ADVANCED CALCULUS - MTH433
LECTURE 4 - FINITE AND INFINITE SETS

1. Cardinal number of a set

The *cardinal number* (or simply cardinal) of a set is a generalization of the concept of the *number of elements* of the set. As long as A is finite according to common sense, $|A|$ is equal to the number of elements of A. When A is infinite (and what does that mean?) the situation is more complicated, since there are many types of infinite cardinals.

- Two sets have the same cardinal if there exists a bijection between them.
Notation: the cardinal of a set A is denoted $|A|$.

- We say that the cardinal of A is less or equal than the cardinal of B and write $|A| \leq |B|$ if either of these conditions are satisfied:
- there exists an injection from A to B
- there exists a surjection from B to A.

- We write $|A| < |B|$ if there exists an injection from A to B, but no bijection.

Theorem 1. *If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.*

Proof. Hard.

2. Finite and infinite sets

Let $\mathbb{N}_n = \{1, \ldots, n\}$ the set of the first n positive integers. Also $|\emptyset| = 0$ and the empty set is considered *finite*.

- A set A is *finite* if either it is empty, or there exists $n \in \mathbb{N}$ and a bijection between A and $\mathbb{N}_n$. The cardinal of A is then exactly $|A| = n$, the number of elements of A.
- A set A is infinite if it is not finite.

Proposition 1. *The set of natural numbers is infinite.*

Proof. Suppose $\mathbb{N}$ was finite. Then there exists $n \in \mathbb{N}$ and there exists a bijection f between $\mathbb{N}$ and $\mathbb{N}_n$. We look at $\{f(1), f(2), \ldots, f(n+1)\}$. By injectivity, these are distinct integers. But at least two must be equal, since there are only n values in $\mathbb{N}_n$. A contradiction, so there is no bijection between them, for any n.

Lemma 1. *A set S is either finite, or there exists an injection from $\mathbb{N}$ into S.*

Proof. It is sufficient to prove that if S is not finite (is infinite), then there exists an injection f from $\mathbb{N}$ into S. We shall define f inductively, using the strong induction principle, over $n \in \mathbb{N}$. Since the set S is not empty, there exists $a \in S$. Define $f(1) = a$ and for the record, denote $a_1 = a$. If $S = \{a_1\}$, then S is finite, which we assumed is not true. Consequently, $S \setminus \{a_1\}$ contains at least one element, denote it $a_2 \in S \setminus \{a_1\}$. Let $f(2) = a_2$. Suppose we constructed $f(k) = a_k$, for $k = 1, 2, \ldots, n$ and we want to construct $f(n+1)$. If $S = \{a_1, a_2, a_3, \ldots, a_n\}$, then it is finite, which would be false. Thus there exists at least one element in $S \setminus \{a_1, a_2, a_3, \ldots, a_n\}$, denoted by a_{n+1}. Put $f(n+1) = a_{n+1}$. We notice that $f(n)$ are all distinct numbers, since at each step they are selected from a set that excluded all previous values. This guarantees that f is injective, which concludes the proof.

Corollary 1. *The cardinal of the natural numbers is* the smallest infinite cardinal.

Proof. We want to show that S infinite implies that $|\mathbb{N}| \leq |S|$. Since $|\mathbb{N}|$ is infinite, we would be done. The Lemma proves exactly that.

Corollary 2. *A set S is infinite if and only if there exists $U \subset S$ with $|U| = |\mathbb{N}|$.*

Proof. We define $U = f(\mathbb{N})$ where f is the bijection from Lemma 1. Then $f : \mathbb{N} \to U$ is bijective.

- Sets in bijection with the natural numbers are said *denumerable*. Sets that are either finite of denumerable are said *countable*.

- The cardinality (or cardinal number) of $\mathbb{N}$ is denoted by $\aleph_0 = |\mathbb{N}|$ (aleph zero).

Sometimes we just say *countable infinite sets* for denumerable. The essence of this definition is that sets A in bijection with $\mathbb{N}$ can be counted, which is the mathematically rigorous way of saying that their elements can be listed in a sequence $a_1 = f(1)$, $a_2 = f(2)$, and so on, where f is the bijection between $\mathbb{N}$ and the set A. In case the set is finite, the sequence is finite as well.

Theorem 2. *The following are equivalent:*
(i) S is infinite;
(ii) there exists an injection that is not surjective form S to S;
(iii) there exists a surjection that is not injective from S to S.

Condition (ii) can be rephrased as: *There exists an injection from S into a proper subset of S.* A subset $S' \subseteq S$ with $S' \neq S$ is said *proper* subset.

Remark. First, notice that if $S = \mathbb{N}$ then (ii) is true, because of the *forward shift* function $f(n) = n + 1$ and (iii) is true because of the *backward shift* function $g(n) = n - 1$ if $n \geq 2$ and $g(1) = 1$. To prove the rest, we have to use the Proposition 1, which was telling us that if S is infinite, then there exists a subset U in bijection with the natural numbers. We can write the elements of U as u_1, u_2, and so on.

Proof. (i) $\Rightarrow$ (iii). We construct $G(s) = s$ if $s \notin U$ and $G(u_n) = u_{n-1}$ for $n \geq 2$ and $G(u_1) = u_1$. This is surjective but not injective since $G(u_2) = G(u_1)$.

(iii) $\Rightarrow$ (ii). Let G be the surjective function from (iii) (it is not necessarily the one described in the previous step). We construct a function F that tries to play the role of inverse of G (which does not properly exist). Since G is surjective, for any $s \in S$ there exists $q \in S$ such that $G(q) = s$. There might be more than one such elements q. Choose only one, for each s. This choice of pre-image of s is denoted by $F(s)$. The function F is injective, since it associates distinct values of q to distinct values of s. To make this clear, we note that $q \in G^{-1}(\{s\})$ and for two distinct s_1 and s_2 we have $G^{-1}(\{s_1\}) \cap G^{-1}(\{s_2\}) = \emptyset$.

On the other hand, F is not surjective. To prove that, we recall that G was not injective. That means that there exists s with at least two q' and q'' having $G(q') = G(q'') = s$. Since there are at least two such pre-images of q, one of them (let it be q'') cannot be equal to $F(s)$. Yet it cannot be $F(s')$ for some other $s' \neq s$, so q'' is not in the range of F, meaning that F is not surjective.

(ii) $\Rightarrow$ (i). It is equivalent to $\neg(i) \Rightarrow \neg(ii)$. In other words, if S is finite with $S = \{s_1, s_2, \ldots, s_n\}$ for some $n \in \mathbb{N}$, then any injective function for S into S is also surjective (onto). If not surjective, then there would exist an injection from a set S with n elements into a set $f(S)$ with $m < n$ elements, which is impossible.

We are done because (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) and we came full circle.

3. DISTINGUISHING BETWEEN INFINITE SETS

As expected, there are finite sets and infinite sets. Corollary 2 and Theorem 2 give various equivalent characterizations of infinite sets. There are many finite sets, from the point of view of cardinality, namely as many as $n \in \mathbb{N}$, since they are the sets with one element, two elements, and so on. How many types of infinities do exist? The answer is infinitely many, due to the following theorem.

Theorem 3. *The set of subsets of a set A, denoted by $\mathcal{P}(A)$, has strictly larger cardinality that A.*

Proof. In the text and in class.

4. Operations with cardinal numbers

- **The sum of two cardinal numbers:** If $A \cap B = \emptyset$ **then** $|A \cup B| = |A| + |B|$**.** Exercise 4 shows that is not possible to define the difference consistently.

- **The product of two cardinal numbers:** $|A| \cdot |B| = |A \times B|$**.** In other words, the cardinal of the cross product of the sets is the product of their cardinal sets. Exercise 10 proves the consistency of this definition.

The set of functions from A to B is denoted by B^A. The set $A^{\mathbb{N}}$ denotes the set of all sequences of elements of A. To understand this notation, think how many functions there are from $\mathbb{N}_n$ to $\mathbb{N}_m$: exactly m^n. This fact is also known as the *counting principle.* A function f is given by values $f(1)$, $f(2)$, ..., $f(n)$ in $\mathbb{N}_m$. We have no restrictions on these choices, so there are m choices for each, giving us m^n. The notation B^A suggests an analogy even when A and B are infinite.

- **The power function for cardinal numbers:** $|B|^{|A|}$ **is the cardinal number of the set of all functions from A to B.**

5. The real numbers versus the natural numbers

- **The cardinality of the real numbers is denoted by $\mathfrak{c} = |\mathbb{R}|$.**

Theorem 4. *The real numbers can be put in bijection with the power set of the natural numbers, or equivalently $\mathfrak{c} = 2^{\aleph_0}$.*

This theorem is very important for understanding the real numbers. A direct approach is to consider real numbers in $(0,1)$ and write them in decimal form

$$x = 0.a_1 a_2 a_3 \ldots$$

where $a_i \in D = \{0, 1, 2, \ldots, 9\}$ (digits). What we have is a correspondence (decimal writing) between real numbers and sequences in D. In other words, $\mathfrak{c} = 10^{\aleph_0}$. With a bit of effort, or accepting that numbers may be written just with two digits 0 and 1 in base two, we have our theorem.

We know that there are no infinite sets *smaller* than the natural numbers. Theorem 4 together with Theorem 3 says that the cardinal of the real numbers is strictly larger than the cardinal of the natural numbers, providing a familiar example of second level of infinite set. Are there infinite cardinals between the two? The *continuum hypothesis* states that there are not. This hypothesis is consistent with the axioms of set theory but cannot be proven within the theory.

Exercise 1. *If $A \cap B = \emptyset$ and both are in bijection with the natural numbers, show that $A \cup B$ is also in bijection with the natural numbers. This says that $\aleph_0 + \aleph_0 = \aleph_0$ yet $\aleph_0 \neq 0$!*

Exercise 2. *Show that* $2\aleph_0 = \aleph_0 + \aleph_0 = \aleph_0$.

Exercise 3. *Show that* $|\mathbb{Z}| = |\mathbb{N}|$.

Exercise 4. *If A and B are both in bijection with the natural numbers, what can one say about $A \setminus B$? Hint: more than one answer, depending on case.*

Exercise 5. *Show that* $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. *Hint: done in the text.*

Exercise 6. *Show that* $|\mathbb{Q}| = |\mathbb{N}|$. *Hint: done in the text.*

Exercise 7. *Show that* $\aleph_0 + \mathfrak{c} = \mathfrak{c}$. *Hint: Find a bijection from the union of the negative integers and $(0, \infty)$ onto $(0, \infty)$.*

Exercise 8. *If* $|A| = |B|$, *then* $|2^A| = |2^B|$.

Exercise 9. *Show that* $|\mathcal{P}(A)| = |2^A|$. *Hint: any subset A' of A has a* characteristic *function* $f_{A'}(x) = 1$ *if* $x \in A'$ *and* $f_{A'}(x) = 0$ *if* $x \notin A'$. *Theorem 3 says that* $|2^A| > |A|$, *with strict inequality (no bijection between them).*

Exercise 10. *If* $|A_1| = |A_2|$ *and* $|B_1| = |B_2|$, *then* $|A_1 \times B_1| = |A_2 \times B_2|$.

Exercise 11. *Show that* $|(0, 1)| = |\mathbb{R}|$.

Exercise 12. *Show directly that there is no bijection between $\mathbb{N}$ and $(0, 1)$, which is the same as* $\mathfrak{c} > \aleph_0$. *This uses the diagonalization argument (in class).*

Exercise 13. *Show that if* $|A_1| \leq |A_2|$ *and* $|B_1| \leq |B_2|$, *then* $|A_1 \times B_1| \leq |A_2 \times B_2|$.

Exercise 14. *Show that if* $|A_1| \leq |A_2|$, *then* $|B|^{|A_1|} \leq |B|^{|A_2|}$.

Exercise 15. *Show that if* $|B_1| \leq |B_2|$, *then* $|A|^{|B_1|} \leq |A|^{|B_2|}$.

Exercise 16. *Show that* $|A||B| = |B||A|$.

Exercise 17. *Show that* $|C|^{|A||B|} = (|C|^{|B|})^{|A|}$. *In other words, what is a bijective function* $\Psi : \{F | F : A \times B \to C\} \to \{f | f : A \to C^B\}$? *Hint: here F is given, so we know $c = F(a, b)$ for all $(a, b) \in A \times B$. For each $a \in A$, define the function $f_a \in C^B$ as follows: The value of $f_a(b)$ will be exactly $F(a, b)$. Then the function Ψ will be:* $\Psi(F)(a) = f_a$ *for all $a \in A$.*

Exercise 18. *Show that* $\aleph_0^{\aleph_0} = \mathfrak{c}$, $\aleph_0^{\mathfrak{c}} = 2^{\mathfrak{c}}$, $\mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}}$.