee$ denote the proper transform on $X_{i+1}$ (and on all further surfaces $X_{i+j+1}$) of the closure in $X_i$ of the affine line $\{y_i = d\}$. Clearly, the rational function $x_{i+1}$ has a first order pole along the curve $F_{i+1}^\vee$. So $(x_{i+1}, y_{i+1})$ are coordinates in the affine chart $U_{i+1}$ as in $(1)_{i+1}$ with axes as in $(2)_{i+1}$ and $(3)_{i+1}$. By construction, the level set $\{y_i = d\}$ of the rational function $y_i$ on $X_{i+1}$ contains both $F_{i+1}$ and $F_{i+1}^\vee$. We show in Lemma 5.2.2 below that $(F_i, F_i^\vee)$ is actually a matching pair as in Definition 3.3.7.

**Type C:** In the next blowup $X_{i+1} \to X_i$ with center at a point $c_{s+1} = (0, c) \in C_s$ the component $C_{s+1}$ is created. In this case $F_{i+1} = F_{i+1}^\vee = \emptyset$ and

$$(x_{i+1}, y_{i+1}) = \left( x_i, \frac{y_i - c}{x_i} \right) \qquad \text{so that} \quad (x_i, y_i) = (x_{i+1}, x_{i+1}y_{i+1} + c) \ . \tag{31}$$

These are coordinates in the affine chart $U_{i+1}$ as in $(1)_{i+1}$. The exceptional curve $C_{s+1}$ is given on $U_{i+1}$ as $\{x_{i+1} = 0\}$. The rational function $x_{i+1}$ has a first order pole along the curve $C_s$, hence $C_s \cap U_{i+1} = \emptyset$.

5.2. **Correspondence fibration revisited.** Let us reverse the zigzag $D = C_0 \cup \ldots \cup C_s$ on the surface $X_i$ by a sequence of inner elementary transformations so that $C_s$ and its previous (new) component $C_{s^\vee - 1}^\vee$ become 0-curves. On the resulting surface $W_i$ the component $C_{s-1}$ is blown down, while the coordinate chart $U_i$ as in (29) remains unchanged, but now

$$U_i = W_i \setminus \left( C_n^\vee \cup \ldots \cup C_{s^\vee - 1}^\vee \cup \bigcup_{j \le i} F_j^\vee \right) \ .$$

The linear systems $|C_s|$ and $|C_{s^\vee - 1}^\vee|$ define morphisms $p_1 : W_i \to \mathbb{P}^1$ and $p_2 : W_i \to \mathbb{P}^1$, respectively. We may suppose that $C_s$ and $C_{s^\vee - 1}^\vee$ are the fibers of $p_1$ and $p_2$ over the points $0 \in \mathbb{P}^1$ and $\infty \in \mathbb{P}^1$, respectively. Here $p_2$ is the correspondence fibration for the pair of curves $(C^\vee_{s^\vee}, C_s)$ (see Definition 3.3.3). We note that this fibration differs from the correspondence fibration for $(C_s, C^\vee_{s^\vee})$ just by a sequence of elementary transformations in the fiber $C^\vee_{s^\vee-1}$ of $p_2$. The following proposition gives an insight into the geometric meaning of the coordinates $(x_i, y_i)$ above.

**Proposition 5.2.1.** *In appropriate coordinates on $\mathbb{A}^1 = \mathbb{P}^1\backslash\{\infty\}$, the maps $p_1$ and $p_2$ restricted to the chart $U_i$ are given by $x_i$ and $y_i$, respectively. In particular $y_i$ is a regular function on $W_i\backslash C^\vee_{s^\vee-1}$. It vanishes on the curve $C^\vee_{s^\vee+1}$ so that*[16]

$$q^\vee = c^\vee_{s^\vee+1} = (\infty, 0) \in C^\vee_{s^\vee}\backslash C^\vee_{s^\vee-1}\,.$$

*Proof.* We assume by induction that both assertions hold for $W_i$, and we show that they hold also for $W_{i+1}$. Suppose first that the blowup $X_{i+1} \to X_i$ is of type (F) so that the feather $F_{i+1}$ attached to the point $(0, d) \in C_s$ is created. Reversing the zigzags, we replace $X_i$ by $W_i$ and $X_{i+1}$ by $W_{i+1}$.

To pass directly from the surface $W_i$ to $W_{i+1}$, we perform first an elementary transformation on $W_i$ by blowing up with center at the point $P \in C^\vee_{s^\vee-1} \cap C^\vee_{s^\vee}$ (this results in a new curve $\tilde{C}^\vee_{s^\vee-1}$) and contracting the proper transform of $C^\vee_{s^\vee-1}$. Then we blow up with center at the point $(0, d) \in C_s$ to create the feather $F_{i+1}$. On the resulting surface $W_{i+1}$ we have $C_s^2 = 0$ and $(\tilde{C}^\vee_{s^\vee-1})^2 = 0$.

Let us consider a coordinate chart on the surface $W_i$ centered at the intersection point $C_s \cap C^\vee_{s^\vee-1}$ with coordinate functions

$$(u, v) = \left(x_i, \frac{1}{y_i - d}\right) \quad \text{and axes} \quad C_s = \{u = 0\}\,, \quad C^\vee_{s^\vee-1} = \{v = 0\}\,.$$

Blowing up $P$ and contracting $C^\vee_{s^\vee-1}$ leads to coordinates

$$(\tilde{u}, \tilde{v}) = (uv, v) = \left(\frac{x_i}{y_i - d}, \frac{1}{y_i - d}\right) = \left(x_{i+1}, \frac{1}{y_{i+1} - d}\right)$$

on the surface $W_{i+1}$ (see (30)) with axes

$$C_s = \{\tilde{u} = 0\} = \{x_{i+1} = 0\} \qquad \text{and} \qquad \tilde{C}^\vee_{s^\vee-1} = \{\tilde{v} = 0\} = \{y_{i+1} = \infty\}\,.$$

Now both assertions follow in this case.

Suppose further that the blowup $X_{i+1} \to X_i$ is of type (C) so that it creates the new component $C_{s+1}$ of the zigzag. Similarly as before, we introduce first a coordinates chart on $W_i$ centered at the point $C_s \cap C^\vee_{s^\vee-1}$, at which $(x_i, y_i) = (0, \infty)$, with coordinates

$$(u, v) = \left(x_i, \frac{1}{y_i - c}\right)\,.$$

Next we create the component $C_{s+1}$ by a blowup at the point $(0, c) \in C_s$ with $(u, v)$-coordinates $(0, \infty)$. After performing an elementary transformation at $P$ as before we get a new surface $\tilde{W}_i$ and a coordinate chart on $\tilde{W}_i$ with coordinate functions

$$(\tilde{u}, \tilde{v}) = (uv, v) = \left(\frac{x_i}{y_i - c}, \frac{1}{y_i - c}\right) \quad \text{and axes} \quad C_s = \{\tilde{u} = 0\}\,, \quad \tilde{C}^\vee_{s^\vee-1} = \{\tilde{v} = 0\}\,.$$

[16]Cf. Lemma 3.3.4(a) and Corollary 3.3.5.

On $\tilde{W}_i$ we have $(\tilde{C}^\vee_{s^\vee-1})^2 = C_s^2 = 0$ and $C_{s+1}^2 = -1$. Around the intersection point $C_s \cap C_{s+1}$ on $\tilde{W}_i$ this leads to coordinates

$$(\tilde{x}, \tilde{y}) = \left(\tilde{u}, \frac{1}{\tilde{v}}\right) = \left(\frac{x_i}{y_i - c}, y_i - c\right) \quad \text{and axes} \quad C_s = \{\tilde{x} = 0\}\,, \quad C_{s+1} = \{\tilde{y} = 0\}\,.$$

To achieve the equality $C_{s+1}^2 = 0$ [17] we have to perform a further elementary transformation by blowing up with center at the point $C_s \cap \tilde{C}^\vee_{s^\vee-1}$ and contracting the proper transform of $C_s$. On the resulting surface $W_{i+1}$ this creates a new component $C^\vee_{s^\vee-2}$ with $(C^\vee_{s^\vee-2})^2 = C_{s+1}^2 = 0$. Proceeding in the same way as before we obtain on $W_{i+1}$ coordinates (see (31))

$$(\hat{u}, \hat{v}) = (\tilde{x}\tilde{y}, \tilde{x}) = \left(x_i, \frac{x_i}{y_i - c}\right) = \left(x_{i+1}, \frac{1}{y_{i+1}}\right)\,,$$

centered at the point $C^\vee_{s^\vee-2} \cap C_{s+1}$, with axes

$$C_{s+1} = \{\hat{u} = 0\} \qquad \text{and} \qquad C^\vee_{s^\vee-2} = \{\hat{v} = 0\}\,,$$

and again the first assertion follows.

Let us finally check that also in this case the intersection point $C^\vee_{s^\vee} \cap \tilde{C}^\vee_{s^\vee-1}$ satisfies the condition $y_{i+1} = 0$. The coordinate chart on $W_{i+1}$ around this intersection point has coordinates

$$(u', v') = \left(\frac{1}{\tilde{u}}, \tilde{v}\right) \quad \text{with axes} \quad \tilde{C}^\vee_{s^\vee-1} = \{v' = 0\} \quad \text{and} \quad C^\vee_{s^\vee} = \{u' = 0\}\,.$$

Since $u' = 1/\tilde{u} = 1/\tilde{x} = y_{i+1}$, the coordinate function $y_{i+1}$ vanishes on $C^\vee_{s^\vee}$. Now the proof is completed. □

The next lemma clarifies the meaning of the curves $F_i^\vee$.

**Lemma 5.2.2.** *If $F_{i+1}$ and $F^\vee_{i+1}$ are nonempty then they form a matching pair on the surface $X = X_N$ as in (3.3.7) above.*

*Proof.* Consider the correspondence fibration $\psi : W \to \mathbb{P}^1$ for the pair $(C_s, C^\vee_{s^\vee})$. With the notation as in the proof of Proposition 5.2.1, $W$ is a blowup of $W_{i+1}$ with centers in points of $C_s \cap U_{i+1}$ and in infinitesimally near points. By construction, the curve $F^\vee_{i+1}$ on $W_{i+1}$ does not meet $C_s$. Hence the proper transform of $F^\vee_{i+1}$ on $W$ does not meet $D^{\geq s}$. By Lemma 3.3.4(b) $F^\vee_{i+1}$ is a feather of $D_{\text{ext}}^{\vee \geq s^\vee}$. Since it meets $F_{i+1}$, the pair $(F_{i+1}, F^\vee_{i+1})$ is a matching pair, as desired. □

## 5.3. **Coordinates on special Gizatullin surfaces.**

**5.3.1.** Let $V$ be a special smooth Gizatullin surface with data $(n, r, t)$ as in Definition 1.0.4. This means that $V$ admits a standard completion $(\bar{V}, D)$ satisfying:

(a) every component $C_i$, $i \geq 3$, of the zigzag $D$ is created by a blowup on $C_{i-1}$;
(b) the extended divisor of $(\bar{V}, D)$ is of the form

$$D_{\text{ext}} = C_0 + \ldots + C_n + F_2 + F_{t1} + \ldots F_{tr} + F_n, \quad n \geq 3\,,$$

where $F_2$, $F_{t\rho}$ and $F_n$ are feathers with mother components $C_2$, $C_t$, and $C_n$, respectively.

[17] That is, to shift two zero weights in the zigzag one position to the right.

Due to Corollary 4.1.6 there exists a presentation

$$\bar{V} = X = X(M_2, c_3, \ldots, c_n, M_n) \tag{32}$$

as in 4.1.1 with $V = X \backslash D$. Reversing the completion $(\bar{V}, D)$, if necessary, we may assume that either

(i) $2 < t < n$, $|M_2| = |M_n| = 1$, $|M_t| = r \geq 0$ and $|M_i| = \emptyset \; \forall i \neq 2, t, n$, or
(ii) $2 < t = n$, $|M_2| = 1$, $|M_n| = r + 1 \geq 1$ and $|M_i| = \emptyset \; \forall i \neq 2, n$.

We suppose in the sequel that the feather $F_2$ is attached to component $C_{k+1}$ with $k \geq 1$. Thus $c_{i+1} \in C_i \cap F_2 \; \forall i = 2, \ldots, k$, while $c_{k+2} \notin F_2$.

We note that in general neither $(\bar{V}, D)$ nor its reversion is of $(-1)$-type. Our main goal is to transform such a presentation into one of $(-1)$-type.

Returning to the procedure as in Definition 4.1.2, we adapt the coordinates in 5.1.1 to our case.

**5.3.2.** In what follows we suppose that $k + 1 \leq t$. To describe coordinate charts of the surface $X$ as in (32) we proceed as follows.

(1) Let $U_1 = X_1 \setminus (C_0 \cup C_1)$ be the affine coordinate chart on the quadric $X_1 = Q$ as in 5.1 with coordinates $(x_1, y_1)$. The feather $F_2$ is created via the blowup $X_2 \to X_1$ of type (F) with center $c_2 = (0, 0) \in C_2$ and $F_2^\vee \subseteq X_2$ is the proper transform of $\{y_1 = 0\}$. In the affine chart $U_2$ as in (29) we have coordinates

$$(x_2, y_2) = (x_1/y_1, y_1) \text{ with axes } C_2 = \{x_2 = 0\} \text{ and } F_2 = \{y_2 = 0\}\,.$$

(2) We perform inner blowups at the subsequent intersection points $c_3 = F_2 \cap C_2$, $c_4 = F_2 \cap C_3, \ldots, c_{k+1} = F_2 \cap C_k$ creating the components $C_3, \ldots, C_{k+1}$ of the zigzag.[18] Since $c_3 = (0, 0)$ in $(x_2, y_2)$-coordinates, its blowup results in new coordinates $(x_3, y_3) = (x_2, y_2/x_2)$. Continuing in this way, in each step $i = 2, \ldots, k$ we obtain the affine chart

$$U_{i+1} = X_{i+1} \backslash (C_0 \cup \ldots \cup C_i \cup F_2^\vee) \tag{33}$$

on the corresponding surface $X_{i+1}$ with coordinates

$$(x_{i+1}, y_{i+1}) = (x_i, y_i/x_i) = \left(x/y, y^i/x^{i-1}\right)\,, \tag{34}$$

the origin $c_{i+2}$ for $i \leq k - 1$ and axes

$$C_{i+1} = \{x_{i+1} = 0\} \quad \text{and} \quad F_2 = \{y_{i+1} = 0\}\,.$$

The converse formulas are:

$$(x, y) = (x_{i+1}^i y_{i+1}, x_{i+1}^{i-1} y_{i+1}), \quad i = 1, \ldots, k\,. \tag{35}$$

(3) For $i = k + 1, \ldots, t - 1$ [19] we perform outer blowups with center at $c_{i+1} \in C_i \setminus C_{i-1}$ creating the components $C_{k+2}, \ldots, C_t$ of the zigzag. In particular, in $(x_{k+1}, y_{k+1})$-coordinates we have $c_{k+2} = (0, c'_{k+2})$ with $c'_{k+2} \neq 0$. To reduce notation we will identify $c_{k+2}$ with its coordinate $c'_{k+2}$. Using this convention also in the following steps, we get the affine chart $U_{i+1} \subseteq X_{i+1}$ as in (33) with coordinates

$$(x_{i+1}, y_{i+1}) = \left(x_i, \frac{y_i - c_{i+1}}{x_i}\right), \quad \text{where} \quad C_{i+1} = \{x_{i+1} = 0\} \tag{36}$$

[18] In the case $k = 1$ this step is absent.

[19] This step is absent if $k + 1 = t$.

The converse formulas are

$$(x_i, y_i) = (x_{i+1}, x_{i+1}y_{i+1} + c_{i+1}) . \tag{37}$$

(4) In particular, for $i = t$ we get the affine chart $U_t \subseteq X_t$ with coordinates $(x_t, y_t)$. We perform outer blowups at the distinct points $d_i \hat{=} (0, d_i)$, $1 \le i \le r$, of $M_t \subseteq C_t \setminus C_{t-1}$ to create the feathers $F_{ti}$, and a further blowup at $c_{t+1} \hat{=} (0, c_{t+1})$ creating the component $C_{t+1}$. On the resulting surface $X_{t+1}$ this leads to the new coordinate system[20]

$$(x_{t+1}, y_{t+1}) = \left(\frac{x_t}{P}, \frac{y_t - c_{t+1}}{x_t} P\right) , \quad \text{where} \quad P = \prod_{i=1}^{r}(y_t - d_i) \in \mathbb{C}[y_t] , \tag{38}$$

in the affine chart

$$U_{t+1} = X_{t+1} \backslash (C_0 \cup \ldots \cup C_t \cup F_2^\vee \cup F_{t1}^\vee \cup \ldots \cup F_{tr}^\vee) \cong \mathbb{A}^2 ,$$

where $C_{t+1} = \{x_{t+1} = 0\}$ and $F_{ti}^\vee$ denotes the proper transform of the closure in $X_{t+1}$ of the affine line $\{y_t = d_i\} \subseteq U_t$, $i = 1, \ldots, r$. If $c_{t+1} \notin M_t$ then $F_{ti} = \{x_{t+1}y_{t+1} = d_i - c_{t+1}\}$, while for $d_i = c_{t+1}$ the feather $F_{ti}$ is given by $y_{t+1} = 0$. The converse formulas are

$$(x_t, y_t) = (x_{t+1}P_{t+1},\ x_{t+1}y_{t+1} + c_{t+1}) , \quad \text{where} \quad P_{t+1} = \prod_{j=1}^{r}(x_{t+1}y_{t+1} + c_{t+1} - d_i) . \tag{39}$$

(5) For $i = t+1, \ldots, n-1$ we blow up at the point $c_{i+1} \in C_i \setminus C_{i-1}$ creating the component $C_{i+1}$, while for $i = n$ we blowup at the point $c_{n+1} \in M_n$ creating the feather $F_n$. In step $i$ with $t+1 \le i \le n-1$ we obtain coordinates $(x_{i+1}, y_{i+1})$ by formula (36), where $c_{i+1} \hat{=} (0, c_{i+1})$ in $(x_i, y_i)$-coordinates. The converse formulas (37) are still available. Obviously $(x_{i+1}, y_{i+1})$ forms a coordinate system in the affine chart

$$U_{i+1} = X_{i+1} \backslash (C_0 \cup \ldots \cup C_i \cup F_2^\vee \cup F_{t1}^\vee \cup \ldots \cup F_{tr}^\vee) \cong \mathbb{A}^2$$

on the corresponding surface $X_{i+1}$, where $t+1 \le i \le n$. Similarly for $i = n$ the same formulas define coordinates $(x_{n+1}, y_{n+1})$ in the affine chart $U_{n+1}$ on the terminal surface $X = X_{n+1}$, where $F_n = \{x_{n+1} = 0\}$.

5.4. **Moving coordinates.** In this subsection we study the effect of an elementary shift[21]

$$h = h_{a,m} : (x, y) \longmapsto (\xi(x, y),\ \eta(x, y)) = (x,\ y + ax^{1+m}) , \tag{40}$$

where $m \ge 0$ and $a \in \mathbb{C}_+$, on the sequence of coordinate systems as in 5.3.2. According to 4.3.1, $h$ transforms the presentation $X$ as in (32) into $X' = h_*(X)$, and yields an isomorphism of the affine surfaces $V$ onto a new one $V' = h_*(V)$. The standard completion can change due to the presence of indeterminacy points of $h$ on $C_0$. We introduce the following coordinates on $h_*(X)$.

Letting $(\xi_1, \eta_1) = (\xi, \eta)$ and performing the sequence of blowups as described in 5.3.1 in the images of the centers under $h$, we obtain a new sequence of coordinates

$$(\xi_i(x_i, y_i), \eta_i(x_i, y_i)), \quad i = 1, \ldots, n+1 .$$

[20]The enumeration differs from those in 5.1.1 since we are performing several blowups in one step.

[21]This corresponds to the elementary shift $h_{a,m+3}$ from Section 3.

Our next aim is to give, for every $i \geq 1$, explicit expressions of the maps $(x_i, y_i) \longmapsto (\xi_i(x_i, y_i),\, \eta_i(x_i, y_i))$ and of the vector field

$$(\bar\xi_i, \bar\eta_i) = \left(\frac{\partial \xi_i}{\partial a}(0),\ \frac{\partial \eta_i}{\partial a}(0)\right).$$

**5.4.1.** According to the cases in 5.3.2, the following recursive formulas hold.

(1) In the blowup $X_2 \to X_1$ we have $(\xi_2, \eta_2) = (\xi_1/\eta_1,\ \eta_1)$.

(2) In the inner blowups $X_{i+1} \to X_i$, $i = 1, \ldots, k$, of 5.3.1(2) we have in view of (34)

$$(\xi_{i+1}, \eta_{i+1}) = \left(\xi_i, \frac{\eta_i}{\xi_i}\right) = \left(\frac{\xi}{\eta}, \frac{\eta^i}{\xi^{i-1}}\right) = \left(\frac{x}{y + ax^{1+m}}, \frac{(y + ax^{1+m})^i}{x^{i-1}}\right).$$

Using (35) we get

$$(\xi_{i+1}(x_{i+1}, y_{i+1}), \eta_{i+1}(x_{i+1}, y_{i+1})) = \left(\frac{x_{i+1}}{1 + ax_{i+1}^{im+1} y_{i+1}^m}, (1 + ax_{i+1}^{im+1} y_{i+1}^m)^i y_{i+1}\right). \tag{41}$$

(3) In the blowups $X_{i+1} \to X_i$, $i = k+1, \ldots, t-1$ with centers at $c_{i+1} \hat{=} (0, c_{i+1}) \in C_i$ we get

$$(\xi_{i+1}, \eta_{i+1}) = \left(\xi_i, \frac{\eta_i - \gamma_{i+1}}{\xi_i}\right), \quad \text{where} \quad \gamma_{i+1} = \eta_i(0, c_{i+1}). \tag{42}$$

In the same range $i = k+1, \ldots, t-1$ we obtain by (37)

$$\xi_{i+1}(x_{i+1}, y_{i+1}) = \xi_i(x_{i+1}, x_{i+1} y_{i+1} + c_{i+1}) \quad \text{and} \tag{43}$$

$$\eta_{i+1}(x_{i+1}, y_{i+1}) = \frac{\eta_i(x_{i+1},\ x_{i+1} y_{i+1} + c_{i+1}) - \gamma_{i+1}}{\xi_i(x_{i+1},\ x_{i+1} y_{i+1} + c_{i+1})}. \tag{44}$$

Moreover

$$\bar\xi_{i+1} = \bar\xi_i(x_{i+1}, x_{i+1} y_{i+1} + c_{i+1}) \quad \text{and} \tag{45}$$

$$\bar\eta_{i+1} = \frac{\bar\eta_i(x_{i+1},\ x_{i+1} y_{i+1} + c_{i+1}) - \bar\eta_i(0, c_{i+1})}{x_{i+1}} - \frac{y_{i+1}}{x_{i+1}} \bar\xi_i(x_{i+1},\ x_{i+1} y_{i+1} + c_{i+1}). \tag{46}$$

(4) Similarly as in (38), with $\gamma_{t+1} = \eta_t(0, c_{t+1})$ and $\delta_\rho = \eta_t(0, d_\rho)$ we have

$$(\xi_{t+1}, \eta_{t+1}) = \left(\frac{\xi_t}{\Pi}, \frac{\eta_t - \gamma_{t+1}}{\xi_t}\Pi\right), \quad \text{where} \quad \Pi = \prod_{\rho=1}^{r} (\eta_t - \delta_\rho). \tag{47}$$

The converse formulas are

$$(\xi_t, \eta_t) = (\xi_{t+1}\Pi, \xi_{t+1}\eta_{t+1} + \gamma_{t+1}), \quad \text{where} \quad \Pi = \prod_{\rho=1}^{r} (\xi_{t+1}\eta_{t+1} + \gamma_{t+1} - \delta_\rho). \tag{48}$$

(5) If $i > t$, the recursion formulas for $(\xi_{i+1}, \eta_{i+1})$ are the same as in step (3).

**Remarks 5.4.2.** 1. In all cases $\xi_i = 0$ is a local equation for $C_i$ and so

$$\xi_i(0, y_i) = \bar\xi_i(0, y_i) = 0. \tag{49}$$

Furthermore,

$$\partial_1 \xi_i(0, y_i) = 1 \quad \text{and} \quad \partial_1 \bar\xi_i(0, y_i) = 0,$$

where $\partial_1 = \frac{\partial}{\partial x_i}$ and $\partial_2 = \frac{\partial}{\partial y_i}$. Indeed, $\partial_1 \xi_i(0, y_i) \neq 0 \ \forall y_i \in C_i \backslash C_{i-1} \cong \mathbb{A}^1$, $\forall a \in \mathbb{A}^1$. Hence $\partial_1 \xi_i(0, y_i)$ is a constant equal to its value at $a = 0$, which equals 1. The second relation follows by differentiating the first one.

2. The map $h_i : y_i \mapsto \eta_i(0, y_i)$ yields a translation

$$\eta_i(0, y_i) = y_i + e_i(a) \quad \text{with} \quad e_i(a) \in \mathbb{C}[a]\,, e_i(0) = 0,$$

of the affine line $C_i \backslash C_{i-1} \cong \mathbb{A}^1$. Indeed, $\eta_i(0, y_i)$ is an automorphism of $\mathbb{A}^1$ hence it has the form $c(a)y_i + e_i(a)$, where $c(a)$, $e(a) \in \mathbb{C}[a]$. Here $c(a) = c(0) = 1$ is constant since it does not vanish for all $a$. Moreover $e_i(0) = 0$ since $h_{0,m} = \mathrm{id}$. In particular

$$\partial_2 \eta_i(0, y_i) = 1 \quad \text{and} \quad \partial_2 \bar{\eta}_i(0, y_i) = 0\,. \tag{50}$$

We can now deduce the following formulas.

**Proposition 5.4.3.** *(a) At step $k+1$ we have*

$$\begin{aligned}
\xi_{k+1}(x_{k+1}, y_{k+1}) &= \frac{x_{k+1}}{1 + a x_{k+1}^{km+1} y_{k+1}^m}\,, \\
\eta_{k+1}(x_{k+1}, y_{k+1}) &= (1 + a x_{k+1}^{km+1} y_{k+1}^m)^k y_{k+1}\,, \\
\bar{\xi}_{k+1}(x_{k+1}, y_{k+1}) &= -x_{k+1}^{km+2} y_{k+1}^m\,, \\
\bar{\eta}_{k+1}(x_{k+1}, y_{k+1}) &= k x_{k+1}^{km+1} y_{k+1}^{m+1}\,.
\end{aligned}$$

*(b) If $k+2 \leq t$ then we have at step $k+2$*

$$\begin{aligned}
\bar{\xi}_{k+2}(x_{k+2}, y_{k+2}) &= -c^m x_{k+2}^{s-k} + \text{h.o.t.}\,, \\
\bar{\eta}_{k+2}(x_{k+2}, y_{k+2}) &= k c^{m+1} x_{k+2}^{s-k-2} + (s-1) c^m x_{k+2}^{s-k-1} y_{k+2} + \text{h.o.t.}\,,
\end{aligned}$$

*where* h.o.t. *stands for higher order terms in the first variable $x_{k+2}$ and*

$$s = k(m+1) + 2 \geq k+2\,, \qquad c = c_{k+2} \neq 0\,. \tag{51}$$

*(c) With $c$ and $s$ as in (51), in the range $k+3 \leq i \leq \min\{s, t\}$ we have*

$$\begin{aligned}
\bar{\xi}_i(x_i, y_i) &= -c^m x_i^{s-k} + \text{h.o.t.} \qquad \textit{and} \\
\bar{\eta}_i(x_i, y_i) &= k c^{m+1} x_i^{s-i} + (s-1) c_{k+3} c^m x_i^{s-i+1} + \text{h.o.t.}
\end{aligned}$$

*Proof.* The first two formulas in (a) are contained in (43) while the remaining two are obtained by differentiation.

Inserting $\bar{\xi}_{k+1}$ and $\bar{\eta}_{k+1}$ as in (a) into the recursive formulas (45)-(46) we get

$$\bar{\xi}_{k+2}(x_{k+2}, y_{k+2}) = -x_{k+2}^{km+2}(x_{k+2} y_{k+2} + c)^m \qquad \text{and} \tag{52}$$

$$\bar{\eta}_{k+2}(x_{k+2}, y_{k+2}) = x_{k+2}^{km}(kc + (k+1) x_{k+2} y_{k+2})(c + x_{k+2} y_{k+2})^m\,. \tag{53}$$

Now the Taylor expansion implies (b).

Finally, starting from (b) and using again the recursion formulas (45)-(46), (c) follows by an easy computation. We leave the details to the reader. □

Proposition 5.4.3 implies the following corollary.

**Corollary 5.4.4.** *Suppose that $m = 0$ so that $s = k + 2$.*

*(a) On component $C_{k+1}$ we have*

$$\xi_{k+1}(x_{k+1}, y_{k+1}) = \frac{x_{k+1}}{1 + ax_{k+1}}\,,$$
$$\eta_{k+1}(x_{k+1}, y_{k+1}) = (1 + ax_{k+1})^k y_{k+1}\,.$$

*(b) For every $i = 2, \ldots, t$ we have*

$$\bar\xi_i(x_i, y_i) = -x_i^2\,,$$
$$\bar\eta_i(x_i, y_i) = (i-1)x_i y_i + (i-2)c_i\,.$$

*In particular, if $s \le t$ then $\bar\eta_i(0, y_i) = 0$ for $i = 2, \ldots, s-1$, while $\bar\eta_s(0, y_s) \neq 0$.*

*Proof.* (a) is just a specialization of Proposition 5.4.3 to the case where $m = 0$. The first two formulas in (b) follow in the case $2 \le i \le k+2$ from (41) by differentiation. In the general case one can proceed by recursion using (43) and (44). The last assertion holds since by our assumptions $c_2 = \ldots = c_{k+1} = 0$ while $c_{k+2} \neq 0$, see 5.3.2(2), (3). □

5.5. **Induced motions.** Let us study the map induced by the shift

$$h = h_{a,m} : (x, y) \mapsto (x, y + ax^{1+m})$$

on the component $C_i = \{x_i = 0\}$. By Remark 5.4.2(2) for every $i = 1, \ldots, n$,

$$h = h|C_i : (0, y_i) \longmapsto (\xi_i(0, y_i), \eta_i(0, y_i)) = (0, y_i + e_i(a)) \quad \text{for some} \quad e_i(a) \in \mathbb{C}[a]$$

with $e_i(0) = 0$. We say that $h$ *generates a motion on component $C_i$* if $\deg e_i > 0$. For instance, using (41) we obtain $\eta_i(0, y_i) = y_i$ for $i = 2, \ldots, k+1$, so there is no motion on components $C_2, \ldots, C_{k+1}$.

In the next lemma we study the motions in the case $m = 0$.

**Lemma 5.5.1.** *Let $X = X_{n+1}$ be as in 5.3.2. Assume as before that $k + 1 \le t$. Then* $\deg e_q = q - k - 1 > 0$ *and* $e_q(0) = 0$ $\forall q \in [k+2, t]$. *In particular, the shift $h = h_{a,0}$ induces a motion on component $C_q$ for every $q$ in this range.*

*Proof.* We let $\tilde x = x_2$ and $\xi = \xi_2$. At each step $i = 2, \ldots, t$ in 5.3.2, only components of the zigzag are created. Hence we have

$$(54) \qquad x_i = \tilde x, \qquad \xi_i = \xi = \frac{\tilde x}{1 + a\tilde x}, \quad \text{and, conversely,} \quad \tilde x = \frac{\xi}{1 - a\xi}\,.$$

Furthermore, by 5.3.2(3) for $k + 2 \le q \le t$

$$(55) \qquad \begin{aligned} y_q &= \frac{y_{q-1} - c_q}{\tilde x} = \frac{y_{q-2} - c_{q-1} - c_q\tilde x}{\tilde x^2} = \ldots = \frac{y_{k+1} - \sum_{j=k+2}^q c_j \tilde x^{j-k-2}}{\tilde x^{q-k-1}} \\ &= \tilde x^{k+1-q} y_{k+1} - \textstyle\sum_{j=k+2}^q c_j \tilde x^{j-q-1}\,. \end{aligned}$$

Conversely,

$$(56) \qquad y_{k+1} = \tilde x^{q-k-1} y_q + \sum_{j=k+2}^q c_j \tilde x^{j-k-2}\,.$$

Similarly as in (55), letting $\gamma_{j+1} = \eta_j(0, c_{j+1})$ we obtain

$$\eta_q(\tilde{x}, y_q) = \xi^{k+1-q}\eta_{k+1}(\tilde{x}, y_{k+1}) - \sum_{j=k+2}^{q} \gamma_j \xi^{j-q-1}\,, \tag{57}$$

where, by virtue of Corollary 5.4.4(a) and (56),

$$\eta_{k+1} = (1 + a\tilde{x})^k y_{k+1} = (1 + a\tilde{x})^k \left( \tilde{x}^{q-k-1} y_q + \sum_{j=k+2}^{q} c_j \tilde{x}^{j-k-2} \right)\,. \tag{58}$$

According to (54) we have $\tilde{x} = \xi(1 - a\xi)^{-1}$ and so $1 + a\tilde{x} = (1 - a\xi)^{-1}$. Inserting (58) into (57) and replacing $\tilde{x}$ by $\xi$ leads to

$$\begin{aligned} \eta_q(\tilde{x}, y_q) &= \xi^{k+1-q}(1 + a\tilde{x})^k y_{k+1} - \sum_{j=k+2}^{q} \gamma_j \xi^{j-q-1} \\ &= (1 - a\xi)^{1-q} y_q + \textstyle\sum_{j=k+2}^{q} c_j (1 - a\xi)^{2-j} \xi^{j-q-1} - \sum_{j=k+2}^{q} \gamma_j \xi^{j-q-1}\,. \end{aligned} \tag{59}$$

Regarding $\eta_q(\tilde{x}, y_q)$ as a function of $(\xi,\, y_q)$ we need to compute the constant term $e_q(a) = \eta_q(0, 0)$ in its Laurent expansion. We note that this function is regular in a neighborhood of the axis $\{\tilde{x} = 0\} = \{\xi = 0\}$. Therefore we can omit the first term and the last sum on the right in (59). Using the binomial expansion the remaining sum can be written as the Laurent series

$$\sum_{j=k+2}^{q} c_j \sum_{\mu=0}^{\infty} \binom{2-j}{\mu} (-1)^\mu a^\mu \xi^{j-q-1+\mu}\,. \tag{60}$$

Its constant term is

$$e_q(a) = \eta_q(0, 0) = \sum_{j=k+2}^{q} \binom{2-j}{q+1-j} (-1)^{q+1-j} c_j a^{q+1-j}\,. \tag{61}$$

Since $c_{k+2} \neq 0$, see 5.3.2(3), this is a polynomial of degree $q-k-1$ in $a$, while $e_q(0) = 0$ by 5.4.2(2), as stated. ☐

**Corollary 5.5.2.** *If $V$ is a Danilov-Gizatullin surface i.e. $r = 0$ in 5.3.2, then in the range $k + 2 \leq q \leq n$ the shift $h = h_{a,0}$ induces a translation $y_q \mapsto y_q + e_q(a)$ on component $C_q$, where $\deg e_q = q - k - 1$ and $e_q(0) = 0$.*

In the general case, where $m \geq 0$ is arbitrary and $X$ is a standard completion of a special smooth Gizatullin surface as in 5.3.2, we have the following lemma.

**Lemma 5.5.3.** *For every $i \geq 2$ with $i \neq t$ the degrees of $e_{i+1}$ and $\partial_1\eta_i(0, c_{i+1})$ as polynomials in $a$ are equal. Moreover the same is true for $i = t$ provided that $c_{t+1} \notin M_t$ i.e., $c_{t+1}$ is not a root of $P$.*

*Proof.* We give the proof only in the case $i = t$; in all other cases the same proof applies by replacing $t$ by $i$ and $P$ by 1. According to (48) and (39)

$$\eta_t(x_{t+1}P_{t+1}, x_{t+1}y_{t+1} + c_{t+1}) = \xi_{t+1}(x_{t+1}, y_{t+1}) \cdot \eta_{t+1}(x_{t+1}, y_{t+1}) + \gamma_{t+1}\,. \tag{62}$$

By virtue of Remark 5.4.2 we have the relations:

$$\xi_{t+1}(0, y_{t+1}) = 0, \qquad \partial_1\xi_{t+1}(0, y_{t+1}) = 1, \quad \text{and} \quad \partial_2\eta_t(0, c_{t+1}) = 1\,.$$

Thus differentiating (62) with respect to $x_{t+1}$ and evaluating at $x_{t+1} = 0$ yields

$$\partial_1\eta_t(0, c_{t+1}) \cdot P(c_{t+1}) + y_{t+1} = \eta_{t+1}(0, y_{t+1})\,.$$

By definition the term on the right is $y_{t+1} + e_{t+1}(a)$. Since $c_{t+1} \notin M_t$ is not a root of $P$, $P(c_{t+1}) \neq 0$, hence the result follows. □

Returning to the special case $m = 0$ (that is, $h = h_{a,0}$) treated in Lemma 5.5.1, let us show that its conclusion can be equally applied to the polynomial $e_{t+1}$.

**Proposition 5.5.4.** *Let as before $k+1 \le t$. If $c_{t+1} \notin M_t$ and $m = 0$, then $\deg_a e_{t+1}(a) = t - k > 0$ and $e_{t+1}(0) = 0$.*

*Proof.* Let again $r = 0$ so that no feather is attached to component $C_t$ and $V = V^{DG}$ is a Danilov-Gizatullin surface. The construction of 5.3.2 yields coordinates $(x_i^{DG}, y_i^{DG})$ and $(\xi_i^{DG}, \eta_i^{DG})$ on $X^{DG} = \bar V^{DG}$. Let us compare these coordinates with those $(x_i, y_i)$ and $(\xi_i, \eta_i)$ on a general surface $X = \bar V$, where $r$ is arbitrary. Since up to step $t$ in 5.3.2 the constructions in the Danilov-Gizatullin case and in the general case are identical, we have

$$(x_t^{DG}, y_t^{DG}) = (x_t, y_t) \quad \text{and} \quad (\xi_t^{DG}, \eta_t^{DG}) = (\xi_t, \eta_t).$$

By Lemma 5.5.3 the degrees of the polynomials $e_{t+1}(a)$ and $\partial_1\eta_t(0, c_{t+1})$ are equal. Moreover

$$\partial_1\eta_t(0, c_{t+1}) = \partial_1\eta_t^{DG}(0, c_{t+1})\,.$$

Applying Lemma 5.5.3 and Corollary 5.5.2, the term on the right is a polynomial of degree $t - k$ with zero constant term, which gives the required result. □

5.6. **Component of first motion.** We continue to study an elementary shift

$$h = h_{a,m} : (x, y) \mapsto (x, y + ax^{1+m})$$

expressed in terms of moving coordinates $(\xi_i, \eta_i)$.

**Proposition 5.6.1.** *(Motion Lemma) Suppose that $k + 1 \le t$ and $c_{t+1} \notin M_t$. Let as before $s = k(m + 1) + 2$. Then the following hold.*

(a) *The first motion under $h_{a,m}$ occurs on component $C_s$ i.e., $e_i = 0$ for all $i < s$, while* $\deg e_s > 0$.

(b) *If $k + 2 \le t$ then after a general coordinate change $(x, y) \to (x, y + bx)$ there is a motion on component $C_{s+1}$ i.e.,* $\deg e_{s+1} > 0$.

The proof of Proposition 5.6.1 is based on the following lemma.

**Lemma 5.6.2.** *If $k + 1 \le t$ and $c_{t+1} \notin M_t$ then the following hold.*

(a) $\bar\eta_i(0, y_i) = 0$ *for all* $i = 1, \ldots, s - 1$ *while* $\bar\eta_s(0, y_s) = \bar\eta_s(0, 0) \neq 0$.

(b) *If $k + 2 \le t$ then after a general coordinate change $(x, y) \to (x, y + bx)$ we have* $\partial_1\bar\eta_s(0, c_{s+1}) \neq 0$.

*Proof of Proposition 5.6.1.* Lemma 5.6.2(a) implies that the derivative $e_s'(0)$ is given by $e_s'(0) = \bar\eta_s(0, 0) = \bar\eta_s(0, y_s) \neq 0$. Hence there is a motion on component $C_s$. For every $i < s$ by Lemma 5.6.2(a), $\bar\eta_i(0, y_i) = 0$. This remains true after a coordinate change $(x, y) \to (x, y + a'x^{1+m})$, which replaces $a$ by $a + a'$. Consequently, $e_i'(a) = \frac{\partial\eta_i}{\partial a}(0, 0) = 0$ for every $a$ and so $e_i(a) = e_i(0) = 0$. This proves (a).

To deduce (b) we note that by Lemma 5.6.2(b), $\partial_1\bar\eta_s(0,c_{s+1})\neq 0$ and so $\partial_1\eta_s(0,c_{s+1})$ is not constant in $a$. In view of Lemma 5.5.3 also $e_{s+1}$ is not constant. Now the assertion follows. □

The rest of this subsection is devoted to the proof of Lemma 5.6.2.

*Proof of Lemma 5.6.2.* By (41) and Proposition 5.4.3, (a) is true if $s\le t$. Let us deduce (b) for $s\le t$. The latter inequality implies that $k+2\le t$. Thus by Lemma 5.5.1 a coordinate change $(x,y)\to(x,y+bx)$ results in a non-trivial translation on component $C_{k+2}$. Applying such a translation we may assume that $c_{k+3}\neq 0$. Now $\partial_1\eta_s(0,c_{s+1})\neq 0$ by the second formula in 5.4.3(c), as required.

We assume in the sequel that $s>t$. We have to distinguish several cases. First we treat in 5.6.3 the case $k+3\le t$. The proof in the remaining cases where $k+2=t$ or $k+1=t$ is given in 5.6.4. □

**5.6.3. The case** $k+3\le t$. According to Proposition 5.4.3(c), in step $t$

$$\begin{aligned}
&(63) && \bar\xi_t(x_t,y_t) = -c^m x_t^{s-k} + \text{h.o.t.} \quad \text{and}\\
&(64) && \bar\eta_t(x_t,y_t) = \alpha x_t^{s-t} + \beta c_{k+3}x_t^{s-t+1} + \text{h.o.t.}, \quad \text{where}\\
&(65) && c=c_{k+2}, \qquad \alpha = kc^{m+1} \quad \text{and} \quad \beta=(s-1)c^m
\end{aligned}$$

are nonzero constants. Let us compute the vector field $(\bar\xi_{t+1},\bar\eta_{t+1})$. By (47),

$$(66)\qquad (\xi_{t+1},\eta_{t+1}) = \left(\frac{\xi_t}{\Pi}, \frac{(\eta_t-\eta_t(0,c_{t+1}))\Pi}{\xi_t}\right), \quad \text{where} \quad \Pi=\prod_{\rho=1}^{r}(\eta_t-\eta_t(0,d_\rho)).$$

Since by our assumption $s-t>0$, by virtue of (64) $\bar\eta_t(0,d_\rho)=0$ $\forall\rho=1,\dots,r$. Consequently $\frac{d}{da}(\Pi)|_{a=0}=P'\cdot\bar\eta_t$, where $P=P(y_t)$ is as in (38). Applying in (66) the derivation $\frac{d}{da}|_{a=0}$ and expressing $x_t,y_t$ by $x_{t+1},y_{t+1}$ as in (39) we get

$$(67)\qquad \begin{aligned}
\bar\xi_{t+1} &= \frac{\bar\xi_t}{P(y_t)} - \frac{x_t}{P^2(y_t)}P'(y_t)\cdot\bar\eta_t\\
&= \frac{\bar\xi_t}{P_{t+1}} - \frac{x_{t+1}}{P_{t+1}}P'_{t+1}\cdot(\alpha x_{t+1}^{s-t}P_{t+1}^{s-t}+\text{h.o.t.}),
\end{aligned}$$

where

$$P_{t+1}:=P(c_{t+1}+x_{t+1}y_{t+1}) \quad \text{and} \quad P'_{t+1}:=P'(c_{t+1}+x_{t+1}y_{t+1}).$$

The first order Taylor expansion of $P_{t+1}^{s-t}$ is

$$(68)\qquad P_{t+1}^{s-t}=P(c_{t+1})^{s-t}+(s-t)P'(c_{t+1})P(c_{t+1})^{s-t-1}x_{t+1}y_{t+1}+\text{h.o.t.}$$

To compute the lowest order term of $\bar\xi_{t+1}$, the first term on the right of (67) is irrelevant by (63). Therefore from (67) and (68) we obtain

$$(69)\qquad \bar\xi_{t+1}=\gamma x_{t+1}^{s-t+1}+\text{h.o.t.}, \qquad \text{where} \quad \gamma=-\alpha P'(c_{t+1})P^{s-t-1}(c_{t+1}).$$

Likewise we can differentiate the expression for $\eta_{t+1}$ in (66) with respect to $a$, then replace $x_t, y_t$ by $x_{t+1}, y_{t+1}$ according to (39), and finally use (63), (64) to obtain

$$
\begin{aligned}
\bar\eta_{t+1} &= \frac{\bar\eta_t}{x_t}P(y_t) + \frac{y_t - c_{t+1}}{x_t}P'(y_t)\bar\eta_t - \frac{y_t - c_{t+1}}{x_t^2}P(y_t)\bar\xi_t \\
&= \alpha x_{t+1}^{s-t-1}P_{t+1}^{s-t} + \beta c_{k+3}x_{t+1}^{s-t}P_{t+1}^{s-t+1} + y_{t+1}\frac{P'_{t+1}}{P_{t+1}}\bar\eta_t + \text{h. o. t.}
\end{aligned}
\tag{70}
$$

Inserting (64) and the Taylor expansion (68) into this formula yields

$$
\begin{aligned}
\bar\eta_{t+1} &= \alpha P(c_{t+1})^{s-t}x_{t+1}^{s-t-1} + (s-t)\alpha P'(c_{t+1})P(c_{t+1})^{s-t-1}y_{t+1}x_{t+1}^{s-t} \\
&\quad +\beta c_{k+3}P(c_{t+1})^{s-t+1}x_{t+1}^{s-t} + \alpha P'(c_{t+1})P(c_{t+1})^{s-t-1}y_{t+1}x_{t+1}^{s-t} + \text{h. o. t.} \\
&= \tilde\alpha x_{t+1}^{s-t-1} + (\tilde\beta c_{k+3} + \tilde\gamma y_{t+1})x_{t+1}^{s-t} + \text{h. o. t.}\,,
\end{aligned}
\tag{71}
$$

where

$$
\tilde\alpha = \alpha P(c_{t+1})^{s-t}, \quad \tilde\beta = \beta P(c_{t+1})^{s-t+1}\,, \quad \text{and} \quad \tilde\gamma = (s-t+1)\frac{P'(c_{t+1})}{P(c_{t+1})}\tilde\alpha
\tag{72}
$$

are constants with $\tilde\alpha$, $\tilde\beta \neq 0$.

In the range $t+1 \le j < s$ we have $x_{j+1} = x_j$ and $\xi_{j+1} = \xi_j$. In view of (69) this yields

$$
\bar\xi_{j+1} = -\gamma x_{j+1}^{s-t+1} + \text{h. o. t.} \quad \text{for} \quad k+1 \le j < s\,.
\tag{73}
$$

To compute $\bar\eta_j$ for $j \ge t+2$ we first consider the step from $t+1$ to $t+2$. Differentiating the recursion formula (42) for $\bar\eta_{t+2}$ we obtain

$$
\bar\eta_{t+2} = \frac{\bar\eta_{t+1}}{x_{t+1}} - \frac{y_{t+1} - c_{t+2}}{x_{t+1}^2}\bar\xi_{t+1}\,.
$$

Using (69) and (71) and replacing $x_{t+1}, y_{t+1}$ by $x_{t+2}, y_{t+2}$ as in (37) we get

$$
\bar\eta_{t+2} = \tilde\alpha x_{t+2}^{s-t-2} + (\tilde\beta c_{k+3} + \tilde\gamma c_{t+2})x_{t+2}^{s-t-1} + \text{h. o. t.}
$$

Recursively the same arguments yield

$$
\bar\eta_j = \tilde\alpha x_j^{s-j} + (\tilde\beta c_{k+3} + \tilde\gamma c_{t+2})x_j^{s-j+1} + \text{h. o. t.} \quad \text{for} \quad t+2 \le j \le s\,.
\tag{74}
$$

Using (74) in the case $s \ge t+2$ and (71) in the case $s = t+1$, assertion (a) of Lemma 5.6.2 follows.

Let us show part (b) of the lemma. For $j = s \ge t+2$ (74) yields

$$
\partial_1\bar\eta_s(0, c_{s+1}) = \tilde\beta c_{k+3} + \tilde\gamma c_{t+2}\,.
\tag{75}
$$

Because of (71) this formula remains valid in the case $s = k+1$. To deduce (b) we have to check that this quantity is nonzero after an appropriate coordinate change

$$
(x, y) \longrightarrow (x(b), y(b)) = (x, y + bx)\,.
$$

For this we perform the sequence of blowups as above with $(x(b), y(b))$ instead of $(x, y)$ so that the centers $c_i(b) \in C_{i-1}$ and $d_\rho(b) \in C_t$ of the blowups now depend on $b$. According to Remark 5.4.2.2 these centers can be written as

$$
c_i(b) = c_i + e_{i-1}(b) \quad \text{and} \quad d_\rho(b) = d_\rho + e_t(b)
$$

with polynomials $e_i(b)$ satisfying $e_i(0) = 0$. Now (75) can be written as

$$\partial_1\bar{\eta}_s(0, c_{s+1})(b) = \tilde{\beta}c_{k+3} + \tilde{\gamma}c_{t+2} + \tilde{\beta}e_{k+2}(b) + \tilde{\gamma}e_{t+1}(b)\,. \tag{76}$$

By Lemma 5.5.1 and Proposition 5.5.4,

$$e_{k+1}(b) = 0, \quad \deg e_{k+2}(b) = 1, \quad \text{and} \quad \deg e_{t+1}(b) = t - k \geq 3\,.$$

In particular, $c_{k+2}(b) = c_{k+2} + e_{k+1}(b) = c_{k+2}$ does not depend on $b$, and hence also the constants $\alpha$ and $\beta$ in (65) do not depend on $b$.

We claim that the polynomial $P$ as in (38) does not depend on $b$ either. Indeed, with $P(b, T) = \prod_{i=1}^{r}(T - d_\rho(b))$ we have

$$P(b, y_t(b)) = \prod_{i=1}^{r}(y_t(b) - d_\rho(b)) = \prod_{i=1}^{r}(y_t - e_t(b) - d_\rho + e_t(b)) = \prod_{i=1}^{r}(y_t - d_\rho) = P(y_t)\,.$$

In particular, $P(b, c_{t+1}(b)) = P(c_{t+1})$ and $P'(b, c_{t+1}(b)) = P'(c_{t+1})$. Hence $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$ in (72) do not depend on $b$.

It follows that (76) is a nonzero polynomial in $b$ of degree 1 if $\tilde{\gamma} = 0$ and of degree $t - k$ otherwise. Anyway, $\partial_1\bar{\eta}_s(0, c_{s+1})(b) \neq 0$ for a suitable choice of $b$, proving (b).

**5.6.4. The cases $k + 2 = t$ and $k + 1 = t$.** If $k + 2 = t$ then according to Proposition 5.4.3(b)

$$\begin{aligned}\bar{\xi}_t(x_t, y_t) &= -c^m x_t^{s-t+2} + \text{h. o. t.}\,,\\ \bar{\eta}_t(x_t, y_t) &= \alpha x_t^{s-t} + \beta x_t^{s-t+1} y_t + \text{h. o. t.}\end{aligned}$$

with $\alpha$ and $\beta$ as before. The formulas (69) for $\xi_{t+1}$, (71) for $\eta_{t+1}$, (73) for $\xi_{j+1}$ and (74) for $\eta_{j+1}$ are applicable again; note that $c_{k+3} = c_{t+1}$ since $t = k + 2$. So the proof of (a) proceeds as before. Also the proof of (b) applies if we take into account that deg $e_{t+1}(b) = t - k = 2$ in our case.

In the case $k + 1 = t$ we only need to establish (a). Therefore it suffices to control the terms of lowest order of $\eta_j$ for $j = t + 1, \ldots, s$. By Proposition 5.4.3(a), instead of (63) and (64) we have to use the formulas

$$\begin{aligned}\bar{\xi}_t(x_t, y_t) &= -x_t^{s-t+1} y_t^m \quad \text{and}\\ \bar{\eta}_t(x_t, y_t) &= k x_t^{s-t} y_t^{m+1}\,.\end{aligned}$$

Proceeding as before we obtain that $\bar{\xi}_{t+1}$ is a multiple of $x_{t+1}^{s-t+1}$, while

$$\bar{\eta}_{t+1}(x_{t+1}, y_{t+1}) = \tilde{\alpha} x_{t+1}^{s-t-1} + \text{h. o. t.}\,,$$

where $\tilde{\alpha}$ is as in (72). By recursion in the range $j = t + 2, \ldots, s$,

$$\bar{\eta}_j(x_j, y_j) = \tilde{\alpha} x_{t+1}^{s-j} + \text{h. o. t.}$$

(cf. (74)) and so (a) follows. Now the proof of Lemma 5.6.2 is completed. □

**Remark 5.6.5.** In case $k+1 = t$ the distinguished feather $F_2$ is attached to component $C_t$. It may happen that $\partial_1\eta_s(0, y_s) = 0$ so that there is no motion on component $C_{s+1}$. Indeed, the formula (71) holds with $\tilde{\alpha}$ as before and $\tilde{\beta} = 0$ while $\tilde{\gamma}$ is a bit more complicated than in (71) since the term $\frac{y_t - c_{t+1}}{x_t^2} P(y_t)\bar{\xi}_t$ in (70), which contributed before only to higher order terms, cannot be ignored any more. More precisely,

$$\tilde{\gamma} = c_{t+1}^m P(c_{t+1})^{km}\left(k(km + 2)c_{t+1}P'(c_{t+1}) + (km + k + 1)P(c_{t+1})\right)$$

and so $\tilde{\gamma}$ vanishes for an appropriate choice of $c_{t+1}$. Note that $\tilde{\gamma}$ also vanishes after a linear change of coordinates as in Lemma 5.6.2(b), since such a coordinate change induces no motion on component $C_t$. Therefore the second highest coefficient of the expansion for $\eta_j$, $j \le s$, vanishes as well in all following steps. Thus $\partial_1 \eta_s(0, c_{s+1}) = \tilde{\gamma} c_{s+1} = 0$ i.e., the derivative vanishes for this choice of $c_{t+1}$.

## 6. Applications

6.1. **Moving feathers.** Given a presentation

$$X = X(M_2, c_3, \ldots, c_t, M_t, c_{t+1}, \ldots, c_n, M_n)$$

of a special smooth Gizatullin surface $V = X \backslash D$ with data $(n, r, t)$ (see 5.3.1), we consider the sequence of coordinate systems $(x_i, y_i)$ as in 5.3.2. For a fixed $i$ the curve $C_i \backslash C_{i-1} \cong \mathbb{A}^1$ is the axis $x_i = 0$. So it is equipped with the coordinate $y_i$ such that $C_i \cap C_{i-1} = \{y_i = \infty\}$. With respect to this coordinate, the data $(M_i, c_{i+1})$ correspond to a collection of complex numbers. We also deal with the reversed presentation

$$X^\vee = X(M_n, c_3^\vee, \ldots, c_{t^\vee}^\vee, M_{t^\vee}, c_{t^\vee+1}^\vee, \ldots, M_2)\,,$$

where as before $t^\vee = n - t + 2$ and the curves $C_i$ and $C_{i^\vee}^\vee$ are identified via the correspondence fibration as in 3.3.3. Under this identification $y_i$ yields a coordinate on $C_{i^\vee}^\vee$ so that the data $(M_{i^\vee}^\vee = M_i, c_{i^\vee+1}^\vee)$ are as well expressed by complex numbers. The reader should keep in mind that according to Proposition 5.2.1 $c_{i^\vee+1}^\vee = 0$.

Let $F_2$, $F_{t\rho}$ $(1 \le \rho \le r)$ and $F_n$ be the feathers of $X$ corresponding to the points of $M_2$, $M_t$ and $M_n$, respectively. The dual feathers $F_2^\vee$, $F_{t\rho}^\vee$ and $F_n^\vee$ have then mother components $C_n^\vee$, $C_{t^\vee}^\vee$ and $C_2^\vee$, respectively.

To study the effect of an elementary shift

$$h_{a,m} : (x, y) \to (x, y + ax^{1+m})$$

on these presentations, we exploit as in 5.4.1 the induced coordinate systems $(\xi_i, \eta_i)$ depending on $a$. The data $c_{i+1}(a)$, $M_i(a)$ and $c_{i^\vee+1}^\vee(a)$ expressed in the coordinate $y_i$ also depend on $a$. The following result is immediate from Proposition 5.2.1 and Remark 5.4.2(2).

**Lemma 6.1.1.** *If $\eta_i(0, y_i) = y_i + e_i(a)$ then*

$$c_{i+1}(a) = c_{i+1} + e_i(a), \quad M_i(a) = M_i + e_i(a), \quad \textit{while} \quad c_{i^\vee+1}^\vee(a) = c_{i^\vee+1}^\vee = 0\ \forall a \in \mathbb{C}\,.$$

Consequently, if $\deg e_i(a) > 0$ then the shift $h = h_{a,m}$ translates $M_i = M_{i^\vee}^\vee$ and $c_{i+1}$ while keeping the point $c_{i^\vee+1}^\vee = 0$ fixed. Thus for general $a$ we have $c_{i+1} \ne c_{i^\vee+1}^\vee$ and $c_{i^\vee+1}^\vee \notin M_{i^\vee}^\vee$. Inspecting Proposition 4.2.2 this means that for every feather $F^\vee$ attached to a component $C_{j^\vee}^\vee$ with $j^\vee > i^\vee$ one has $\mu^\vee > i^\vee$, where $C_{\mu^\vee}^\vee$ is the mother component of $C_{j^\vee}^\vee$. In other words, feathers cannot "cross" $C_{i^\vee}^\vee$. Using this idea to remove jumping feathers we can prove the following lemma.

**Lemma 6.1.2.** *Assume that $F_2$ sits on component $C_{k+1}$ and $F_n^\vee$ on component $C_{l^\vee+1}^\vee$. Applying suitable elementary shifts and backward shifts we can achieve that*

*(a) $k + 1 < \max(3, t)$ and, dually, $l^\vee + 1 < \max(3, t^\vee)$;*
*(b) all feathers $F_{t\rho}$ of $X$ and $F_{t^\vee\rho}^\vee$ of $X^\vee$ are attached to their mother components $C_t$ and $C_{t^\vee}^\vee$, respectively.*

*Proof.* By Corollary 4.2.3 we have $k+1 \le l-1$ or, equivalently, $k+1+l^\vee+1 \le n+2 = t+t^\vee$. Thus $k+1 \le t$ or $l^\vee+1 \le t^\vee$. By symmetry we may suppose that $k+1 \le t$. We proceed in several steps.

(1) *After a suitable shift we can achieve that* $l^\vee + 1 \le t^\vee$. Indeed, if $k+1 = t$ then by the above inequality we have $l^\vee+1 \le t^\vee$. If $k+1 < t$ then using Lemma 5.5.1, after a shift $c^\vee_{t^\vee+1} = 0 \ne c_{t+1}$. Inspecting Lemma 4.2.2(b) this shows that $l^\vee+1 \le t^\vee$.

(2) *After suitable shifts and backward shifts (b) holds.* If $k+1 = t$ then applying Corollary 4.2.3 to the pairs $(F_2, F^\vee_{t\rho})$, all feathers $F^\vee_{t\rho}$ are attached to their mother component $C^\vee_{t^\vee}$. If $k+1 < t$ then by Lemma 5.5.1, after a suitable shift $h_{a,0}$ we have $c^\vee_{t^\vee+1} = 0 \notin M_t$. Thus by Lemma 4.2.2(b) all feathers $F^\vee_{t\rho}$ are attached to their mother component $C^\vee_{t^\vee}$. Applying the same arguments to the reversion, after suitable backward shifts all feathers $F_{t\rho}$ are attached to their mother component $C_t$.

(3) *If* $t^\vee \ne 2$ *then after a suitable shift* $l^\vee + 1 < t^\vee$. Indeed, using (b) we have $c_{t+1} \notin M_t$ so that Proposition 5.5.4 can be applied. The shift $h_{a,0}$ creates a motion on component $C_{t+1}$. After such a motion we may assume that $c_{t+2}$ is nonzero, i.e. $c_{t+2} \ne c^\vee_{t^\vee}$. By Lemma 4.2.2(b) this forces $l^\vee+1 < t^\vee$.

Applying now (3) and its dual statement, (a) follows. □

**Corollary 6.1.3.** *Given a presentation $X$ of a special smooth Gizatullin surface, by performing shifts and backward shifts we can transform $X$ into a $(-1)$-presentation.*

*Proof.* After applying suitable shifts and backward shifts we may assume that (a) and (b) in Lemma 6.1.2 are fulfilled. Interchanging $X$ and $X^\vee$, if necessary, we may assume that $k \le l^\vee$. We may also suppose that $k \ge 2$, since otherwise $X$ has $(-1)$-type and we are done. We choose now $m$ in such a way that

$$n \ge s = k(m+1) + 2 \ge n - l^\vee + 1;$$

this is always possible because of our assumption $k \le l^\vee$. By Proposition 5.6.1, the elementary shift $h_{a,m}$ induces a motion on the curve $C_s$. After this motion we will have $c_{s+1} \ne c^\vee_{s^\vee+1} = 0$. Inspecting Lemma 4.2.2(b), on the new surface $X^\vee$ the feather $F^\vee_n$ will sit on a component $C^\vee_{\tilde{l}^\vee}$ with $\tilde{l}^\vee < l^\vee$. Thus after several such shifts we can achieve that $l^\vee < k$. Interchanging now $X$ and $X^\vee$ and continuing as before we obtain after a finite number of steps that $k = l^\vee = 1$, as required. □

Combining this with Proposition 4.4.1 leads to the following result.

**Corollary 6.1.4.** *Assume that $V$ and $V'$ are smooth special Gizatullin surfaces such that the zigzags of standard completions are equal up to reversion. Then $V$ and $V'$ are isomorphic if and only if the configuration invariants of $V$ and $V'$ coincide.*

*Proof.* The 'only if' part follows from Theorem 3.4.1. To prove the converse, let $(\bar V, D)$ and $(\bar V', D')$ be standard completions of $V$ and $V'$, respectively. Reversing one of them, if necessary, we may suppose that the dual graphs of $D$ and $D'$ are equal. By Proposition 4.1.6 both surfaces admit presentations. Moreover by Corollary 6.1.3 we may assume that both presentations are of $(-1)$-type. Applying Proposition 4.4.1 the result follows. □

6.2. **Main theorem and its corollaries.** Let us now deduce the Isomorphism Theorem 1.0.7(c) in the Introduction.

**Theorem 6.2.1.** *Given a special smooth Gizatullin $\mathbb{C}^*$-surface $V = \operatorname{Spec}\mathbb{C}[u][D_+, D_-]$, the isomorphism type of $V$ is uniquely determined by the unordered pair of numbers $(\deg\{D_+\}, \deg\{D_-\})$ and the configuration of points*

$$\operatorname{supp}(\lfloor -D_+ - D_-\rfloor) = \{p_1, \dots, p_r\}, \tag{77}$$

*up to the natural action of the automorphism group $\operatorname{Aut}(\mathbb{A}^1)$ on such configurations.*

*Proof.* This follows immediately from Corollary 6.1.4. Indeed, according to Corollary 3.5.5 the configuration invariant is given by $\{p_1, \dots, p_r\} \in \mathfrak{M}_r^+$, whereas the boundary zigzag is up to reversion uniquely determined by the numbers $\deg\{D_+\}$ and $\deg\{D_-\}$. □

The next result and Corollary 6.2.5 below yield the first assertion of Theorem 1.0.5 in the Introduction.

**Theorem 6.2.2.** *Every special smooth Gizatullin surface carries a $\mathbb{C}^*$-action. Moreover, if $V$ is of type I then the conjugacy classes of $\mathbb{C}^*$-actions on $V$ form in a natural way a one-parameter family, while in case of type II they form a two-parameter family.*

*Proof.* By Corollaries 4.1.6 and 6.1.3, $V$ admits a presentation of $(-1)$-type $X = X(M_2, c_3, \dots, M_t, \dots M_n)$ as in 5.3.1, where $M_t = \{p_1, \dots, p_r\}$. After reversing the presentation, if necessary, we may suppose that $t < n$.

First assume that also $t > 2$. Let us consider the $\mathbb{C}^*$-surface $V' = \operatorname{Spec}\mathbb{C}[u][D_+, D_-]$ with

$$D_+ = -\frac{1}{t-1}[p_+] \quad \text{and} \quad D_- = -\frac{1}{n-t+1}[p_-] - \sum_{\rho=1}^{r}[p_\rho],$$

where $p_+, p_-$ are different and not contained in $M_t$. Applying Corollary 3.5.5(a) the configuration invariant of this surface is given by $M_t \in \mathfrak{M}_r^+$. Moreover by (12) the boundary zigzags of $V$ and $V'$ coincide. Now by virtue of Corollary 6.1.4, $V$ and $V'$ are isomorphic.

Furthermore, the equivariant isomorphism type of $V'$ depends on the position of $p_+$, $p_-$ and $M_t$ while the abstract isomorphism type only depends on the configuration $M_t$. Thus in the case $r \geq 2$ the family of conjugacy classes of $\mathbb{C}^*$-actions on the surface $V$ depends on two parameters, while in the case $r = 1$ it depends on just one parameter. Since for $r \geq 2$ the surface is of type II while for $r = 1$ it is of type I, the result follows in this case.

Finally if $t = 2$ and $r \geq 2$ then the above reasoning together with Corollary 4.4.2 shows again that the $\mathbb{C}^*$-actions on $V$ form a one-parameter family. □

**6.2.3.** Any Danilov-Gizatullin surface $V$ admits a presentation

$$X = X(M_2, c_3, \dots, c_n, M_n) \text{ with } n \geq 2,\ |M_2| = |M_n| = 1 \text{ and } M_i = \emptyset \text{ otherwise.} \tag{78}$$

Indeed, let $V = \Sigma_k \backslash S$, where $S$ is an ample section in the Hirzebruch surface $\Sigma_k \to \mathbb{P}^1$ with $S^2 = n > k$. Blowing up a point of $S$ successively yields a semistandard boundary zigzag $[[0, -1, (-2)_{n-1}]]$ and so a standard zigzag $[[0, 0, (-2)_{n-1}]]$. Hence by Lemma 4.1.5, $V$ admits a presentation as in (78). Applying Corollary 6.1.4 we recover the theorem of Danilov and Gizatullin [DaGi] cited in the Introduction[22].

[22]See also [CNR, $\text{FKZ}_4$].

**Corollary 6.2.4.** *The isomorphism type of a Danilov-Gizatullin surface* $V = \Sigma_k \backslash S$ *depends only on* $n = S^2$.

The following corollary completes the proof of Theorem 1.0.5 in the Introduction.

**Corollary 6.2.5.** *A special smooth Gizatullin surface* $V$ *of type I admits a one-parameter family of pairwise non-equivalent* $\mathbb{A}^1$*-fibrations* $V \to \mathbb{A}^1$*, while for type II it admits such a family depending on two parameters.*

*Proof.* Every $\mathbb{C}^*$-action on $V$ extends to two mutually reversed equivariant standard completions $(\bar{V}, D)$ and $(\bar{V}^\vee, D^\vee)$. On each of them there is an associated $\mathbb{A}^1$-fibration of $V$ induced by the linear system $|C_0|$. By Lemma 5.12 in [FKZ$_3$], if these $\mathbb{A}^1$-fibrations are conjugated then the associated extended divisors are isomorphic as reduced curves. Moreover, by Proposition 5.12 in [FKZ$_2$] and its proof, the latter holds if and only if the associated $\mathbb{C}^*$-actions are conjugated. Consequently, there are at least as many conjugacy classes of $\mathbb{A}^1$-fibrations as of $\mathbb{C}^*$-actions. □

Every $\mathbb{A}^1$-fibration on $V$ arising as in Corollary 6.2.5 is compatible with a certain $\mathbb{C}^*$-action. In the next subsection we exhibit further $\mathbb{A}^1$-fibrations that do not appear in this way.

6.3. **Applications to $\mathbb{C}_+$-actions and $\mathbb{A}^1$-fibrations.** Every $\mathbb{A}^1$-fibration $V \to \mathbb{A}^1$ on a Gizatullin surface $V$ arises as the orbit map of a $\mathbb{C}_+$-action $\Lambda$ on $V$, see e.g. Lemma 1.6 in [FlZa$_2$]. Of course, $\Lambda$ is not unique. If, say, $\partial$ is the locally nilpotent derivation associated to $\Lambda$ and $a \in \operatorname{Ker} \partial$ is non-zero then $\partial' = a\partial$ is also locally nilpotent and generates a $\mathbb{C}_+$-action with the same general orbits. It is well known that this is the only ambiguity in associating a $\mathbb{C}_+$-action to a given $\mathbb{A}^1$-fibration, see e.g., [KML]. Thus to classify $\mathbb{C}_+$-actions up to conjugation is essentially equivalent to determining all $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$ up to conjugation.

According to Theorem 5.2 in [FKZ$_3$] for a wide class of Gizatullin surfaces there are at most two conjugacy class of $\mathbb{A}^1$-fibrations; see also Section 6.4 below. However, for surfaces arising from a presentation the assumptions of this theorem are almost never satisfied. Thus we concentrate below on surfaces of the latter class and even of a subclass called quasi-special surfaces. In Theorem 6.3.18 we give a complete classification of $\mathbb{A}^1$-fibrations on such surfaces up to conjugation. Our methods can be applied as well to other classes of Gizatullin surfaces.

To start with, let us recall some notation. Let $V$ be a smooth Gizatullin surface and $(\bar{V}, D)$ be a standard completion of $V$. The linear system $|C_0|$ defines a $\mathbb{P}^1$-fibration $\Phi_0 : \bar{V} \to \mathbb{P}^1$ called the *standard fibration.* Its restriction $\varphi_0 = \Phi_0|V : V \to \mathbb{A}^1$ will be called the *associated* $\mathbb{A}^1$*-fibration* on $V$. A key observation is the following result, which summarizes 5.11, 5.12 in [FKZ$_3$] and their proofs.

**Proposition 6.3.1.** *Let* $V$ *be a normal Gizatullin surface equipped with an* $\mathbb{A}^1$*-fibration* $\pi : V \to \mathbb{A}^1$*. Then* $V$ *admits a standard completion* $(\bar{V}, D)$ *such that* $\pi$ *is induced by the standard fibration* $\Phi_0 : \bar{V} \to \mathbb{P}^1$*. Furthermore, if* $V \not\cong \mathbb{A}^1 \times \mathbb{A}^1_*$ *and* $\Phi_0' : \bar{V}' \to \mathbb{P}^1$ *is a second extension of* $\pi$ *to another standard completion* $(\bar{V}', D')$ *of* $V$ *then there exists an isomorphism* $\psi : \bar{V} \setminus C_0 \to \bar{V}' \setminus C_0'$ *with* $\psi|V = \mathrm{id}_V$ *and* $\Phi_0' \circ \psi = \Phi_0$*. In particular, the extended graphs of both completions coincide.*

Thus every $\mathbb{A}^1$-fibration $V \to \mathbb{A}^1$ arises as a standard fibration from a standard completion $(\bar{V}, D)$, which is unique up to a modification at $C_0$.

The pair of linear systems $|C_0|$, $|C_1|$ defines a birational morphism $\Phi = (\Phi_0, \Phi_1) : \bar{V} \to Q = \mathbb{P}^1 \times \mathbb{P}^1$ to the quadric $Q$ called the standard morphism in 3.2.1. In suitable coordinates $(x, y)$ on $Q \setminus (C_0 \cup C_1) \cong \mathbb{A}^2$ we have $\Phi_0 = x \circ \Phi$ on $Q \setminus (C_0 \cup C_1)$. Moreover

$$C_0 \cong \{\infty\} \times \mathbb{P}^1, \quad C_1 \cong \mathbb{P}^1 \times \{\infty\}, \quad \text{and} \quad C_2 \cong \{0\} \times \mathbb{P}^1\,,$$

while the curves $C_3, \ldots, C_n$ are contained in the preimage of the origin.

**6.3.2.** Let as in 4.3.1 $\mathrm{Aut}_y(\mathbb{A}^2)$ stand for the group of all automorphisms of $\mathbb{A}^2$ stabilizing the $y$-axes, and let $\mathrm{Aut}_{y,0}(\mathbb{A}^2) \subseteq \mathrm{Aut}_y(\mathbb{A}^2)$ denote the stabilizer of the origin. Every automorphism $\alpha \in \mathrm{Aut}_{y,0}$ can be written as

$$\alpha.(x, y) = (\lambda_1 x, \lambda_2 y + q(x)) \quad \text{with} \quad \lambda_1, \lambda_2 \in \mathbb{C}^* \text{ and } q \in \mathbb{C}[x],\ q(0) = 0\,,$$

cf. (17). Hence $\mathrm{Aut}_{y,0}(\mathbb{A}^2) = H \rtimes \mathbb{T}$ is a semidirect product of the torus $\mathbb{T} = \mathbb{C}^{*2}$ acting on $\mathbb{A}^2$ by

$$\lambda.(x, y) = (\lambda_1 x, \lambda_2 y), \quad \lambda = (\lambda_1, \lambda_2) \in \mathbb{T}\,, \tag{79}$$

and the abelian group $H$ of all triangular automorphisms

$$h_q : (x, y) \mapsto (x, y + q(x)), \qquad q \in \mathbb{C}[x],\ q(0) = 0\,. \tag{80}$$

Here $\mathbb{T}$ acts on $H$ by conjugation $(\lambda, \mu).h_{q(x)} = h_{\mu^{-1} q(\lambda x)}$.

**6.3.3.** Given a standard completion $(\bar{V}, D)$ of a Gizatullin surface $V$, we consider the restriction $\Psi = \Phi|_{\bar{V} \setminus (C_0 \cup C_1)} : \bar{V} \setminus (C_0 \cup C_1) \to \mathbb{A}^2$ of the standard morphism. Clearly $\Psi$ is the contraction of the feathers and the components $C_3, \ldots, C_n$ of the zigzag. Every automorphism $\alpha \in \mathrm{Aut}_{y,0}(\mathbb{A}^2)$ extends to $\bar{V} \setminus (C_0 \cup C_1)$ inducing a commutative diagram

$$\begin{array}{ccc} \bar{V} \setminus (C_0 \cup C_1) & \xrightarrow{\tilde{\alpha}} & \bar{V}' \setminus (C_0' \cup C_1') \\ \downarrow \Psi & & \Psi' \downarrow \\ \mathbb{A}^2 & \xrightarrow{\alpha} & \mathbb{A}^2, \end{array} \tag{81}$$

where $\bar{V}'$ is a completion of another Gizatullin surface $V'$ isomorphic to $V$, and $\Psi'$ is defined similarly as $\Psi$. In fact $\tilde{\alpha}$ can be extended to an automorphism $\psi : \bar{V} \setminus C_0 \to \bar{V}' \setminus C_0'$. Note that $\alpha \in \mathrm{Aut}_{y,0}(\mathbb{A}^2)$ in (81) is compatible with the standard fibration $\Phi_0$ on $V$ as it preserves the $x$-coordinate up to a multiple. Proposition 6.3.1 implies the following result.

**Corollary 6.3.4.** *Every isomorphism $\psi : \bar{V} \setminus C_0 \to \bar{V}' \setminus C_0'$ as in Proposition 6.3.1 is induced by an automorphism $\alpha \in \mathrm{Aut}_{y,0}(\mathbb{A}^2)$.*

In terms of presentations this leads to the following proposition.

**Proposition 6.3.5.** *Let $V$ be a smooth Gizatullin surface, and let*

$$X_n = X(M_2, c_3, \ldots, M_i, \ldots, c_n, M_n) \ \text{ and } \ X_n' = X(M_2', c_3', \ldots, M_i', \ldots, c_n', M_n')$$

*be two presentations of $V$. Then the associated $\mathbb{A}^1$-fibrations $\Phi_0|V, \Phi_0'|V : V \to \mathbb{A}^1$ are conjugated if and only if there is an automorphism $\alpha \in \mathrm{Aut}_{y,0}(\mathbb{A}^2)$ with*

$$X_n' = \alpha_*(X_n).$$

The Motion Lemma 5.6.1 was stated for presentations of special Gizatullin surfaces. However, part (a) remains true more generally for quasi-special surfaces which we introduce below.

**Definition 6.3.6.** A presentation $X = X(M_2, c_3, \ldots, M_n)$ as in 4.1.1 will be called *quasi-special of type* $(n, k)$ (or simply quasi-special) if $0 \in M_2$ and $c_3 = \ldots = c_{k+1} = 0$ but $c_{k+2} \neq 0$ and $c_{i+1} \notin M_i$ for $i = k+1, \ldots, n-1$. As usual, here the data $(M_i, c_{i+1})$ are considered as collections of complex numbers expressed in the coordinates introduced in Section 5.1. It will be convenient to complete these data by introducing also the point $c_{n+1}$ as the center of mass of $M_n$.

Thus all components $C_{i+1}$, $i \geq 2$ of the zigzag $D$ are of +-type i.e., $C_{i+1}$ is created by blowing up a point $c_{i+1} \in C_i \setminus C_{i-1}$. Furthermore, for $k \geq 2$ the extended divisor $D_{\text{ext}}$ of $(\bar{V}, D)$ has exactly one feather, denoted $F_2$, with $F_2^2 \leq -2$; the mother component of this feather is $C_2$ and the neighbor in the zigzag is $C_{k+1}$. Otherwise the presentation can be arbitrary.

In particular, every $(-1)$-presentation is quasi-special, cf. Definition 4.1.2 and Corollary 4.1.6. By Corollary 6.1.3 every special surface admits quasi-special presentations. However there exist also presentations of special surfaces which are not quasi-special.

**Lemma 6.3.7.** *Given a quasi-special presentation $X = X(M_2, c_3, \ldots, M_n)$, the first motion under the elementary shift $h_{a,m}$ as in (40) occurs on component $C_s$, where as in Section 5 $s = k(m+1) + 2$, i.e., $e_i = 0$ for all $i < s$, while $\deg e_s > 0$.*

*Proof.* The map $a \mapsto h_{a,m}$ yields a $\mathbb{C}_+$-action on $\mathbb{A}^2$. We can lift its infinitesimal generator $\partial$ to the surface $X$ as a meromorphic vector field. The curves on which $h_{a,t}$ is constant are characterized by the fact that $\partial$ is regular and identically zero on them. Moreover the curve of first motion $C_l$ is characterized by the property that $\partial$ (tangent to $C_l$) is regular and nonzero in the general points of $C_l$. However this property is independent of blowdowns of $(-1)$-feathers. Thus it is enough to find the curve of first motion in the case where $X$ has no $(-1)$-feathers, which is just the Danilov-Gizatullin case. Applying Proposition 5.6.1(a), the result follows. □

We let below $m_0(n,k) = \lfloor \frac{n-2}{k} \rfloor$.

**Corollary 6.3.8.** *With $X = X_n$ and $h = h_{a,m}$ as in Lemma 6.3.7, if $m \geq m_0(n,k)$ then $X_n = h_*(X_n)$ i.e., $h$ generates an automorphism of $X_n \setminus C_0$.*

*Proof.* By the Motion Lemma 6.3.7 the data $M_i, c_{i+1}$ remain unchanged for $i \leq k(m+1) + 2$, in particular for $i \leq n$ so that $X_n = h_*(X_n)$. □

Given $m \in \mathbb{N}$, we let $H_m$ denote the subgroup of $H$ generated by all elements $h_q$ as in (80) with $\deg q(x) \leq m$.

**Definition 6.3.9.** A presentation $X = X_n$ of type $(n, k)$ will be called *semi-canonical* if it is quasi-special with $c_{k+2} = 1$ and $c_{j+1} = 0$ for all $j = k+2,\ 2k+2, \ldots, m_0k+2$, where $m_0 = m_0(n,k)$. [23]

**Lemma 6.3.10.** *Every quasi-special presentation $X = X_n$ of type $(n, k)$ can be transformed into a semi-canonical one by applying a suitable automorphism $h_0 \circ \lambda \in \mathrm{Aut}_{y,0}(\mathbb{A}^2)$, where $\lambda \in \mathbb{T}$ and $h_0 \in H_{m_0}$. Furthermore, such an element $h_0 \in H_{m_0}$ is unique.*

*Proof.* The torus $\mathbb{T}$ acts non-trivially on $C_{k+1} \setminus C_k \cong \mathbb{A}^1$ with the fixed point 0. Hence a suitable $\lambda \in \mathbb{T}$ sends $c_{k+2} \neq 0$ to the point 1.

[23]In particular, for $m_0 = \frac{n-2}{k}$ the center of mass $c_{n+1}$ of $M_n$ should be also 0.

Consider further an elementary shift $h_q$ as in (80) with $q(x)=ax^{m+1}$, where $m\le m_0$. By the generalized Motion Lemma 6.3.7 it does not change the data $(M_i,c_{i+1})$ for $i\le s-1$, where $s=k(m+1)+2$, and induces a nontrivial translation on $C_s\backslash C_{s-1}\cong\mathbb{A}^1$. So $h_q$ sends $c_{s+1}$ to 0 for a suitable value of $a\in\mathbb{C}_+$. Applying such actions repeatedly for $m=0,1,\dots,m_0$ we obtain the desired semi-canonical presentation. The uniqueness part is easy and left to the reader. □

**6.3.11.** Given an arbitrary presentation $X_n=X(M_2,c_3,\dots,M_n)$ we can adopt the construction of canonical coordinate systems on $X_n$ in 5.3.2 as follows. Starting with the affine coordinates $(x_2,y_2)=(x,y)$ on the quadric $Q$, at the first step we blow up $M_2\cup\{c_3\}$ getting a coordinate chart $(x_3,y_3)$, at the second one we blow up $M_3\cup\{c_4\}$ getting $(x_4,y_4)$, and so forth. [24]

Under this procedure the coordinates are given recursively by the formulas in 5.3.2(4). So letting $M_i=\{d_{i1},\dots d_{is_i}\}$ for $i=2,\dots,n$, by virtue of (38) we have

$$(82)\qquad (x_{i+1},y_{i+1})=\left(\frac{x_i}{P},\frac{y_i-c_{i+1}}{x_i}P\right),\quad\text{where}\quad P=\prod_{j=1}^{s_i}(y_i-d_{ij})\in\mathbb{C}[y_i]\,.$$

Let us investigate the torus action in terms of these coordinates. In the following, for $m=(m_1,m_2)\in\mathbb{Z}^2$ and $\lambda=(\lambda_1,\lambda_2)\in\mathbb{T}$ the power $\lambda^m$ stands for $\lambda_1^{m_1}\lambda_2^{m_2}$.

**Lemma 6.3.12.** *(a) $x_i$ and $y_i$ ($i=2,\dots,n+1$) are quasi-invariant functions under the induced $\mathbb{T}$-action, i.e. there are vectors $a_i$, $b_i\in\mathbb{Z}^2$ with*

$$(83)\qquad \lambda.x_i=\lambda^{a_i}x_i\quad\textit{and}\quad\lambda.y_i=\lambda^{b_i}y_i\quad\textit{for}\quad\lambda\in\mathbb{T}$$

*In particular, this action leaves the point on $C_i$ with $y_i$-coordinate zero unchanged.*

*(b) If we form the matrix $A_i=(a_i,b_i)$ with column vectors $a_i,b_i$ then* $\det A_i=1$ *for $2\le i\le n+1$.*

*(c) $\mathbb{T}$ acts on the set of all quasi-special presentations.*

*Proof.* (a) is certainly true for $i=2$ with $a_2$, $b_2$ being the standard basis of $\mathbb{Z}^2$. Assume that for $i\ge 2$ this action can be expressed in $(x_i,y_i)$-coordinates as in (83). In particular, with the notations as in 6.3.11, $\lambda.d_{ij}=\lambda^{b_i}d_{ij}$ and $\lambda.c_{i+1}=\lambda^{b_i}c_{i+1}$. Hence $\lambda.P=\lambda^{s_ib_i}P$ and so

$$\lambda.x_{i+1}=\frac{\lambda.x_i}{\lambda.P}=\lambda^{a_i-s_ib_i}\frac{x_i}{P}=\lambda^{a_i-s_ib_i}x_{i+1}$$

and similarly

$$\lambda.y_{i+1}=\frac{\lambda.(y_i-c_{i+1})}{\lambda.x_i}\lambda.P=\lambda^{b_i-a_i+s_ib_i}\frac{y_i-c_{i+1}}{P}=\lambda^{b_i-a_i+s_ib_i}y_{i+1}\,.$$

Thus

$$(84)\quad \lambda.(x_{i+1},y_{i+1})=(\lambda^{a_{i+1}}x_{i+1},\lambda^{b_{i+1}}y_{i+1})\text{ with }a_{i+1}=a_i-s_ib_i\,,\,b_{i+1}=b_i(1+s_i)-a_i\,,$$

proving (a). To deduce (b) we note that the recursion (84) can be expressed as

$$A_{i+1}=B_iA_i=B_i\cdot B_{i-1}\cdots B_2,\quad\text{where}\quad B_i:=\begin{pmatrix}1&-s_i\\-1&1+s_i\end{pmatrix}\quad 2\le i\le n\,.$$

[24]In the case of special surfaces the enumeration is slightly different from 5.3.2 since we perform here the first two blowups in one step.

Since $A_2$ is the identity matrix and $\det B_i = 1$ we obtain by induction that also $\det A_i = 1$ for $i = 2, \ldots, n+1$. Finally (c) was already observed in 4.3.1. □

**6.3.13.** Given a semi-canonical presentation of type $(n, k)$ we let $\mathbb{T}' \subseteq \mathbb{T}$ denote the stabilizer subgroup of the point $c_{k+2} = 1$. According to (83)

$$\mathbb{T}' = \{\lambda \in \mathbb{T} \mid \lambda^{b_{k+1}} = 1\}\,.$$

Hence $\mathbb{T}' \cong \mathbb{C}^*$; indeed, by Lemma 6.3.12(c) the columns of the matrix $A_i$ with $\det A_i = 1$ are unimodular i.e., primitive lattice vectors. Due to Lemma 6.3.12(c) the 1-torus $\mathbb{T}'$ is independent of the choice of a semi-canonical presentation as soon as the sequence $(s_i = |M_i|)_{i=2,\ldots,n}$ is fixed. Furthermore, it acts on the set of all such presentations.

It also acts non-trivially on every component $C_i$ of the zigzag except for the component $C_{k+1}$. To show this, let us first grow in the blowup process the zigzag $D$ together with the feather $F_2$. At this point our $\mathbb{T}' = \mathbb{C}^*$-action lifts to the resulting surface, say $X'$ with $C_t$ as the unique parabolic component. Since our surface $X = X(M_2, c_3, \ldots, M_n)$ is obtained from $X'$ by blowing up the remaining $(-1)$-feathers, $\mathbb{T}'$ will act also non-trivially on $C_i$, $i \neq k+2$, when considered as curves in $X$.

Combining now Lemmas 6.3.10, 6.3.12 and Proposition 6.3.5 leads to the following corollary.

**Corollary 6.3.14.** *Given two semi-canonical presentations $X_n$ and $X'_n$ of the same Gizatullin surface $V$, the associated $\mathbb{A}^1$-fibrations $\varphi_0$, $\varphi'_0 : V \to \mathbb{A}^1$ are conjugated if and only if $X'_n = \lambda_*(X_n)$ for some $\lambda \in \mathbb{T}'$.*

We can normalize the presentation further using the action of $\mathbb{T}'$.

**Definition 6.3.15.** A semi-canonical presentation $X_n$ (see Definition 6.3.9) will be called $c_{l+1}$-*canonical*, if $c_{i+1} = 0$ for $k+2 \le i \le l-1$ and $c_{l+1} = 1$. Such a $c_{l+1}$-canonical presentation can only exist for $k+2 \le l \le n$ and $l \neq k+2,\ 2k+2, \ldots, m_0k+2$.

It can happen that $c_{i+1} = 0$ for every $i \neq k+1$ so that the presentation is not $c_{l+1}$-canonical whatever $l$ is. In this case we let $l$ $(2 \le l \le n)$ be the minimal index with $l \neq k+1$ such that $M_l^* := M_l \backslash \{0\} \neq \emptyset$. [25] The presentation $X_n$ will be called $M_l$-*canonical* if

$$m_l := \prod_{m \in M_l^*} m = 1\,.$$

In the remaining case, where

$$c_{i+1} = 0 \text{ for all } i = k+2, \ldots, n \quad \text{and} \quad M_l^* = \emptyset \quad \text{for all } l \neq k+1 \text{ with } 2 \le l \le n,$$

$X_n$ will be called $*$-*canonical.*

A presentation will be called for short *canonical* if it is $a$-canonical for some $a \in \{c_{l+1}, M_l, *\}$.

The next lemma is immediate from the fact that the 1-torus $\mathbb{T}'$ acts in a nontrivial way on each component $C_i$, $i \neq k+1$, with the only fixed points 0 and $\infty$.

**Lemma 6.3.16.** *Every semi-canonical presentation can be transformed into a canonical one by an element $\lambda \in \mathbb{T}'$.*

We also need below the following simple lemma.

[25] We note that $M_l^* = M_l$ if $3 \le l \le n-1$ and $l \neq k+1$.

**Lemma 6.3.17.** *Let $X_n$ be a canonical presentation of type $(n,k)$. If it is $c_{l+1}$-canonical or $M_l$-canonical for some $l \le n$, then the subgroup*

$$G_{kl} = \{\lambda \in \mathbb{T} \mid \lambda^{b_{k+1}} = \lambda^{b_l} = 1\} \subseteq \mathbb{T}'$$

*is a finite cyclic group of order $|\det(b_{k+1}, b_l)| \neq 0$.*

*Proof.* Since $G_{kl}$ is contained in $\mathbb{T}' \cong \mathbb{C}^*$ it is either cyclic or equal to $\mathbb{T}'$. As observed in 6.3.13, $\mathbb{T}'$ acts non-trivially on $C_l$ for $l \neq k+1$. As by (83) $G_{kl}$ is the subgroup of all elements of $\mathbb{T}'$ acting trivially on $C_l$, it is finite. □

We now come to our main classification results. By Proposition 6.3.1, if the standard $\mathbb{A}^1$-fibrations associated to two different presentations of the same Gizatullin surface $V$ are conjugated, then the corresponding extended divisors are isomorphic. Therefore, if these presentations are quasi-special then they are of the same type $(n,k)$.

**Theorem 6.3.18.** *Let $X_n$ be an $a$-canonical and $X'_n$ an $a'$-canonical presentation of the same type $(n,k)$ of a smooth Gizatullin surface $V$, where $a \in \{c_{l+1}, M_l, *\}$ and $a' \in \{c_{l'+1}, M_{l'}, *\}$ are as in Definition 6.3.15. Let $\varphi_0$, $\varphi'_0 : V \to \mathbb{A}^1$ be the associated $\mathbb{A}^1$-fibrations.*

*(a) If $a \neq a'$ then $\varphi_0$ and $\varphi'_0$ are not conjugated.*

*(b) If $a = a' = c_{l+1}$ or $a = a' = M_l$ then $\varphi_0$ and $\varphi'_0$ are conjugated if and only if $X'_n = \lambda_*(X_n)$ for some $\lambda \in G_{kl}$.*

*(c) If $X_n$ and $X'_n$ are both $*$-canonical, then $\varphi_0$ and $\varphi'_0$ are conjugated if and only if $X_n = X'_n$. Furthermore $\mathbb{T}'$ leaves $X_n$ invariant and yields a $\mathbb{C}^*$-action on $V$. Conversely, if a canonical presentation $X_n$ admits an effective $\mathbb{C}^*$-action then it is $*$-canonical.*

*Proof.* Assume that $\varphi_0$ and $\varphi'_0$ are conjugate. Then by Corollary 6.3.14 there exists $\lambda \in \mathbb{T}'$ transforming $X_n$ into $X'_n$. To show (a) let us first suppose that $X_n$ is $c_{l+1}$-canonical and $X'_n$ is $c_{l'+1}$-canonical with $l < l'$. Then $c_{l+1} = 1$, which implies by (83) that $c'_{l+1} = \lambda^{b_{l+1}} \neq 0$. The latter contradicts the assumption that $X'_n$ is $c_{l'+1}$-canonical, see Definition 6.3.15. A similar argument yields the other cases in (a).

In (b), assuming again that $\varphi_0$ and $\varphi'_0$ are conjugated, in the case of $c_{l+1}$-canonical presentations we obtain as before that $X'_n = \lambda_*(X_n)$ for some $\lambda \in \mathbb{T}'$. However, since $c_{l+1} = c'_{l+1} = 1$, by (83) we have $\lambda^{b_{l+1}} = 1$. Hence $\lambda \in G_{kl}$. The proof in the case of $M_l$-canonical presentations is similar. The converse can be easily deduced along the same lines.

Finally, if both presentations are $*$-canonical then $\mathbb{T}'$ leaves all points of $M_i$ and $c_{i+1}$ fixed. Hence it induces a $\mathbb{C}^*$-action on $X_n$ and on $V$. Conversely, if $X_n$ is canonical and admits an effective $\mathbb{C}^*$-action then this action (which has just one parabolic component $C_{k+1}$) must stabilize $M_i$ and $c_{i+1}$ for all $i \neq k+1$. This can occur only if $X_n$ is $*$-canonical. □

**Corollary 6.3.19.** *Let $\mathcal{X}(V)$ denote the set of all canonical presentations of a smooth Gizatullin surface $V$ and $\mathcal{Y}(V)$ denote the set of all conjugacy classes of $\mathbb{A}^1$-fibrations $V \to \mathbb{A}^1$. Then the natural correspondence $\mathcal{X}(V) \to \mathcal{Y}(V)$, which sends a canonical presentation into the conjugacy class of its standard fibration, is surjective and finite-to-one.*

In the case of Danilov-Gizatullin surfaces this leads to the following complete classification of $\mathbb{A}^1$-fibrations; cf. 1.0.2 in the Introduction.

**Corollary 6.3.20.** *The $\mathbb{A}^1$-fibrations on the Danilov-Gizatullin surface $V(n)$ are completely classified by the following canonical presentations.*

*(a) For every type $(n,k)$, $k=1,\ldots,n-1$, there is exactly one $*$-canonical presentation of $V(n)$.*
*(b) There is no $M_l$-canonical presentation on $V(n)$.*
*(c) For every type $(n,k)$ with $1\le k\le n-1$ the surface $V(n)$ has a $c_{l+1}$-canonical presentation if and only if*

$$k+2\le l\le n \quad \text{and} \quad l\neq k+2, 2k+2, \ldots, m_0k+2\,, \tag{85}$$

*where as before $m_0=\lfloor\frac{n-2}{k}\rfloor$. Furthermore, if $l\le n$ and $ak+2<l<(a+1)k+2$ with $1\le a\le m_0$, then these $c_{l+1}$-canonical presentations form a family of dimension $r(n)=(n-l)-(m_0-a)$.*

*Proof.* Every surface admitting a presentation with $s_2=s_n=1$ and $s_i=0$ for $i\neq 2,n$, is a Danilov-Gizatullin surface (see 6.2.3). Clearly one can choose such a presentation of any given type $(n,k)$. In view of Theorem 6.3.18 this proves (a).

(b) and also the first part of (c) are immediate from Definition 6.3.15. The second part of (c) is a consequence of the fact, that for a $c_{l+1}$-canonical presentation the positions of the remaining points $c_{i+1}$, $l<i\le n$ and $i\neq 2k+2,\ldots,m_0k+2$, can be freely chosen and so give rise to a family of the claimed dimension. □

**Example 6.3.21.** In particular, choosing $k=2$ and $l=5$ in Corollary 6.3.20(c) any Danilov-Gizatullin surface $V(n)$ of index $n\ge 7$ carries continuous families of pairwise non-conjugated $\mathbb{A}^1$-fibrations with the number of parameters being an increasing function of $n$.

If $n<6$ then according to Corollary 6.3.20 all canonical presentations of $V(n)$ are $*$-canonical and so related to a $\mathbb{C}^*$-action. If $n=6$ then besides the five $*$-canonical presentations there are only two further canonical presentations, one for $k=2$, $l=5$ and another one for $k=3$, $l=6$. Thus for any $n\ge 6$ there exist $\mathbb{A}^1$-fibrations on $V(n)$ not related to $\mathbb{C}^*$-actions.

With a similar reasoning we obtain the following result for special surfaces.

**Corollary 6.3.22.** *Let $V$ be a special surface of type I or II. Then there are families of pairwise non-conjugated $\mathbb{A}^1$-fibrations $V\to\mathbb{A}^1$ depending on $r(n)\ge 1$ parameters with* $\lim_{n\to\infty} r(n)=\infty$.

*Proof.* By Corollary 6.2.5 $V$ admits at least a one-parameter family of pairwise non conjugated $\mathbb{A}^1$-fibrations. To construct other such families we proceed similarly as in Corollary 6.3.20 for Danilov-Gizatullin surfaces. We assume first that $V$ is special of type II so that there is a presentation $X=X(M_2,c_3,\ldots,M_t,\ldots c_n,M_n)$ with $|M_2|=|M_n|=1$, $s_t=|M_t|\ge 2$ and $M_i=\emptyset$ otherwise, where $2<t<n$. According to Corollary 6.1.4 the isomorphism class of the affine part $V$ is uniquely determined by its configuration invariant $[M_t]\in\mathfrak{M}^+_{s_t}$.

This implies in particular that for any $k=1,\ldots,n-1$ there are quasi-special presentations of type $(n,k)$ of $V$. Furthermore, different choices of the points $c_{i+1}$ lead to the same surface as long as we keep the configuration $[M_t]\in\mathfrak{M}^+_{s_t}$ fixed. Thus one can construct families of $\mathbb{A}^1$-fibrations depending on the same number $r(n)$ of parameters as in Corollary 6.3.20(c).

In the case of special surfaces of type I the reasoning is similar; we leave the details to the reader. □

Let us give another application to Danielewski-Fieseler surfaces; see [$\mathrm{Du}_1$]. By this we mean a smooth affine surface $V$ equipped with an $\mathbb{A}^1$-fibration $\pi : V \to \mathbb{A}^1$ such that all scheme-theoretic fibers $\pi^{-1}(a)$, $a \neq 0$, are affine lines while $\pi^{-1}(0)$ is a disjoint union of affine lines.[26] We provide below an alternative proof of the following result of A. Dubouloz, see Corollary 4.13 in [$\mathrm{Du}_2$].

**Proposition 6.3.23.** *For a Danielewski-Fieseler surface $V$ the following are equivalent.*

*(a) Its $\mathbb{A}^1$-fibration is uniquely determined up to conjugation;*
*(b) either $V$ is not Gizatullin or $V$ is isomorphic to a hypersurface in $\mathbb{A}^3$ given by an equation $\{xy = P(z)\}$, where $P \in \mathbb{C}[z]$ has only simple roots.*

*Proof.* By the result of Gizatullin [$\mathrm{Gi}_2$] a non-Gizatullin surface $V$ carries at most one $\mathbb{A}^1$-fibration $V \to \mathbb{A}^1$ up to an automorphism of the base. Moreover, by Proposition 6.4.1 below (see also [Dai, ML]) the surface $V = \{xy = P(z)\}$ in $\mathbb{A}^3$ has only one $\mathbb{A}^1$-fibration $V \to \mathbb{A}^1$ up to conjugation. This proves (b)⇒(a).

To show the converse we may assume that $V$ is Gizatullin and not the surface $\{xy = P(z)\}$ in $\mathbb{A}^3$. By Proposition 6.3.1 there is a standard completion $(\bar{V}, D)$ of $V$ such that $\pi$ extends to the standard fibration on $\bar{V}$.

The surface $\{xy = P(z)\}$ in $\mathbb{A}^3$ has DPD presentation with $D_+ = 0$ and $D_- = -\mathrm{div}(P)$, see e.g., [$\mathrm{FlZa}_1$]. Hence inspecting Proposition 3.5.1 the boundary zigzag $D$ is of length $n + 1 = 3$, and every surface with a boundary zigzag of length 3 arises in this way. Hence under our assumptions $n > 2$.

The feathers of the extended divisor $D_{\mathrm{ext}}$ are just the components of the fiber $\pi^{-1}(0) \subseteq V$. Since they are all reduced in the fiber, to create the surface $\bar{V}$ from the quadric only outer blowups can occur. Thus $\bar{V}$ arises form a presentation of $(-1)$-type say, $X_n = X(M_2, c_3, \ldots, M_n)$. As before we suppose that the data $M_i$ and $c_i$ are represented by complex numbers in the standard coordinates of Section 5.1. Applying an elementary shift we may as well assume that $0 \in M_n$. Inspecting Proposition 4.2.2, the reversion $X_n^\vee$ is then not any longer of $(-1)$-type. Thus the extended divisors of $X_n$ and $X_n^\vee$ cannot be isomorphic. In particular, by Proposition 6.3.1 its associated $\mathbb{A}^1$-fibration cannot be conjugated to $\pi$. Now the proposition follows. □

**Remark 6.3.24.** 1. It is interesting to compare our results with those in the recent paper [GMMR] devoted to the study of affine lines on smooth Gizatullin surfaces. Due to Theorems 2.3 and 2.4 in [GMMR], given $\rho \geq 1$ there exists such a surface $V$ with Picard rank $\rho(V) = \rho$ (i.e., there are $\rho + 1$ feathers), containing an affine line $\mathbb{A}^1 \hookrightarrow V$ which is not a component of a fiber of any $\mathbb{A}^1$-fibration $V \to \mathbb{A}^1$. Consequently, there is no analogue of the Abhyankar-Moh-Suzuki Theorem for such surfaces, and the classification of affine lines on them cannot be deduced from that of $\mathbb{A}^1$-fibrations $V \to \mathbb{A}^1$.

2. Let $V$ be a Gizatullin surface admitting continuous families of $\mathbb{A}^1$-fibrations $V \to \mathbb{A}^1$. Let us deduce the existence on $V$ of a continuous family of affine lines

[26] In particular, all fibers are reduced.

$\mathbb{A}^1 \hookrightarrow V$ such that for any two of them, there is no automorphism of $V$ sending one into another.

Indeed, let $\varphi : V \to \mathbb{A}^1$ and $\varphi' : V \to \mathbb{A}^1$ be two non-conjugated $\mathbb{A}^1$-fibrations and let $\ell$ and $\ell'$ be smooth fibers of $\varphi$ and $\varphi'$, respectively. Let us show that these two affine lines on $V$ are not conjugated in the automorphism group. Assuming the contrary, after an automorphism we may suppose that $\ell = \ell'$ is a smooth fiber of two different $\mathbb{A}^1$-fibrations. Let $(\bar{V}, D)$ be a completion of $V$ such that $\varphi$ is the restriction of the associated standard fibration $\Phi_0$. If $\ell'' = \varphi'^{-1}(a)$ is a general fiber of $\varphi'$ then $\ell''$ cannot be a fiber component of $\varphi$. Hence its closure $\bar{\ell}''$ in $\bar{V}$ is horizontal with respect to $\Phi_0$ and so intersects every fiber of $\Phi_0$. In particular, $\bar{\ell}''$ meets both $C_0 \subseteq D$ and $\bar{\ell}'$. Since $\ell'$ and $\ell''$ are disjoint, this shows that $\bar{\ell}''$ meets $D$ in two different points, which is impossible.

6.4. **Uniqueness of $\mathbb{A}^1$-fibrations on singular surfaces.** Here we consider more generally normal Gizatullin $\mathbb{C}^*$-surfaces, which are not necessary smooth. Using the technique developed in the previous sections we are able to strengthen our previous uniqueness result for $\mathbb{A}^1$-fibrations on such surfaces, see Corollary 5.13 in [FKZ$_3$].

We recall the following notation from [FKZ$_3$]. Let us consider a DPD presentation $V = \operatorname{Spec} A_0[D_+, D_-]$ of a Gizatullin $\mathbb{C}^*$-surface $V$, where $A_0 = \mathbb{C}[u]$. We let $(\bar{V}, D)$ be a $\mathbb{C}^*$-equivariant standard completion of $V$. Such a completion is unique up to reversion of the boundary zigzag $D = C_0 \cup \ldots \cup C_n$, cf. (7). The linear system $|C_0|$ defines an $\mathbb{A}^1$-fibrations $\Phi_0 : V \to \mathbb{A}^1$, and similarly the linear system $|C_0^\vee|$ on the reversed equivariant completion $(\bar{V}^\vee, D^\vee)$ provides a second $\mathbb{A}^1$-fibration $\Phi_0^\vee : V \to \mathbb{A}^1$.

In [FKZ$_3$] we introduced the following two conditions:

($\alpha_*$) $\operatorname{supp}\{D_+\} \cup \operatorname{supp}\{D_-\}$ *is empty or consists of one point $p$, where either $D_+(p) + D_-(p) \le -1$ or both fractional parts $\{D_+(p)\}$, $\{D_-(p)\}$ are nonzero.*

($\alpha_+$) $\operatorname{supp}\{D_+\} \cup \operatorname{supp}\{D_-\}$ *is empty or consists of one point $p$, where either $D_+(p) + D_-(p) = 0$ or*

$$D_+(p) + D_-(p) \le -\max\left(\frac{1}{m^{+2}}, \frac{1}{m^{-2}}\right),$$

*where $\pm m^\pm$ denote the minimal positive integers such that $m^\pm D_\pm(p) \in \mathbb{Z}$.*

Corollary 5.13 in [FKZ$_3$] asserts the uniqueness of the $\mathbb{A}^1$-fibration $V \to \mathbb{A}^1$, up to conjugation and reversion, under condition ($\alpha_+$). In the next proposition we show that the latter uniqueness holds as well under the weaker condition ($\alpha_*$).

**Proposition 6.4.1.** *Let $V$ be a Gizatullin $\mathbb{C}^*$-surface as above. If condition ($\alpha_*$) is fulfilled then the following hold.*

*(a) Every $\mathbb{A}^1$-fibration $V \to \mathbb{A}^1$ is conjugated either to $\Phi_0$ or to $\Phi_0^\vee$.*
*(b) Suppose that $V$ is non-toric. Then $\Phi_0$ and $\Phi_0^\vee$ are conjugated if and only if $\{D_+(p)\} = \{D_-(p)\}$.*

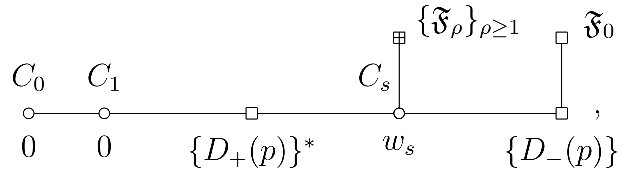
**6.4.2.** The proof of (b) is the same as in [FKZ$_3$, Corollary 5.13]. To deduce (a) we start with some preliminary observations. Comparing ($\alpha_*$) and ($\alpha_+$) it is enough to suppose that $\operatorname{supp}\{D_+\} = \operatorname{supp}\{D_-\} = \{p\}$ and

$$0 < -(D_+ + D_-)(p) < \max\left(\frac{1}{m^{+2}}, \frac{1}{m^{-2}}\right) < 1\,. \tag{86}$$

Indeed, assuming (86), condition $(\alpha_*)$ is fulfilled, but $(\alpha_+)$ fails. We precede the proof with some necessary preliminaries.

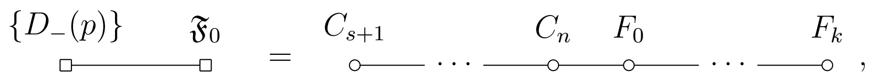
Let $D_{\text{ext}}$ and $D^\vee_{\text{ext}}$ denote the extended divisors of $(\bar{V}, D)$ and $(\bar{V}^\vee, D^\vee)$, respectively. According to Corollary 3.26 in [FKZ$_3$], in our case at least one of these divisors is rigid[27]. We may assume that $D_{\text{ext}}$ is. By Proposition 3.10 in [FKZ$_3$], its dual graph is

(87) $D_{\text{ext}}$ :




where $\{\mathfrak{F}_\rho\}_{\rho\geq 1}$ is a collection of $A_{k_\rho}$-feathers. Since $D_{\text{ext}}$ is rigid, by Proposition 2.15 in [FKZ$_3$] the bridge curve $\tilde{O}_p^-$ of the feather $\mathfrak{F}_0$ is a $(-1)$-curve. We recall (see Definition 3.20 in *loc.cit.*) that the tail $L = L_{s+1}$ of $D_{\text{ext}}$ is the linear chain

(88)

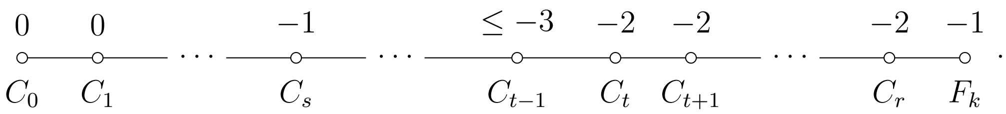



where the feather $\mathfrak{F}_0$ is formed by the curves $F_0, \ldots, F_k$ with bridge curve $F_0 = \tilde{O}_p^-$, cf. Remark 3.3.9. By our assumption, $F_0^2 = -1$. According to Lemmas 3.21 and 3.22(c) in *loc.cit.* we have

1. $L$ is not contractible;
2. some subtail $L_t$ of $L$, $s+2 \leq t \leq n$ can be contracted to a smooth point, where $L_t$ is the chain starting with $C_t$ to the right in (88).

Let $C_r$ be the mother component of $F_k$. By (2) $r \geq t$. Furthermore, since the subtail $L_t$ is contractible, after contracting the divisor $L_r \setminus F_k$ and all $(-1)$-feathers $\mathfrak{F}_\rho$ $(\rho \geq 1)$ we obtain the linear chain

(89)



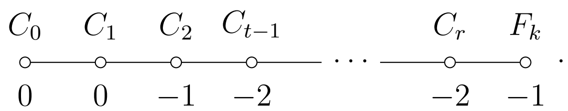

Here $C_{t-1}^2 \leq -3$ since by our minimality assumption the subtail $L_{t-1}$ is not contractible. Moreover, in this chain necessarily $C_s^2 = -1$, since otherwise it cannot be contracted to $[[0,0,0]]$.

The mother component of $C_{t-1}$ is $C_2$, since otherwise, blowing down successively $(-1)$-vertices in (89), we arrive at a chain $[[0,0,w_2,\ldots,w_l,-1]]$ with $l \geq 2$, $w_i \leq -2\ \forall i$, which cannot be contracted to $[[0,0,0]]$.

Therefore (89) can be obtained starting from the chain $C_0 \cup C_1 \cup C_2$ on the quadric $Q$ and performing a sequence of outer blowups, which create the component $C_{t-1}, \ldots, C_r$, $F_k$, followed by a sequence of inner blowups to create the components $C_3, \ldots, C_{t-2}$. Thus we can obtain the completion $\bar{V}$ from $Q$ via the following 3 steps.

**Step 1.** Performing a sequence of outer blowups, we create first the components $C_{t-1}, C_t, \ldots, C_r, F_k$ getting a surface $\bar{V}_{(1)}$ together with a linear chain

(90)




[27] See the terminology in [FKZ$_3$].

Applying on $Q$ a sequence of suitable shifts $h_{a,m}$ as in Lemma 4.3.3 we can subsequently move the centers of outer blowups in $\bar{V}_{(1)} \to Q$ into the fixed points of the torus action on $\bar{V}_{(1)}$ induced, step by step, by the standard $\mathbb{T}$-action on $Q$.

**Step 2.** Starting from $\bar{V}_{(1)}$ we create the components $C_3, \ldots, C_t$ and also $C_{r+1}, \ldots, C_n$, $F_1, \ldots, F_{k-1}$ by a sequence of inner blowups, which results in a surface $\bar{V}_{(2)}$ and a $\mathbb{T}$-equivariant morphisms $\bar{V}_{(2)} \to \bar{V}_{(1)} \to Q$. We note that the $\mathbb{T}$-action on $\bar{V}_{(2)}$ restricts to a non-trivial $\mathbb{T}$-action on the component $C_s$.

**Step 3.** Starting from $\bar{V}_{(2)}$ we create the $A_{k_\rho}$-feathers $\mathfrak{F}_\rho$, $\rho \geq 1$, attached to the component $C_s$. This requires outer blowups, one at each point where a feather is attached. The remaining blowups are inner, resulting in the completion $\bar{V}$.

*Proof of Proposition 6.4.1.* Suppose that we are given an $\mathbb{A}^1$-fibration $\varphi : V \to \mathbb{A}^1$. By Proposition 6.3.1 there exists a standard completion $(\bar{V}', D')$ of $V$ such that $\varphi = \Phi'_0|V$ is defined by the linear system $|C'_0|$ on $\bar{V}'$. After replacing $(\bar{V}, D)$, if necessary, by the reversed completion we may assume that $(\bar{V}', D')$ is obtained from $(\bar{V}, D)$ via a symmetric reconstruction, see Lemma 2.2.2.

Let us now look at the blowup process as in 3.2.1 which creates $\bar{V}$ and $\bar{V}'$, respectively, starting from the quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$. It suffices to prove the following claim.

*Claim. There is an automorphism* $\alpha \in \operatorname{Aut}(Q \setminus C_0) \cong \operatorname{Aut}(\mathbb{A}^1 \times \mathbb{P}^1)$ *which stabilizes the curves* $C_1 = \mathbb{P}^1 \times \{\infty\}$, $C_2 = \{0\} \times \mathbb{P}^1$, *fixes the point* $(0, 0)$ *and maps the centers of successive blowups in the decomposition of the standard morphism* $\Phi : \bar{V} \to Q$ *into the respective centers of blowups in the decomposition of* $\Phi' : \bar{V}' \to Q$.

Assuming the claim, such an automorphism $\alpha$ preserves the $\mathbb{A}^1$-fibration associated to the first projection $\mathrm{pr}_1 : Q \to \mathbb{P}^1$. Hence it can be lifted to an automorphism of $V$ which conjugates the standard $\mathbb{A}^1$-fibrations $\Phi_0 : V \to \mathbb{A}^1$ and $\varphi = \Phi'_0 : V \to \mathbb{A}^1$, as required.

Thus it remains to prove the claim.

*Proof of the claim.* We decompose the standard morphisms $\Phi, \Phi'$ following Steps 1-3:

$$\Phi : \bar{V} \to \bar{V}_{(2)} \to \bar{V}_{(1)} \to Q \qquad \text{and} \qquad \Phi' : \bar{V}' \to \bar{V}'_{(2)} \to \bar{V}'_{(1)} \to Q\,.$$

On Step 1 the $\mathbb{T}$-equivariant morphisms $\bar{V}_{(1)} \to Q$ and $\bar{V}'_{(1)} \to Q$ coincide, since they have the same centers of blowups at the fixed points of the $\mathbb{T}$-action. Hence $\bar{V}_{(1)} = \bar{V}'_{(1)}$. Then also $\bar{V}_{(2)} = \bar{V}'_{(2)}$, since Step 2 involves only inner blowups. We let further

$$P_1, \ldots, P_k \in C_s, \qquad P'_1, \ldots, P'_k \in C'_s = C_s$$

denote the base points of the $A_{k_\rho}$-feathers collections $\{\mathfrak{F}_{s,\rho}\}_{\rho \geq 1}$ and $\{\mathfrak{F}_{s',\rho'}\}_{\rho \geq 1}$ attached to the components $C_s \subseteq \bar{V}_{(2)}$ and $C'_s \subseteq \bar{V}'_{(2)} = \bar{V}_{(2)}$, respectively. Since $D_{\mathrm{ext}}$ is rigid, by Proposition 5.3 in [FKZ$_3$] the configurations

$$\{P_1, \ldots, P_k\} \qquad \text{and} \qquad \{P'_1, \ldots, P'_k\}$$

of points in $C_s \setminus (C_{s-1} \cup C_{s+1}) \cong \mathbb{C}^*$ must be equivalent under the $\mathbb{C}^*$-action on $C_s \setminus (C_{s-1} \cup C_{s+1})$. Hence, changing suitably the enumeration and using the induced $\mathbb{T}$-action on $C_s$ in $\bar{V}_{(2)}$ we can achieve that $P_i = P'_i$ for all $i = 1, \ldots, k$ (cf. the proof of Proposition 4.4.1). Now the surfaces $\bar{V}$ and $\bar{V}'$ become isomorphic via an isomorphism which conjugates the induced $\mathbb{A}^1$-fibrations on $V$, as required. □

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