HYDRODYNAMIC LIMIT FOR THE BAK-SNEPPEN BRANCHING DIFFUSIONS

ILIE GRIGORESCU[1,2], YISHU SONG[1]

Abstract. We prove a hydrodynamic limit for a system of N particles moving in an open domain $D \subseteq \mathbb{R}^d$ according to a diffusion and undergoing branching when one particle reaches the boundary. The particle at the boundary and another random particle are eliminated and replaced with two new particles created instantaneously at a random point with distribution $\gamma(dx)$ in D. The mechanism represents a hybrid between the Fleming-Viot branching and a mean-field version of the Bak-Sneppen fitness model where the absorbing boundary represents the *minimal* configuration, seen as biologically not viable. The limiting profile is the normalization of the solution of a heat equation with mass creation, a PDE with non-standard boundary conditions which was studied independently in [12]. Under stronger conditions, the limit solves a semi-linear parabolic equation of reaction-diffusion type with a reaction term depending directly on the flux balance that determines the mass creation. An outline of the tagged particle limit included.

1. Introduction

Let $D \subseteq \mathbb{R}^d$ an open domain with smooth boundary ∂D and a diffusion on D generated by a second order, strongly elliptic differential operator L with sufficiently smooth coefficients up to the boundary. The corresponding Dirichlet kernel determines a diffusion called the *underlying diffusion*. n addition, let $\gamma \in M_1(D)$ be a probability measure on D. It is important that γ does not charge the boundary, i.e. $\gamma(D) = 1$. These are the building blocks of a finite particle system (3.1) constructed as follows.

1.1. **The BSBD process.** Let $N \geq 2$ be an arbitrary but fixed positive integer. We start with N particles, each moving independently according to L until the first, say of index i, $1 \leq i \leq N$, hits ∂D. Instantaneously we choose particle $j \neq i$, $1 \leq j \leq N$ with probability

Date: June 24, 2019.
1991 *Mathematics Subject Classification.* Primary: 60K35; Secondary: 60J70, 60J80, 35K91, 92D25.
Key words and phrases. Key words and phases: Fleming-Viot, branching processes, super-critical, Bak-Sneppen, reaction-diffusion, nonstandard boundary conditions, quasi stationary distribution.
[1] Department of Mathematics, University of Miami, 1365 Memorial Drive, Coral Gables, FL 33124-4250
[2] email: igrigore@math.miami.edu.

$1/(N-1)$ and the two particles i and j *jump at the same random point chosen with distribution* γ. We emphasize that the number of particles N does not change after jump. The independent motion of the system is restarted afresh from the new locations of the particles, following a new cycle, until the next particle reaches ∂D; at that point the same mechanism redistributes the N particle system inside the domain D and continues with a new iteration. A discussion of the consistency of the construction is given in Subsection 1.3.

In distribution, the construction is equivalent to the killing of the pair (i, j) with instantaneous birth at the new location. The branching approach is consistent to the birth and death dynamics, but we prefer a finite system with simpler particle labelling, suitable when studying the evolution and scaling of the empirical measure (3.4), the main goal of this paper (Theorem 2). The resulting N - particle system $X_t^N = (X_t^{N,1}, \ldots, X_t^{N,N})$, defined more formally in (3.1) will be called the *Bak-Sneppen branching diffusions* (BSBD).

The branching/redistribution mechanism may choose an arbitrary number $K > 2$ of particles, with practically no change in the proof. The case K random, as well as subcritical can be handled with minor modifications.

1.2. **Motivation and relation to other particle dynamics.** The BSBD is a hybrid between the Fleming-Viot (FV) particle system (e.g. [5, 9, 15, 6, 1]) and the Bak-Sneppen self-organizing fitness model from [2, 3], explaining the name.

First, from the FV model we adopt the motion in between jumps and the trigger of the branching events. Like here, independent particles move in D until one hits the absorbing boundary (which can be regarded as a hard catalyst for branching). It is killed and jumps to the location of one of the remaining particles (redistribution). The process continues from the new locations until a new boundary hit.

Second, from the Bak-Sneppen dynamics we adopt the redistribution mechanism. The original model looks at $D = (0, 1)$ and N points in D. The one with minimum value is chosen, *together with a "neighbor"*, deleted, and replaced by three (or another number greater than two, according to some variants) independent particles with some continuous distribution γ (e.g. uniform) in D. Absolute continuity prevents ties in choosing the minimum.

We pick the first particle at the boundary (a substitute for the minimum value). The second particle (neighbor in extended sense) is randomly chosen among all others, emphasizing the *mean field character of the BSBD.* One of the most striking features of the Bak-Sneppen model is that strong correlation between neighbors (taken as exactly label i and $i+1$, for example) renders the asymptotic study of the distribution almost intractable analytically, in spite of the simplicity of the problem.

In broader sense, the BSBD can also be interpreted as an evolutionary model for the genome population. The particles undergo *mutation* represented by a diffusive term (Brownian), *selection*, represented by drift (in the probabilistic, not geneticists' sense) and *recombination*, represented by branching/redistribution at a random point $\sim \gamma(dx)$ where the new mass is born. Genetic recombination can be seen as a repair mechanism to damaged DNA. If artificial, it is under the effect of a *catalyst*, here seen as contact with the absorbing boundary.

The main results are the hydrodynamic limits stated in Theorems 2 and 3. In a nutshell, as $N \to \infty$, the FV process converges to the normalized density of the dissipative heat equation (subcritical), whereas the BSBD converges to the normalized density of the accretive heat equation (2.2)-(2.3) introduced in [12, 18], referred to as *the heat equation with mass creation*, which is super-critical. This comparison is discussed in more detail in Subsection 3.3.

1.3. **The underlying diffusion.** The underlying process is the diffusion driven by a second order, strongly elliptic operator L given by

$$Lu(x) = \frac{1}{2} \sum_{1 \le i,j \le d} a_{ij}(x) \partial^2_{x_i x_j} u(x) + \sum_{1 \le j \le d} b_j(x) \partial_{x_j} u(x) \,, \qquad u \in C_c^\infty(\mathbb{R}^d)\,, \tag{1.1}$$

where $a = (a_{ij})$ is a symmetric $d \times d$ matrix, $b = (b_j)$ a d - dimensional vector, both with $C^\infty(\mathbb{R}^d)$ components having bounded derivatives and there exists $a_0 > 0$ such that $\langle az, z \rangle \ge a_0 ||z||^2$, $z \in \mathbb{R}^d$ (uniform ellipticity). The diffusion is killed at τ^D, the hitting time of the boundary of the bounded domain $D \in C^2$.

Denote the sets of test functions

$$\mathcal{D}(L)\,(\text{resp.}\, \mathcal{D}_c(L)) = \{\phi \in C^2(\bar{D}) \,|\, \phi \in (BC)\,(\text{resp.}\,(BC)_c)\}\,, \tag{1.2}$$

where $\phi \in (BC)$, $(\in (BC)_c)$ if $\phi(x)$ vanishes (is constant) when $x \in \partial D$. The constant boundary conditions are needed later on in Section 4 starting with eq. (4.2) and have no bearing on the definition of the underlying diffusion. Since the coefficients are smooth and $\partial D \in C^2$, this process is the unique solution to the martingale problem $(L, \mathcal{D}(L))$.

Analytically, the underlying diffusion defines a *strongly continuous Feller-Dynkin semi-group* in the sense of [16], Chapter III.6, that is, the transition probabilities satisfy for $\phi \in C_b(D)$

$$(1.3) \qquad S_t^D \phi(x) = E_x[\phi(x_t)] = \int_D p^D(t, x, dy)\phi(y) \in C(\bar{D}) \qquad \text{(strong Feller property)}\,.$$

Due to the smoothness of the domain and coefficients, the Dirichlet kernel is smooth up to the boundary $p^D(\cdot, \cdot, \cdot) \in C^{1,2}((0, \infty) \times \bar{D} \times \bar{D})$.

Since D is bounded, the hitting time τ^D of the boundary ∂D has a positive exponential moment, i.e. there exits $\alpha_- < 0$ such that

$$(1.4) \qquad \sup_{x \in \bar{D}} = E_x[e^{\alpha_- \tau^D}] = c_D < \infty\,.$$

The distribution function $F_D(t, x)$ of τ^D, when starting at $x \in \bar{D}$ has a density $f_D(t, x)$ for $t > 0$ satisfying

$$(1.5) \qquad P_x(\tau^D > t) = 1 - F_D(t, x) = S_t^D 1(x) = \int_t^\infty f_D(s, x)\, ds\,.$$

Remark. The conditions needed to define the BSBD process can be relaxed substantially. In fact, a strongly Feller process with a finite moment for the hitting time of the boundary and γ not charging the boundary would be sufficient. Proposition 1 will show that the assumptions made in this subsection are sufficient to define the BSBD dynamics as a non-explosive Markov process. The stronger assumptions on the Dirichlet heat kernel given in subsection 2.1 are needed to solve the PDE from Definition 1.

2. The heat equation with mass creation

The limit of the empirical measures (Theorem 2) is determined via a weak solution of a partial differential equation with non-classical boundary conditions called *the heat equation with mass creation.* Its analytic properties are proven in [12]. These results are summarized in Theorem 1 and discussed in the Appendix. There is no overlap between the proofs in [12] and this paper, beyond the existence, uniqueness and smoothness of the solution to this PDE. To distinguish from $M_1(D)$, the space of probability measures on D, the space of finite measures is denoted by $M_F(D)$.

We shall write $\langle m, \psi \rangle$ for the integral of any bounded function ψ against a finite measure $m(dx)$ on D. Define the time-space set of test functions

$$\phi(\cdot,\cdot) \in \mathcal{D} = C^{1,2}([0,\infty) \times \bar{D}; \mathbb{R})\,, \tag{2.1}$$

where we shall sometimes consider functions independent of time with the same notation $\phi(\cdot) \in C^2(\bar{D})$. Alternatively, we could adopt $\mathcal{D} = C_c^{1,2}(\mathbb{R} \times \mathbb{R}^d)$, the space of smooth functions with compact support in both (t,x) variables, without loss of generality.

Definition 1 (Heat equation with mass creation)**.** *We shall say that $(\nu_t)_{t\geq 0} \in C([0,\infty), M_F(D))$ is a weak solution to the heat equation for $(L, \mathcal{D}(L))$ with mass creation at $\gamma \in M_1(D)$ of intensity $\bar{K} > 1$ and initial value $\nu_0 \in M_F(D)$ if*

$$\langle \nu_t, \phi(t,\cdot)\rangle - \langle \nu_0, \phi(0,\cdot)\rangle - \int_0^t \langle \nu_s, \frac{\partial}{\partial s}\phi(s,\cdot) + L\phi(s,\cdot)\rangle\, ds = 0 \tag{2.2}$$

for any test function ϕ from the class $\mathcal{D}$ defined in (2.1) satisfying the boundary condition

$$\phi(t,y) = \bar{K}\langle \gamma, \phi(t,\cdot)\rangle \qquad \forall t > 0,\, \forall y \in \partial D\,. \tag{2.3}$$

In case $\nu_0 = \delta_x$ the solution is denoted $(\nu_t^x)_{t\geq 0}$.

Remark. In this paper we are only interested in $\bar{K} = 2$, corresponding to the BSBD.

For sufficiently small $\delta > 0$, we denote

$$D_\delta = \{x \in D \,|\, dist(x, \partial D) > \delta\}\,, \quad \delta > 0\,. \tag{2.4}$$

A *separation condition* between the "boundaries" involved in the jump/branching mechanism. We assume $supp(\gamma) \subset D$. Since D is open, there exists $d_a > 0$ such that

$$dist(\partial D, supp(\gamma)) \geq d_a\,, \qquad \text{equivalently} \qquad supp(\gamma) \subset D_\delta\,, \quad \delta < d_a\,. \tag{2.5}$$

where $supp(q)$ is the topological support of the measure $q(dx)$.

Definition 2 (Regular solution)**.** *The solution is said* regular*, if it has bounded total variation and has a density away from the support of γ which is locally uniformly bounded, i.e. for any pair (t_0, T) with $0 < t_0 < T < \infty$ and any compact set $F \subseteq \bar{D} \setminus supp(\gamma)$, there* *exists a constant $C(t_0,T)$, possibly dependent on F such that*

$$\sup_{0\le t\le T} ||m_t|| < \infty \qquad \textit{and} \qquad \sup_{t_0\le t\le T} |m_t(F)| \le C(t_0,T)\, Leb(F)\,. \tag{2.6}$$

2.1. **The solution to the PDE.** To solve eq. (2.2) with the boundary conditions (2.3), we need the following additional assumptions, which are not necessarily minimal. On the other hand, Theorem 1 (iv) gives sufficient conditions.

The first is an off-diagonal bound. Given $\beta > 0$, for all $x, y \in \bar{D}$ with $dist(x,y) \ge \beta$, there exists $C_D(t,\beta) > 0$, depending on $t > 0$ and $\beta > 0$ uniformly bounded on $(0,T]$, $T > 0$ such that

$$|p^D(t,x,y)| \le C_D(t,\beta)\,, \quad \lim_{t\to 0} C_D(t,\beta) = 0\,. \tag{2.7}$$

The second is a smoothness property of the density of the hitting time. The density f_D must belong to $C([0,\infty)\times\bar{D})\cap C^{1,2}((0,\infty)\times\bar{D})$, has bounded derivatives and is a classical solution to the heat equation

$$\partial_t f_D = L f_D\,, \quad (t,x)\in(0,\infty)\times D\,, \quad f(0+,x) = 0\,. \tag{2.8}$$

We note that such a condition would be satisfied for sufficiently regular boundary. Even though technical, it is used in the proof of uniqueness of the PDE (Theorem 1 (i)), hence is essential in the identification of the hydrodynamic limit (Theorem 2).

The following is a summary of the analytic results proved in [12] needed in the proof of the hydrodynamic limit (Theorem 2).

Theorem 1. (Summary of main results in [12]) *Assume (2.7), (2.8) and (2.5) in addition to the conditions from Subsection 1.3. Then:*

(i) (Existence and uniqueness, Theorem 1 in [12]) *The heat equation with mass creation has a unique weak regular solution $(\nu_t)_{t\ge 0}$ such that, if ν_0 is positive, then the solution is positive and the total mass $n_t = \langle \nu_t, 1\rangle$ satisfies $0 < n_t \le C\exp(\alpha^* t)$, $t > 0$, where C depends on ν_0 only and $\alpha^* > 0$ is the unique solution of $E_\gamma[e^{\alpha^*\tau^D}] = 1/\bar{K}$ (see eq. (3.12)).*

(ii) (Regularity properties, Theorem 2 in [12]) *On $\bar{D}\setminus supp(\gamma)$, the solution $(\nu_t^x)_{t\ge 0}$ starting at $\nu_0 = \delta_x$, $x\in D$, has a density $\nu_t^x(dy) = v^x(t,y)dy$ that is locally bounded, i.e. $0 \le v^x(t,y) \le C(t_0,T)$ for any $T \ge t_0 > 0$ on any compact subset, where the constant depends on the compact.*

(iii) (Backward equation, Theorem 4 in [12]) *The function* $(t,x) \to w(t,x) = \langle \nu_t^x, g\rangle$, $g \in C(\bar{D})$, *belongs to* $C([0,\infty)\times \bar{D};\mathbb{R}) \cap C^{1,2}((0,\infty)\times\bar{D};\mathbb{R})$, *has bounded derivatives on bounded time intervals, and solves the equation* $\partial_t w = Lw$ *with initial condition* $w(0,x) = g(x)$ *and boundary conditions (2.3).*

(iv) (Sufficient conditions, Proposition 1 in [12]) *When* $L = \frac{1}{2}\Delta$, *the conditions (2.7) and (2.8) are verified.*

Proof. The theorem is an immediate collection of results stated in brackets. □

They key to proving these conditions is that the solution admits the representation $\langle \nu_t, \phi\rangle = E_{\nu_0}[\langle \zeta_t, \phi\rangle]$, for any sufficiently smooth ϕ and $t \geq 0$, where $(\zeta_t)_{t\geq 0}$ is the auxiliary super-critical branching process $(\zeta_t)_{t\geq 0}$ presented in the Appendix. See the remark at the end of Subsection 3.3 and the discussion of the related particle models.

3. Main results

The BSBD process described in Subsection 1.1 can be constructed by piecing together an countable sequence of independent processes driven by L and, independently, a countable sequence of i.i.d. pairs of relocation random points in D with distribution γ. This construction is essentially the same as for the Fleming-Viot particle system [5, 9]. Henceforth we consider the N-particle process

$$(X_t^N(\omega))_{t\geq 0}\,, \quad X_t^N(\omega) = (X_t^{N,1}(\omega), \ldots, X_t^{N,N}(\omega)) \tag{3.1}$$

defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $\omega \in \Omega$, where $\mathcal{F}_t$ satisfies the usual conditions. By construction, $(X_t^N(\omega))_{t\geq 0}$ is a jump-diffusion on the Skorokhod space $\mathbb{D}^N([0,\infty), D^N)$ of right continuous with left-side limits paths.

Denote the number of times particle i hits ∂D up to time $t \geq 0$

$$A_t^{N,i}(\omega) = \int_0^t \mathbf{1}_{\partial D}(X_{s-}^{N,i})\, ds \tag{3.2}$$

and the average number of boundary hits

$$A_t^N(\omega) = \frac{1}{N-1}\sum_{i=1}^N A_t^{N,i}(\omega)\,, \tag{3.3}$$

where the normalization constant $(N-1)^{-1}$ is chosen for convenience and asymptotically consistent to the total number N of particles. These processes are naturally adapted to the filtration $\mathcal{F}_t$. We shall omit ω unless absolutely necessary.

The *hydrodynamic limit* from Theorem 2 is a a Law of Large Numbers for the time-dependent empirical measure process

$$t \longrightarrow \mu_t^N(dy,\omega) = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{N,i}(\omega)}(dy) \, \in \mathbf{D}([0,\infty), M_1(D))\,, \tag{3.4}$$

where $\mathbb{D}([0,\infty), M_1(D))$ denotes the Skorokhod space of probability measure - valued paths on D. To simplify notation, the random element ω will be omitted unless absolutely necessary.

Definition 3. *Let $(\mathbb{X}, ||\cdot||)$ be a Polish space. A sequence of processes $(Y_\cdot^N)_{N\geq 1}$ supported on the Skorokhod space $\mathbf{D}([0,\infty);\mathbb{X})$* converges in probability *to $(Y_\cdot)$, locally uniformly in time, if $(Y_\cdot^N)_{N\geq 1}$ is a tight family and for any $T>0$,*

$$\forall \epsilon > 0 \qquad \lim_{N\to\infty} P\Big(\sup_{t\in[0,T]} ||Y_t^N - Y_t|| > \epsilon\Big) = 0\,. \tag{3.5}$$

Definition 4. *The process $(\mu_\cdot^N)_{N>0}$* converges weakly in probability *to $(\mu_\cdot)$ if for any function $\phi \in C(\bar{D})$, the process $t \to \langle \mu_t^N, \phi(\cdot)\rangle_{t\geq 0}$ converges in probability to $t \to \langle \mu_t, \phi(\cdot)\rangle_{t\geq 0}$ in the sense of the Definition 3.*

3.1. **The general solution.** Assume $\mu_0(dx)$ is a non-random measure in $M_1(D)$ and the initial condition

$$\mu_0^N \text{ converges weakly in probability to } \mu_0 \text{ as } N\to\infty\ . \tag{3.6}$$

Let ν_t be the unique solution to the heat equation with particle creation (2.2) with coefficient $\bar{K}=2$, boundary conditions (2.3) and initial value $\nu_0 = \mu_0$. Write

$$\mu_t = \frac{\nu_t}{\langle \nu_t, 1\rangle}\,, \qquad n_t = \langle \nu_t, 1\rangle \quad \text{with} \quad n_0 = 1 \quad \text{and} \quad A_t = \ln n_t\,, \tag{3.7}$$

where the logarithm is justified by the lower bound given in Theorem 1 (i).

Theorem 2. *Assume the heat equation with mass creation from Definition 1 has a unique regular solution. Then, under the initial condition (3.6), as $N\to\infty$, the empirical* *measure process (3.4) converges weakly in probability to the deterministic trajectory* $\mu_\cdot \in \mathbf{C}([0,\infty); M_1(D))$*, and* $A^N_\cdot$ *converges in probability to* $A_\cdot \in C([0,\infty); \mathbb{R}_+)$ *from (3.7).*

The trajectory $t \to (\mu_t, A_t)$ is unique, because $t \to \nu_t$ is unique and (3.7) determines ν_t from the pair (μ_t, A_t). It also has all the regularity and boundary properties inherited from Theorem 1, being absolutely continuous for $t > 0$ with density $\rho(t,y)$ and continuous for $t \geq 0$ in the topology of convergence in distribution.

3.2. **The strong solution.** In general, the limit $\mu_\cdot$ is the normalization of a weak solution of a parabolic equation. The boundary conditions (2.3) are expressed in weak form as well.

Let $w \in C^1(\bar{D} \setminus \{c\})$. Define the *inward flux from* c the limit

$$\Phi(w,c) = \lim_{\epsilon \to 0} \int_{\partial B(c,\epsilon)} \nabla w(y) \cdot \mathbf{n}\, dS\,, \tag{3.8}$$

where $\mathbf{n}$ is the outward normal to the sphere centered at c, whenever the limit exists and is finite. We shall write $\Phi(w, \partial D)$ for the total flux of w over the boundary ∂D.

Remark. The flux is said *inward* because it *enters* the set at c; it can be considered asymptotically equal to the opposite of the outward flux seen from the interior of the punctured set through a small ball centered at c.

For such a function $w(t,y)$, we say it satisfies the *flux balance condition* if

$$\Phi(w(t,\cdot),c) = \bar{K}\Phi(w(t,\cdot),\partial D)\,. \tag{3.9}$$

Theorem 3. *Assume* $L = \frac{1}{2}\Delta$*,* $\gamma(dx) = \delta_c(dx)$ *for some* $c \in D$*,* $\bar{K} = 2$ *and* $\mu_0(dy) = \rho_0(y)dy$*,* $\rho_0 \in C(\bar{D})$*. Let* $(\nu_t)_{t\geq 0}$ *and* $(\mu_t)_{t\geq 0}$ *be the solutions from Theorems 1, respectively 2 with common initial value* μ_0*. Then* ν_t *has density* $v(t,y)$ *and* μ_t *has density* $\rho(t,y) = v(t,y)/n_t$*,* $t \geq 0$*, where* $\rho \in C([0,\infty) \times \overline{D} \setminus \{c\}) \cap C^{1,2}((0,\infty) \times D \setminus \{c\})$ *and has integral one in the space variable for any* $t \geq 0$*. The total mass takes the form* $n_t = \langle v(t,\cdot), 1\rangle = \exp(A_t)$ *with* $A_t = \int_0^t a(\rho,s)ds$ *and* $\rho(t,y)$ *verifies the reaction-diffusion equation*

$$\partial_t \rho(t,y) = \frac{1}{2}\Delta\rho(t,y) - a(\rho,t)\rho(t,y) \quad \text{with} \quad v(0,\cdot) = \rho_0(\cdot) \tag{3.10}$$

with simultaneous Dirichlet and flux balance boundary conditions (3.9). The factor $a(\rho,t)$ *is linear in* ρ *being equal to*

$$a(\rho,t) = \frac{1}{2}\langle \Delta\rho(t,y), 1\rangle = -\frac{1}{2}\Phi(\rho(t,\cdot),\partial D)\,, \quad t > 0\,. \tag{3.11}$$

Remarks. 1) The boundary condition (3.9) has a simple interpretation: The flux entering the set at the source c is double the flux exiting the set through the boundary. In addition, $v(t,y)$ behaves like a Green function, and has a singularity of the same type $v(t,y) \sim ||y-c||^{2-d}$ as $y \to c$.

2) The constant $\phi \equiv 1$ does not satisfy the boundary conditions (2.3), which makes relation (3.11) nontrivial. In the more general case when the two particles redistributed are replaced by, say, a random number K with mean $\bar{K} \geq 2$, the factor in front of the flux becomes $-(\bar{K}-1)/2$. Note that the 1/2 part comes from the half-Laplacian. The super-critical nature of the underlying process is transparent here, as $a(\rho,t)$ is strictly positive and inversely proportional to the dissipation of mass manifested through the outward flux.

Proof. Let $v(t,y)$ be the density of the solution in Theorem 1 (i) and (ii), with initial value μ_0. In full detail, its existence is guaranteed by Theorem 5 (Appendix) everywhere outside $supp(\gamma)$, that is, on $D \setminus \{c\}$, and extends continuously up to ∂D. Theorem 2 shows that we have the normalization formula $\mu_t(dy) = \rho(t,y)dy$, $t > 0$ and $\rho(t,y) = v(t,y)/n_t$, with total mass satisfying the bound $1 < n_t \leq Ce^{\alpha^* t}$, starting at $n_0 = 1$ and differentiable for $t > 0$ with continuous derivative. This justifies the logarithm in $\ln n_t$ and the definition of $a(\rho,t)$ as its derivative in time. Equation (3.10) follows by elementary differentiation. The boundary conditions are satisfied because $v(t,y)$ satisfies them and ρ is obtained by dividing by n_t, a function of time only. To justify (3.11), we integrate the equation against the constant function one. The second equality is a consequence of Greene's theorem and the condition (3.9). We formally keep $\bar{K}$ for clarity in relation to (3.9)

$$\begin{aligned}\langle \Delta\rho(t,y), 1\rangle &= \Phi(\rho(t,\cdot),\partial D) - \Phi(\rho(t,\cdot),c)\\ &= -(\bar{K}-1)\Phi(\rho(t,\cdot),\partial D) = (1-\frac{1}{\bar{K}})\Phi(\rho(t,\cdot),c)\,, \qquad \text{where} \quad \bar{K}=2\,.\end{aligned}$$

□

3.3. **Self-organizing criticality: Comparison with the Bak-Sneppen and Fleming-Viot models.** As mentioned briefly in Subsection 1.2, the Bak-Sneppen fitness model introduced in the seminal works [2, 3] consists in a system of N *fitness columns* on $D = (0,1)$ and the long time behavior (ergodicity, stationary profile) are investigated. One difference is that we look at one neighbor only, so the number of individuals branching in the BSBD is $K = 2$; that is not significant qualitatively, noting that the proofs would remain almost identical for a fixed number of neighbors $K > 2$. In general, K may be random and its mean value, the branching intensity, is denoted as $\bar{K}$. To unify notation, we keep the notation $\bar{K}$ for the number of particles killed upon reaching the boundary.

The second difference is the random choice of the "neighbor". This makes our current model mean-field, and, as such, closable. Another important difference is, of course, that instead of a dynamic value of the "minimum", we trigger branching only by contact with the boundary, an absolute extreme value. Nonetheless, the most important feature of the Bak-Sneppen model remains present: *self-organizing criticality*, in the sense that the relaxation limit (as $t \to \infty$) of the macroscopic profile is equal to the quasi-stationary distribution of the supercritical branching system described in the Appendix, Subsection 8.1, as explained below.

Under the conditions in Subsection 1.3, denote the resolvent of the Dirichlet kernel semigroup

$$R_\alpha^D \phi(x) = \int_0^\infty e^{-\alpha t} S_t^D \phi(x) dt$$

and the Laplace transform of the exit time

$$\hat{f}(\alpha, x) = E_x[e^{-\alpha \tau^D}] = \int_0^\infty e^{-\alpha t} f(t, x)\, dt$$

with $f(t, \gamma) = \langle \gamma, f(t, \cdot) \rangle$. If τ^D has an exponential moment (1.4), there exists a largest number $\tilde{\alpha} < 0$ such that $\hat{f}(\tilde{\alpha}, \gamma) = +\infty$. Since $\hat{f}(+\infty, \gamma) = 0$ by monotone convergence, there is an increasing bijection $\alpha^* : (0, \infty) \to (-\tilde{\alpha}, +\infty)$ where

$$\alpha^* = \alpha^*(\bar{K}) \quad \text{solves} \qquad 1 - \bar{K} \hat{f}(\alpha^*, \gamma) = 0 \,. \tag{3.12}$$

To fix ideas, let K be binomial with probability of success $p = \bar{K}/N$.

Let ν_t be the solution of the heat equation with mass creation, with its representation $\nu_t = E[\zeta_t]$ and total mass $n_t = \langle n_t, 1 \rangle$. The Yaglom limit $\nu_t / n_t \to \nu_{\bar{K}}$ is exactly a quasi-stationary distribution for the semigroup $t \to S_t \phi(x) = E_x[\langle \zeta_t, \phi \rangle]$ (see [13] for the detailed derivation). With a normalization constant $C > 0$,

$$\nu_{\bar{K}}(dy) = C^{-1} \int_D R^D_{\alpha^*(\bar{K})}(x, dy)\, \gamma(dx) \,.$$

Then, since $\mu_t = \nu_t / n_t$, for a given redistribution with intensity $\bar{K}$, we obtain a limit $\lim_{t \to \infty} \mu_t \to \nu_{\bar{K}}$.

Like in the Bak-Sneppen model, $\nu_{\bar{K}}$ *is a function of* $\bar{K}$. We obtained a bijection between interaction intensities and the stationary profiles; this feature is a manifestation of self-organizing equilibrium.

There is another important feature of this correspondence, in that it produces a particle representation of the resolvent R_α of $(L, \mathcal{D}(L))$ for $\alpha \in \mathbb{R} \cap Res(L)$.

In the FV case [5, 9, 17, 15, 6, 1] one particle is killed and resampled, so there $\bar{K} = 1$. The hydrodynamic limit is the normalization of the solution to heat equation with Dirichlet boundary conditions, i.e. a dissipative solution (varying in time), divided by its total mass in order to have mass one. Before normalization, this total mass decreases with exponentially fast at rate $e^{\lambda_1 t}$, with $\lambda_1 < 0$. For example, when $(L, \mathcal{D}(L))$ is a Brownian Motion killed at the boundary, λ_1 is exactly the first eigenvalue for the Dirichlet Laplacian.

In the BSBD case, $\bar{K} > 1$ is the number of individuals re-sampled (here $\bar{K} = 2$). There is no classical boundary condition ensuring a solution to the needed super-critical mass heat equation. Yet, the hydrodynamic limit is the normalization of ν_t, the transition function of a process accruing mass exponentially fast at rate $e^{\alpha_* t}$, where $\alpha_* > 0$ when $\bar{K} > 1$. This process replaces the one-particle killed diffusion from the FV model. In [12] we construct the necessary analytical solution as the transition function of the super-critical branching process (ζ_t).

To summarize, in order to prove the hydrodynamic limit of the BSBD, a conservative system, we need to represent its solution with a non-conservative Markov measure valued process and then normalize to obtain mass one. We need a non-conservative process in either models. While the dissipative case allows a representation with a single particle as in [9, 1], the mass creation can only be modeled stochastically using a Markov semigroup of a measure-valued process cf. Appendix, eq. (8.2).

4. Martingales and Ito formula

We prove a preliminary result on the non-explosive behavior of the system.

Proposition 1. *Assume that $\gamma \in M_1(D)$. Then, with probability one, the visits to the boundary can occur only one particle at a time. If $\tau_1 < \tau_2 \ldots < \tau_n \ldots$ is the strictly increasing sequence of such visits and $\tau^* = \lim_{n\to\infty} \tau_n$, then $P(\tau^* = +\infty) = 1$.*

Remark. The condition that $\gamma(\partial D) = 0$ is sufficient for non-explosiveness and thus for the definition of the BSBD process. Later we shall require a stronger condition (2.5) needed for the regularity of the solution of the hydrodynamic limit.

Proof. Simultaneous visits cannot occur because the hitting times of two independent particles (between boundary hits) have absolutely continuous distributions, implying that almost surely, the times τ_n, $n \geq 1$ can be ordered in a strictly increasing sequence. It is shown in [10], Subsection 4.2, that as long as the underlying diffusion $(L, \mathcal{D}(L))$ has a heat kernel (density) satisfying three conditions (given below), then the process is non-explosive and only one particle branches at a time, almost surely. The conditions are the following.

(i) A Doeblin recurrence condition on the driving process, in this case guaranteed by the lower bound of (4.1). Due to the regularity of D, for sufficiently small δ, the set D_δ can be taken with smooth boundary. The maximum principle will guarantee that for any time interval $0 < t_0 < T < \infty$ there exist positive constants c_- and c_+, such that

$$c_- \leq p^D(t,x,y) \leq c_+ \,, \qquad \forall t \in [t_0, T] \,, \quad x, y \in \bar{D}_\delta \,. \tag{4.1}$$

(ii) The lifetime τ^D has finite positive expectation a.s., here implied by (1.4).

(iii) The distribution γ does not charge the boundary, - as required in the hypothesis. □

Denote the set of test functions

$$\mathcal{D}_N = C^{1,2}([0,\infty) \times \bar{D}^N; \mathbb{R}) \,. \tag{4.2}$$

the class of N - dimensional time-space test functions $F(t,x)$ *continuous up to the boundary.*

Denote $L^{\otimes N}$ the direct sum of the one variable operator L, and by F^{ij} (defined precisely below) the configuration under F after redistribution of the particle i.

This is obtained as particle i has reached ∂D, has chosen particle $j \neq i$ uniformly, and both are created anew at the same random point with distribution $\gamma(dx)$. Using the vector notation $X = (x^1, x^2, \ldots, x^N)$,

$$L^{\otimes N} F(s,X) = \sum_{i=1}^{N} L_{x_i} F(s, \ldots, x_i, \ldots) \tag{4.3}$$

$$F^{ij}(s,X) = 2 \int_D \int_D \mathbf{1}(x_i = x)\mathbf{1}(x_j = x) F(s, \ldots, x_i, \ldots, x_j \ldots) \gamma(dx) \,, \qquad i \neq j \tag{4.4}$$

where the identical entries are on position i and j.

The boundary ∂D can be assimilated to the cemetery state $\mathfrak{b}$ and a function $\phi \in (BC)_c$ will take constant value $\phi(\mathfrak{b})$ on ∂D, and that constant will be zero if $\phi \in (BC)$.

Let $(A_t^{N,i})_{t\geq 0}$ be the number of hits of particle i to the absorbing boundary ∂D from (3.2). Notice that $X_{t-}^{N,i} = \mathfrak{b}$ if and only if the counting process $A_t^{N,i}$ has a discontinuity, with probability one.

The joint set of interacting processes $(X_t^{N,i}, A_t^{N,i})_{t\geq 0}$, for $1 \leq i \leq N$, was defined constructively in Section 1, based on the strong Markov property. We note the fact that there are no simultaneous boundary hits due to the absolute continuity of the distributions. For a similar construction in more detail, more details, see [9].

For $F \in \mathcal{D}_N$, using the superscripts c and J designate the continuous, respectively jump parts, we write the process $\mathbf{X}_t^F = F(t, X_t^N)$ as a semimartingale

$$\mathbf{X}_t^F = \mathbf{X}_0^F + \mathbf{X}_t^{F,c} + \mathbf{X}_t^{F,J}\,,$$

$$\mathbf{X}_t^{F,c} = \int_0^t L^{\otimes N} F(s, X_s^N)\,ds + \mathcal{M}_t^{F,c} \tag{4.5}$$

$$\mathbf{X}_t^{F,J} = \sum_{i=1}^N \int_0^t \Big(\frac{1}{N-1}\sum_{j\neq i} F^{i,j}(s, X_{s-}^N) - F(s, X_{s-}^N)\Big)\,dA_s^{N,i} + \mathcal{M}_t^{F,J}\,. \tag{4.6}$$

Here the first terms on the right hand side are the finite variation part. The second terms $(\mathcal{M}_t^{F,c})$ and $(\mathcal{M}_t^{F,J})$ are $\mathcal{F}_t$ - martingales (cf. Proposition 2). Moreover, the processes $\mathcal{N}_t^{F,c}$, respectively $\mathcal{N}_t^{F,J}$ are also $\mathcal{F}_t$ - martingales, where

$$\mathcal{N}_t^{F,c} = (\mathcal{M}_t^{F,c})^2 - \sum_{i=1}^N \int_0^t (L_{x_i}F^2 - 2\langle F, L_{x_i}F\rangle)(s, X_s^N)ds \tag{4.7}$$

$$\mathcal{N}_t^{F,J} = (\mathcal{M}_t^{F,J})^2 - \sum_{i=1}^N \int_0^t \frac{1}{N-1}\sum_{j\neq i}(F^{i,j}(s, X_{s-}^N) - F(s, X_{s-}^N))^2\,dA_s^{N,i}\,. \tag{4.8}$$

Set $F(t,x) = \frac{1}{N}\sum_{i=1}^N \phi(t, x_i)$, for $\phi(t,\cdot) \in \mathcal{D}_c(L)$ and denote the corresponding process $\mathbf{X}_t^\phi$. Then, the expressions (4.5) show that $\mathcal{M}_t^\phi = \mathcal{M}_t^{c,\phi} + \mathcal{M}_t^{J,\phi}$, then

$$\mathcal{M}_t^\phi = \langle \mu_t^N, \phi(t,\cdot)\rangle - \langle \mu_0^N, \phi(0,\cdot)\rangle - \int_0^t \langle \mu_s^N, L\phi(s,\cdot)\,ds \tag{4.9}$$

$$- \int_0^t \Big[(2\langle\gamma, \phi(s,\cdot)\rangle - \phi(s,\mathfrak{b})) - \langle \mu_{s-}^N, \phi(s,\cdot)\rangle\Big] - \frac{2}{N}\langle\gamma, \phi(s,\cdot)\rangle\,dA_s^N\,. \tag{4.10}$$

Proposition 2. *The processes* $(\mathcal{M}_t^{F,c})$ *and* $(\mathcal{M}_t^{F,J})$ *defined in (4.5) are* $\mathcal{F}_t$ *- martingales. Moreover, there exists a constant* $C(\gamma)$*, independent of* t *and* N *but dependent on the initial limiting profile* μ_0*, such that, for all* $t \geq 0$ *and* $N \in \mathbb{Z}_+$*,*

$$E\Big[\frac{1}{N}\sum_{i=1}^{N} A_t^{N,i}\Big] \leq 1 + C(\gamma)t\,. \tag{4.11}$$

Remark. As *Step 1* below shows it, it is not hard to see that the processes in the statement are local martingales. In fact, all the processes in Proposition 2 are proper martingales, which is equivalent to showing that $E[A_t^{N,i}] < \infty$ for all components $1 \leq i \leq N$ and $t \geq 0$.

Proof. Step 1. The process (X_t^N) is non-explosive, as shown in Subsection 4.2 in [10]. We then know that $\lim_{t\to\infty} A_t^{N,i} = +\infty$ a.s., which implies, due to the boundedness of all integrand terms in the martingales, that setting T_m, $m \geq 1$ the first hitting time of the positive integer m by the sum $\sum_{i=1}^{N} A_t^{N,i}$, the processes (4.5), (4.7), (4.8) are local martingales by setting $t \to t \wedge T_m$, in other words with localization sequence T_m.

Step 2. We prove the processes are martingales. Set $F(t,X) = \frac{1}{N}\sum_{i=1}^{N}\phi(x_i)$ for a function $\phi \in (BC)$, $0 \leq \phi \leq 1$ with $c_\gamma = 2\langle\gamma,\phi\rangle - 1 > 0$. Such a function exists since γ has integral one and ϕ can be taken as a smooth function approximating the indicator function of a compact set in D. In that case, the integrand of the dA_t^N term in (4.9) is greater or equal to c_γ. More precisely $\phi(s,\mathfrak{b}) = 0$ and the integrand equals

$$c_\gamma + \langle\mu_s^N, 1-\phi(s,\cdot)\rangle - \frac{2}{N}\langle\gamma,\phi(s,\cdot)\rangle \geq c_\gamma - \frac{2}{N} \geq \frac{c_\gamma}{2} > 0$$

for large enough N. We obtain, almost surely,

$$\frac{c_\gamma}{2}A_{t\wedge T_m}^N \leq -\mathcal{M}_{t\wedge T_m}^F + F(t\wedge T_m, X_{t\wedge T_m}^N) - F(0,X_0^N) - \int_0^{t\wedge T_m} L^{\otimes N}F(s,X_s^N)\,ds\,. \tag{4.12}$$

Taking the expected value, we see that there exists a constant $C(\gamma)$, independent of t and N because it is simply a uniform bound on the function ϕ and its derivatives, such that $E[A_{t\wedge T_m}^N] \leq 1 + C(\gamma)t$. Since $\lim_{m\to\infty} T_m = +\infty$ a.s. we obtain by monotone convergence the same bound for $E[A_t^N]$, proving the proposition. $\square$

5. Tightness

The main result of the section is Theorem 4, which show tightness for both $(\mu_\cdot^N)_{N>0}$ and $(A_\cdot^N)_{N>0}$. The important part is to show the tightness for the latter, and the former will follow by standard arguments using the generalized Ito formula.

Definition 5 (Tightness)**.** *A sequence of processes* $(Y_\cdot^N)_{N>0}$ *on a Polish space* $(\mathbb{X}, ||\cdot||)$ *is* C *- tight if*

(i) For any $t \geq 0$*,* $(Y_t^N)_{N>0}$ *is a tight family, and*

(ii) For any $T > 0$*, the process*

$$\forall \epsilon > 0 \qquad \lim_{\delta \to 0} \limsup_{N \to \infty} P\left(\sup_{0 \leq s < t \leq T, |t-s| < \delta} ||Y_t^N - Y_s^N|| > \epsilon \right) = 0\,. \tag{5.1}$$

The Polish space for A_t^N is $\mathbb{R}$ with the Euclidean distance. We prove a stronger condition (i) by showing (5.10) in Proposition 3. For μ_t^N we keep in mind that the time indexed process is supported on the Skorokhod space $\mathbf{D}([0,\infty); M_1(D))$, where $M_1(D)$ is endowed with the weak topology of convergence in distribution and that C-tightness means that any sequential limit belongs to $C([0,\infty); M_1(D))$. We have to prove (i) uniform boundedness for any $t \geq 0$ and (ii) equicontinuity. These are implied by (5.3), which shows that any limit point is indeed in $M_1(D)$, and not simply in $M_1(\bar{D})$, and then by (5.1) when $Y_t^N = \langle \mu_t^N, \phi(t,\cdot)\rangle$, $t \geq 0$, for each $\phi \in \mathcal{D}$.

We start with two lemmas. One shows that the number of particles near the absorbing boundary remains small, uniformly in N, provided that it was small at time zero. The other one shows that even though the particles are not independent, the duration between visits to the absorbing boundary cannot be very short, provided the starting point is, in some sense, distributed away from the boundary, like is the case with the point with distribution $\gamma(dx)$. Since the time of return is controllable, uniformly in N, there cannot be too many boundary visits in a short time interval.

Denote $D(\gamma) = D \setminus supp(\gamma)$. Given a sufficiently small $\delta > 0$, the set within distance δ from the boundary is included in $D(\gamma)$. Let $U_t^N(\delta)$ be the number of particles within distance δ from the absorbing boundary at time t, $\nu^N \Rightarrow \nu$ denote the convergence in distribution for a sequence $(\nu^N) \in M_1(D)$.

Lemma 1. *Assume* μ_0^N *converges weakly in probability to* $\mu_0 \in M_1(D)$ *as* $N \to \infty$*. Let* $T > 0$ *and* $0 < t_0 < T$ *and* F *be a compact set* $F \subseteq \bar{D} \setminus D(\gamma)$*. Then, for any* $g \in C(\bar{D})$

supported on F there exists a constant $C(t_0,T)$, depending on t_0, T but independent of N, such that,

$$\limsup_{N\to\infty} \sup_{t\in[t_0,T]} E[\langle \mu_t^N, g\rangle] \le C(t_0,T)\int_D g(y)dy\,. \tag{5.2}$$

As a consequence,

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \sup_{t\in[t_0,T]} E[\frac{U_t^N(\delta)}{N}] = 0\,. \tag{5.3}$$

Proof. Let g be a smooth function and $w(t,x) = E_x[\langle \zeta_t, g\rangle]$, with the notations from Theorem 1 (iii). For a fixed $t>0$, the function $\bar w(s,x) = w(t-s,x)$, $s\in[0,t]$, satisfies the backward equation $\partial_s \bar w + L\bar w = 0$ with terminal condition $\bar w(t,x) = g(x)$ and the boundary conditions from (1.2), together with $2\langle \gamma, \bar w(s,\cdot)\rangle = \bar w(s,\mathfrak{b})$ from (2.3).

Setting $\bar w \to \phi$ in (4.9) we obtain that $s\to \langle \mu_s^N, \bar w(s,\cdot)\rangle$ is a super-martingale. Let $0<\epsilon<t_0$. The expected values at $s=0$ and $s=t-\epsilon$ give the inequality

$$E[\langle \mu_{t-\epsilon}^N, \bar w(t-\epsilon,\cdot)\rangle] \le E[\langle \mu_0^N, w(t,\cdot)\rangle] = E[\int_D w(t,x)\mu_0^N(dx)] = E[E_{\mu_0^N}[\langle \zeta_t, g\rangle]] \tag{5.4}$$

Because the variables in the process are rcll, the function $\bar w$ is continuous, hence bounded, we use dominated convergence to obtain

$$\lim_{\epsilon\downarrow 0} E[\langle \mu_{t-\epsilon}^N, \bar w(t-\epsilon,\cdot)\rangle] = E[\langle \mu_t^N, \bar w(t,\cdot)\rangle] = E[\langle \mu_t^N, g\rangle]\,. \tag{5.5}$$

Combined with (5.4), we obtain

$$E[\langle \mu_t^N, g\rangle] \le E[\langle \mu_0^N, w(t,\cdot)\rangle] = E[\int_D w(t,x)\mu_0^N(dx)] = E[E_{\mu_0^N}[\langle \zeta_t, g\rangle]] \tag{5.6}$$

If g has support in a closed $F\subseteq D(\gamma)\cup\partial D$, the solution $w(t,x) = E_x[\langle\zeta_t, g\rangle]$ has a kernel representation equal to $\langle v^x(t,\cdot), g\rangle$.

We recall that for $t\in[t_0,T]$, $t_0>0$,

$$\begin{aligned} E_{\mu_0^N}[\langle \zeta_t, g\rangle] &= \int_D\int_D v^x(t,y)g(y)\mu_0^N(dx) \\ &\le \Big(\sup_{t\in[t_0,T],(x,y)\in D\times F} |v^x(t,y)|\Big)\int_D g(y)dy \le C(t_0,T)\int_D g(y)dy \ \text{ a.s.}\,, \end{aligned} \tag{5.7}$$

where we used Theorem 1 (ii). Taking the supremum over $t\in[t_0,T]$ in both (5.4) and then (5.7) we obtain (5.2).

The last assertion follows from taking a smooth approximation of the indicator function of the complement of the compact set $\bar{D}_{2\delta}$, as in (2.4), which is well defined since D has smooth boundary. $\square$

The underlying process with generator L is a *one-particle process* and its hitting time of the boundary is τ^D. In the following, we shall need the same hitting time τ^D_X for the *tagged particle process* $(X^{N,i}_t)_{t\geq 0}$, i.e. the process with fixed label $1 \leq i \leq N$. For simplicity, we suppress the index i, since in Lemma 2 and in its applications the label will never change, and it would be redundant.

Lemma 2. *Assume supp*$(q) \subset D$ *for a probability measure* $q(dx)$*, hence a lower bound (2.5) holds for some* $d_a > 0$*. We recall that the coefficients of* L *are continuous up to the boundary, hence bounded.*

Let $1 \leq i \leq N$ *be a fixed index of one of the particles. We assume the* N *- component vector process* X^N_t *starts at a finite stopping time* τ *from a configuration with marginal distribution of particle* i *equal to* $q(dx) \in M_1(D)$*. Then there exists a constant* $c(q)$*, dependent only on* $q(dx)$ *only, and a fortiori independent of* N*, such that, for any* $\eta > 0$

$$P_{X^N_\tau}(\tau^D_X \leq \tau + \eta) \leq c(q)\eta\,. \tag{5.8}$$

Remarks. 1) Inequality (5.8) is valid pointwise, holding simply due to the distribution $\gamma(dx)$ of the tagged particle.

2) This lemma will be applied twice, once for $\tau = 0$ and q equal to the distribution of $X^{N,i}_0$, in order to prove tightness for the tagged particle, and another time with τ a time when $X^{N,i}_{\tau-} \in \partial D$ and $q = \gamma$. In the second case it will be essential that $q(dx)$, and consequently $c(q)$, do not depend on τ, N or the index i.

3) Lemma 2 is the only place in the hydrodynamic limit where we use that *supp*$(\gamma) \subset D$, more precisely that the topological support of the redistribution measure is at a positive distance from the absorbing boundary. The condition is needed otherwise for the existence and uniqueness of the macroscopic solution of the PDE.

Proof. We construct a coupling between two processes, one without jumps, and then use a small ball estimate based on Doob's maximal inequality.

Step 1. Let $\psi \in C^2(\bar{D}, \mathbb{R})$ be a test function with the properties

1) $0 \leq \psi(x) \leq 1$,

2) $\psi(x) = 1$ on $supp(\gamma)$ and $\psi(x) = 0$ if and only if $x \in \partial D$,

3) There exists $0 < \delta < \frac{d_a}{2} \wedge 1$, such that $\psi(x) = dist(x, \partial D)$ on $D \setminus D_\delta$.

Define $y_t = \psi(X_t^{N,i})$, $t \geq \tau$. Notice that by construction, at any τ', a jump time of $X_t^{N,i}$, y_t jumps $y_{\tau'} - y_{\tau'-} \geq 0$, a non-negative jump. This is because the values on the support of γ, where it jumps, are guaranteed to equal the maximum value of ψ over the full set $\bar{D}$. We notice that $(y_t) \in [0,1]$ is a semi-martingale, adapted to $(\mathcal{F}_{t\wedge\tau})$, driven by the *full process* (X_t^N), not just the particle i, due to the jumps it undergoes at times when $X_t^{N,i}$ is chosen randomly by another particle hitting the absorbing boundary, in addition to its own jumps triggered by hitting the absorbing boundary. This process will be coupled with a new process denoted $(z_t)_{t\geq\tau}$, with the same initial value, driven by the same equations between jumps, only *with all jumps suppressed.* Then

$$0 \leq z_t \leq y_t \leq 1 \qquad a.s.$$

and $(z_t)_{t\geq\tau}$ is an Ito process $dz_t = \alpha_t dt + \beta_t dw_t$, with coefficients given by

$$dz_t = L\psi(X_t^{N,i})dt + (\nabla\psi)(X_t^{N,i}) \cdot [\sigma(s, X_t^{N,i})dw_t] , \quad z_0 = y_0 = \psi(X_\tau^{N,i}) ,$$

if the underlying diffusion is given by $L\phi = \sum b_k \partial_k \phi + \frac{1}{2}\sum(\sigma^*\sigma)_{kl}\partial_{kl}\phi$ and B_t is the d - dimensional Brownian motion used in the construction of (X_t^N). We can see that the times to hit zero are ordered a.s. for the three processes $\tau_z^0 \leq \tau_y^0 \leq \tau_X^D$, where τ_X^D is the hitting time of the absorbing boundary by the process $X_t^{N,i}$.

In the next estimate we use he boundedness of the coefficients of the operator L. Let $\alpha_0 \geq 0$ and $\beta_0 \geq 0$ be bounds for the coefficients

$$\alpha_0 = \sup_{x\in D} |L\psi(x)| , \quad \beta_0^2 = \sup_{x\in D} ||\sigma^*\sigma|| \, ||D^2\psi(x)|| \tag{5.9}$$

where the norms are the sum of the maximum of all elements of a matrix/vector, depending on ψ and its derivatives, and L.

It remains to evaluate, for an initial value $X_\tau^{N,i}$ as prescribed in the lemma, the sequence of upper bounds

$$P(\tau_X^D \leq \tau + \eta | X_\tau^{N,i}) \leq P(\tau_z^0 \leq \tau + \eta | X_\tau^{N,i}) \leq P(\inf_{t\in[\tau,\tau+\eta]} z_t \leq 1 - d_a | z_\tau = 1)$$

$$\leq P(\sup_{t\in[\tau,\tau+\eta]} |z_t - 1| \geq d_a | z_\tau = 1)$$

$$\leq P(\sup_{t\in[\tau,\tau+\eta]} |\int_\tau^t \beta_s dw_s| \geq d_a - \alpha_0\eta) \leq \left(\frac{\beta_0}{d_a-\alpha_0\eta}\right)^2 \eta \leq \frac{4\beta_0^2}{d_a^2}\eta$$

as soon as $0 < \eta < \frac{\beta_0}{2\alpha 0}$. Taking $c(q) = \frac{2\alpha_0}{\beta 0} \vee \frac{4\beta_0^2}{d_a^2}$ we conclude the proof. □

We move on to prove the tightness for both the empirical measure and the number of boundary hits. Additionally, we shall prove that for each fixed index i, $A_t^{N,i}$ is tight.

Naturally $A_t^{N,i}(\omega)$, $1 \leq i \leq N$ and their average $A_t^N(\omega)$, $\omega \in \Omega$, are random variables for all $t \in [0,\infty)$ and we omitted the sample space element ω to simplify notation. If (5.10)-(5.11) are satisfied, then a limit point $(A_t(\omega))_{t\geq 0}$ is a stochastic process with almost surely continuous paths. We can also verify that in this particular case, it is non-decreasing.

Proposition 3. *Assume $\mu_0^N \Rightarrow \mu_0$ and $\mu_0 \in M_1(D)$. Then, for any arbitrary but fixed $T > 0$,*

$$\limsup_{N\to\infty} E[A_T^N] < +\infty \tag{5.10}$$

$$\lim_{\eta\to 0} \limsup_{N\to\infty} \sup_{t\in[0,T]} P(A_{t+\eta}^N - A_t^N > \epsilon) = 0\,. \tag{5.11}$$

Remarks. 1) Evaluating (5.11) is based on the argument from line (5.13), which is a form of Wald's theorem for non-iid random variables $(\tau_X^D)_i$, $i \geq 1$, the waiting times between visits to the absorbing boundary. Independence is replaced by the condition in Lemma 2 and the strong Markov property.

2) Condition (5.11) is stronger than Aldous's criterion. It says cf. [14] that $(A_\cdot^N)$ is C - tight in the Skorokhod space, i.e. tight and that any limit point is continuous in time. Alternatively, if tightness is shown in the Skorokhod space, we recall that the *maximum jump size* $J_T(\omega(\cdot))$ of a path in D is a continuous functional in the *Skorokhod* J_1 - norm (not the same as the notation used below for the first jump). Since the jumps of $A_\cdot^N$ are at most of size $1/N$, it follows that a limit point $A.$ is continuous. This approach would prove immediately that $\mu.(dx)$ is also continuous in time.

Proof. Let $t \in [0,T]$, $\eta > 0$ and $J_1 < J_2 < \ldots$ be the ordered jump times after t. Then

$$A_{t+\eta}^{N,i} - A_t^{N,i} = [1 + m^\gamma(J_1, t+\eta)]\mathbf{1}_{\{J_1\leq t+\eta\}}\,, \tag{5.12}$$

with $m^\gamma(s,t)$ denoting the number of episodes when $X_\cdot^i$ travels from the redistribution point with distribution γ to the absorbing boundary, observed in the time interval $(s,t]$,

$0 \le s \le t$. Recall that τ^D is the hitting time of the boundary at $x = 0$ by the underlying diffusion process. Applying the Markov property, we can start at the vector configuration X_t^N.

$$E[A_{t+\eta}^{N,i} - A_t^{N,i}] = \sum_{k=1}^{\infty} P(A_{t+\eta}^{N,i} - A_t^{N,i} \ge k)$$
$$\le E[P_{X_t^N}(\tau_X^D \le \eta)] + \sum_{k=1}^{\infty} E[P_{X_{J_1}^N}(m^\gamma(J_1, t+\eta) \ge k)]\,.$$

Notice that $\{X_{J_1}^{N,i} \sim \gamma\}$ has probability one. We condition on this event in order to emphasize the label i that undergoes a jump. The general term of the infinite sum can be bounded

$$P_{X_{J_1}^N}(m^c(J_1, t+\eta) \ge k \,|\, X_{J_1}^{N,i} \sim \gamma) \le P_{X_{J_1}^N}((\tau_X^D)_1 + \ldots (\tau_X^D)_k \le \eta \,|\, X_{J_1}^{N,i} \sim \gamma) \tag{5.13}$$
$$\le P_{X_{J_1}^N}(\max_{1\le l\le k}(\tau_X^D)_l \le \eta \,|\, X_{J_1}^{N,i} \sim \gamma) \le P_{X_{J_1}^N}((\tau_X^D)_k \le \eta \,|\, \mathcal{A}_{k-1}) P_{X_{J_1}^N}(\mathcal{A}_{k-1})\,.$$

where $\mathcal{A}_{k-1} = \{\max_{1\le l\le k-1}(\tau_X^D)_l \le \eta\}$. In our count, $J_2 - J_1 = (\tau_X^D)_1$, ending with the k-th episode between jumps $J_{k+1} - J_k = (\tau_X^D)_k$. Taking the expectation under the initial condition X_t^N and using the strong Markov property recursively, we get the further bound

$$E_{X_t^N}[\Pi_{l=1}^k P_{X_{J_l}^N}((\tau_X^D)_l \le \eta)] \le [c(\gamma)\eta]^k\,, \tag{5.14}$$

This is due to the fact that $X_{J_l}^{N,i}$, $l \ge 1$ starts with distribution γ, which allows using Lemma 2 recursively. Summarizing (5.13)-(5.14) we see that *independently of the configuration* X_t^N,

$$P_{X_t^N}(m^c(J_1, t+\eta) \ge k) \le [c(\gamma)\eta]^k\,, \qquad k \ge 1\,. \tag{5.15}$$

We obtained

$$E[A_{t+\eta}^{N,i} - A_t^{N,i}] \le E[P_{X_t^i}(\tau^D \le \eta)] + \frac{c(\gamma)\eta}{1 - c(\gamma)\eta}\,. \tag{5.16}$$

After summation and division by $N-1$,

$$E[A_{t+\eta}^N - A_t^N] \le \frac{1}{N-1}\sum_{i=1}^N E[P_{X_t^i}(\tau^D \le \eta)] + (\frac{N}{N-1})\frac{c(\gamma)\eta}{1 - c(\gamma)\eta} \tag{5.17}$$

To prove (5.10) we pick $\eta = [2c(\gamma)]^{-1}$. Then we put back to back at most $[\frac{T}{\eta}]+1$ intervals of length η to see that

$$E[A_T^N] \leq 2(\frac{N}{N-1})([\frac{T}{\eta}]+1) \leq 8c(\gamma)T\,. \tag{5.18}$$

We now turn to (5.11). Let $\delta > 0$ be an arbitrary number not exceeding $d_a/2$. Working on the first term

$$\begin{aligned} &\frac{1}{N}\sum_{i=1}^{N} E[P_{X_t^i}(\tau^D \leq \eta)] \\ &\leq \left[\sup_{dist(X_t^{N,i},\partial D)\geq\frac{\delta}{2}} P_{X_t^N}(\tau_X^D \leq \eta)\right] E[1-\frac{U_t(\frac{\delta}{2})}{N}] + E[\frac{U_t(\frac{\delta}{2})}{N}] \\ &\leq c(\delta)\eta + E[\frac{U_t(\frac{\delta}{2})}{N}]\,, \end{aligned} \tag{5.19}$$

where $c(\delta)$ refers to the constant corresponding to an initial value away from the absorbing boundary at least by δ.

To finalize the proof, we turn to (5.11). Let $0 < \eta_0 < \eta$, momentarily fixed. We split the interval $[0,T]$, to calculate

$$\begin{aligned} \sup_{t\in[0,\eta_0]} E[A_{t+\eta}^{N,i} - A_t^{N,i}] &\leq E[A_{2\eta_0}^{N,i} - A_0^{N,i}] = E[A_{2\eta_0}^{N,i}] \\ &\leq \frac{1}{N-1}\sum_{i=1}^{N} E[P_{X_0^i}(\tau^D \leq 2\eta_0)] + (\frac{N}{N-1})\frac{c(\gamma)(2\eta_0)}{1-c(\gamma)(2\eta_0)} \end{aligned} \tag{5.20}$$

and

$$\sup_{t\in[\eta_0,T]} E[A_{t+\eta}^{N,i} - A_t^{N,i}] \leq \sup_{t\in[\eta_0,T]} \left(\frac{1}{N-1}\sum_{i=1}^{N} E[P_{X_t^i}(\tau^D \leq \eta)]\right) + (\frac{N}{N-1})\frac{c(\gamma)\eta}{1-c(\gamma)\eta} \tag{5.21}$$

The first term on the right-hand side of these inequalities is reduced to a bound on the number of particles within $\delta > 0$, for (5.20), respectively $\delta' > 0$ for (5.21), as we did in

(5.19). Taking $\eta c(\gamma) < \frac{1}{2}$ and $N \geq 2$, we obtain

$$\begin{aligned}(5.22) \qquad \sup_{t\in[0,T]} E[A^{N,i}_{t+\eta} - A^{N,i}_t] &\leq \sup_{t\in[0,\eta_0]} E[A^{N,i}_{t+\eta} - A^{N,i}_t] + \sup_{t\in[\eta_0,T]} E[A^{N,i}_{t+\eta} - A^{N,i}_t] \\ &\leq [4c(\gamma) + 2c(\delta')](2\eta_0) + 2E[\frac{U_0(\frac{\delta'}{2})}{N}]\,, \\ &+ [4c(\gamma) + 2c(\delta)]\eta + 2\sup_{t\in[\eta_0,T]} E[\frac{U_t(\frac{\delta}{2})}{N}]\,.\end{aligned}$$

Lemma 1 (5.3) concludes the proof, by having the limits over $N \to \infty$, $\eta \to 0$, $\delta \to 0$, $\eta_0 \to 0$, and finally $\delta' \to 0$, in this order. □

In fact, we can prove more than (5.10).

Proposition 4. *For any $T > 0$, $\beta > 0$*

$$(5.23) \qquad M(\beta, T) = \limsup_{N\to\infty} E[e^{\beta A^N_T}] < \infty\,.$$

Proof. From Hölder's inequality we see that it is sufficient to prove the exponential bound for each tagged particle, where $i \leq N$ is fixed, i.e.

$$(5.24) \qquad M_i(\beta, T) = \limsup_{N\to\infty} E[e^{\beta A^{N,i}_T}] < \infty\,.$$

Let $\eta > 0$ be such that $\eta < (c(\gamma)e^\beta)^{-1}$. Assume, for a moment, that there exists a number $\bar{M}(\beta, \eta) > 0$, independent of N, such that for any $t \geq 0$, *independently of* X^N_t,

$$(5.25) \qquad E_{X^N_t}[e^{\beta A^{N,i}_\eta}] \leq \bar{M}(\beta, \eta)\,.$$

The uniformity in the initial condition is inherited from (5.15), which, in its turn, comes from Lemma 2.

The Markov property shows that

$$\begin{aligned}E[e^{\beta A^{N,i}_T}] &= E[E[e^{\beta(A^{N,i}_T - A^{N,i}_{T-\eta})} \,|\, \mathcal{F}_{T-\eta}]e^{\beta A^{N,i}_{T-\eta}}] \\ &= E[E_{X^N_{T-\eta}}[e^{\beta A^{N,i}_\eta}]e^{\beta A^{N,i}_{T-\eta}}] \leq \bar{M}(\beta, \eta)^{[\frac{T}{\eta}]+1} < \infty\,,\end{aligned}$$

an upper bound independent of N, proving that $M_i(\beta, T) < \infty$. It remains to show (5.25). Recall that (5.15) holds uniformly in the initial state X^N_t. Since

$$P_{X^N_t}\Big(A^{N,i}_\eta > \frac{\ln s}{\beta}\Big) \leq (c(\gamma)\eta)^{[\frac{\ln s}{\beta}]} \leq (c(\gamma)\eta)^{\frac{\ln s}{\beta} - 1} \leq (c(\gamma)\eta)^{-1} s^{\beta^{-1}\ln(c(\gamma)\eta)}$$

that

$$E_{X_t^N}[e^{\beta A_\eta^{N,i}}] = \int_1^\infty P_{X_t^N}\Big(A_\eta^{N,i} > \frac{\ln s}{\beta}\Big)\, ds \le (c(\gamma)\eta)^{-1} \int_1^\infty s^{-\beta^{-1}\ln(\frac{1}{c(\gamma)\eta})}\, ds < +\infty\,,$$

due to the choice of η. $\square$

Theorem 4. *Under the same conditions of Theorem 2, the pair $(\mu_\cdot^N, A_\cdot^N)_{N>1}$ is C - tight on $D([0,\infty), M_1(D)\times\mathbb{R}_+)$, i.e. is tight and the limit is continuous in time.*

Proof. We can apply (4.9) for $\phi(s,\cdot)\in\mathcal{D}_c(L)$ for two times $s=t$, $s=t'$ in $[0,T]$ with $0<t'-t<\eta$. There exist constants $K(c,\phi)$, $K(J,\phi)$, independent of t, N such that the squares of the martingales are bounded by $N^{-1}K(c,\phi)T$ for the continuous part and $N^{-1}K(J,\phi)A_T^N$ for the jump part. In similar fashion, the integrands of dt and dA_t^N parts are bounded by $K(c,\phi)\eta$, respectively $K(J,\phi)(A_{t'}^N - A_t^N)$. Due to Proposition 3, part (ii) of Definition 5 is satisfied. To obtain (i) we turn to (5.3) for g a smooth approximation of the indicator function of the complement of a compact set in D. The bound we need to prove is pointwise in t. For any $t>0$ we note that (5.3) is valid for any $0<t_0<T$. It is only the uniform bound that may not hold all the way to $t_0=0$, but that is not necessary for condition (i) of tightness. At $t=0$ the tightness comes from the initial condition (3.6) because μ_0 charges D and not the boundary. The C - tightness is true because conditions (i), (ii) in Proposition 3 imply that any limit point has continuous paths almost surely. $\square$

6. The Rescaled Process and identification of the limit

Define the pair $(\nu_\cdot^N, n_\cdot^N)$, obtained by the transformation

$$\nu_t^N = e^{A_t^N}\mu_t^N\,,\quad n_t^N = e^{A_t^N}\,,\qquad t\ge 0\,. \tag{6.1}$$

Proposition 5. *For any $T>0$ and any $\phi\in\mathcal{D}$ satisfying the boundary condition (2.3), the transformed process $(\nu_t^N)_{t\ge0}$ satisfies*

$$\lim_{N\to\infty} E\Big[\sup_{0\le t\le T}\Big|\langle\nu_t^N,\phi(t,\cdot)\rangle - \langle\nu_0^N,\phi(0,\cdot)\rangle - \int_0^t \langle\nu_s^N, L\phi(s,\cdot)\rangle\, ds\Big|\Big] = 0\,. \tag{6.2}$$

Proof. According to [14], let $\mathbf{Y}(t) = (Y_1(t),\cdots,Y_m(t))$ be an $m-$dimensional semimartingale and G a smooth function on $\mathbb{R}^m$. Denote

$$\tilde{\Delta}Y(t) = \sum_{0\le s\le t}\Big(Y(s) - Y(s-)\Big)$$

and $\langle (Y_k)^c, (Y_l)^c\rangle(s)$ the cross variation of the continuous martingale parts of $Y_k(t)$ and $Y_l(t)$. Then we have:

$$G(\mathbf{Y}(t)) - F(\mathbf{Y}(0)) = \sum_{l=1}^{m}\int_0^t \partial_l G(\mathbf{Y}(s-))dY_l(s) \tag{6.3}$$
$$+\frac{1}{2}\sum_{k,\,l=1}^{m}\int_0^t \partial_{kl}G(\mathbf{Y}(s-))d\langle (Y_k)^c, (Y_l)^c\rangle(s)$$
$$+\sum_{0\le s\le t}\Big[G(\mathbf{Y}(s)) - G(\mathbf{Y}(s-)) - \sum_{k=1}^{m}\partial_k G(\mathbf{Y}(s-))\tilde{\Delta}Y_k(s)\Big]\,.$$

Now fix N and let $\phi \in \mathcal{D}$ satisfying the boundary condition (2.3). Then apply the Ito Formula above with $m = 2$ and $\mathbf{Y}(t) = (Y_1(t), Y_2(t)) = (A_t^N, \langle \mu_t^N, \phi(t,\cdot)\rangle)$. The function G we are going to use is defined as $G(Y_1, Y_2) = e^{Y_1}Y_2$, for the purpose of eliminating dA_t^N term. Denoting $\nu_t^N = G(Y_1(t), Y_2(t))$, we want to prove (6.2).

Notice that the pure jump part subtracted in the third line of (6.3) can be omitted if we replace $dX_l(s)$ in the first line by the continuous part only. Since A_t^N is a pure jump process (the average of counting measures), with the notations from (4.9), the generalized Ito formula gives

$$\langle \phi(x), \nu_t^N\rangle - \langle \phi(x), \nu_0^N\rangle = \int_0^t e^{A_s^N} d\mathbf{X}_s^{\phi,c} + \tilde{\Delta}(\langle \nu_t^N, \phi(s,\cdot)\rangle)\,.$$

Since

$$\int_0^t e^{A_s^N} d\mathbf{X}_s^{\phi,c} = \int_0^t e^{A_s^N}\langle \mu_s^N, L\phi(s,\cdot)\rangle ds + \int_0^t e^{A_s^N} d\mathcal{M}_s^{\phi,c}\,,$$

the expression to be evaluated form (6.2) can be written

$$\nu_t^N, \phi(t,\cdot)\rangle - \langle \nu_0^N, \phi(0,\cdot)\rangle - \int_0^t \langle \nu_s^N, L\phi(s,\cdot)\rangle\, ds \tag{6.4}$$
$$= \int_0^t e^{A_s^N} d\mathcal{M}_s^{\phi,c} + \tilde{\Delta}(\langle \nu_t^N, \phi(s,\cdot)\rangle)\,. \tag{6.5}$$

To evaluate the pure jump part we write

$$\tilde{\Delta}(\langle \nu_t^N, \phi(s,\cdot)\rangle) = \sum_{0\le s\le t}\Big[G\Big(A_{s-}^N + \Delta A_s^N,\, \mathbf{X}_{s-}^{\phi,J} + \Delta\mathbf{X}_s^{\phi,J}\Big) - G\Big(A_{s-}^N, \mathbf{X}_{s-}^{\phi,J}\Big)\Big]\,.$$

Taylor's formula evaluates the jump

$$G(A+\Delta A, \mathbf{X}+\Delta\mathbf{X}) - G(A,\mathbf{X}) = e^A\mathbf{X}\Delta A + e^A\Delta\mathbf{X} + \frac{1}{2}\Big[e^{A'}\mathbf{X}'(\frac{1}{N})^2 + 2e^{A'}(\frac{1}{N})(\Delta\mathbf{X})\Big]$$

with $A = A_{s-}^N$, A' an intermediate point between A and $A + \frac{1}{N}$, and $\mathbf{X} = \mathbf{X}_{s-}^{\phi,J}$, $\mathbf{X}'$ an intermediate point between $\mathbf{X}$ and $\mathbf{X} + \Delta\mathbf{X}$. Notice that $\Delta A_s^N = 1/N$ if there is a jump and zero otherwise, while $|\Delta\mathbf{X}_s^{\phi,J}| \leq 4||\phi||/N$ and $|\mathbf{X}| \leq ||\phi||$. This implies that there exists a constant C_1 independent of N, $t$such that the last two terms in brackets are uniformly bounded by $N^{-2}C_1\|\phi\|\exp(A_s^N)$. Summing up over all jumps we obtain a term of order at most

$$\frac{C_1\|\phi\|}{N} A_t^N \exp(A_t^N)\,,$$

where a factor of $1/N$ was absorbed into A_s^N.

At this point we emphasize that ϕ satisfies the boundary condition (2.3). Formula (4.5) in the context of the empirical measure, appearing in (4.9), shows the pure jump part equals

$$\Delta\mathbf{X}_s^{\phi,J} = \Big[\Big(2\langle\gamma,\phi(s,\cdot)\rangle - \phi(s,\mathfrak{b}) - \mathbf{X}_{s-}^{\phi,J}\Big) - \frac{2}{N}\langle\gamma,\phi(s,\cdot)\rangle\Big]\Delta A_s^N + \Delta\mathcal{M}_s^{\phi,J}\,.$$

The linear part

$$e^A\mathbf{X}\Delta A + e^A\Delta\mathbf{X} = e^{A_{s-}^N}\mathbf{X}_{s-}^{\phi,J}\Delta A_s^N + e^{A_{s-}^N}\Delta\mathbf{X}_s^{\phi,J}$$

equals

$$-\frac{2}{N}\langle\gamma,\phi(s,\cdot)\rangle e^{A_s^N}\Delta A_s^N + e^{A_s^N}\Delta\mathcal{M}_s^{\phi,J}$$

showing that

$$|\tilde{\Delta}(\langle\nu_t^N,\phi(t,\cdot)\rangle)| \leq \frac{2\|\phi\|}{N}\int_0^t e^{A_{s-}^N} dA_s^N + \Big|\int_0^t e^{A_{s-}^N} d\mathcal{M}_s^{\phi,J}\Big| + \frac{C_1\|\phi\|}{N} A_t^N e^{A_t^N}\,. \tag{6.6}$$

To evaluate the supremum over $0 \leq s \leq t$ of the martingale part (now sub-martingale due to the absolute value) we use (4.8) with the observation that all jumps are of size at most $4\|\phi\|N^{-1}$; in the quadratic variation they are squared and summed up over $i = 1\ldots N$, yielding a term of order N^{-1}. More precisely, employing Doob's L^2 maximal inequality we have

$$E\Big[\sup_{0\leq s\leq T}\Big|\int_0^T e^{A_{s-}^N} d\mathcal{M}_s^{\phi,J}\Big|^2\Big] \leq \frac{C_2\|\phi\|^2}{N} E\Big[\int_0^T e^{2A_{s-}^N} dA_s^N\Big]\,.$$

Let $C_3(T) = E[A_T^N e^{2A_T^N}]$, the largest integral appearing in the three terms of the right hand side of (6.6). It is finite due to Proposition 4. We proved that the jump part (6.6) satisfies

$$E\Big[\sup_{0\leq s\leq T}|\tilde{\Delta}(\langle\nu_t^N,\phi(t,\cdot)\rangle)|\Big] \leq \frac{\|\phi\|}{\sqrt{N}}\left(\frac{2C_3(T)}{\sqrt{N}} + \sqrt{C_2C_3(T)} + \frac{C_1C_3(T)}{\sqrt{N}}\right) \leq \frac{C_4(T,\phi)}{\sqrt{N}}\,. \tag{6.7}$$

The last bound depends explicitly only on ϕ and N. Finally, it is easy to check that the quadratic variation of the continuous martingale is bounded above by $\frac{C_5 \cdot T \sup \|\nabla \phi\|}{\sqrt{N}}$, where the constant depends on the coefficients of the operator L.

We have proven that, in absolute value, the right hand side of (6.4) satisfies a uniform bound of the form $\frac{C_6(T,\phi)}{\sqrt{N}}$, with the constant depending only on T, L, and the suprema of ϕ and its derivatives. By letting $N \to \infty$ we conclude the proof. $\square$

Proposition 6. *The pair $(\nu_\cdot^N, n_\cdot^N)$ defined in (6.1) is C - tight and $\nu_\cdot^N$ has hydrodynamic limit, in the sense of Definition 4, equal to the solution $\nu_\cdot$ to (2.2)-(2.3) and $n_\cdot^N$ converges in probability to the total mass $n_\cdot$.*

Proof. Tightness. The exponentials $\exp \beta A_t^N$, $0 \leq t \leq T$, $\beta > 0$ will remain bounded in expectation due to Proposition 4. All possible integrands in (6.4), including in the quadratic variations of the martingales, are dominated by constant multiples of $\exp A_T^N$ or $A_T^N \exp A_T^N$, both bounded above by $\exp 2A_T^N$. Then, the tightness of the pair $(\mu_\cdot^N, A_\cdot^N)$ (Theorem 4) implies tightness of the transformed variables (6.1).

Identification of the limit using the Portmanteau theorem. Denote a generic element of $\mathbb{D}([0,\infty), M_F(D))$ by σ. Given any $\phi \in \mathcal{D}$ satisfying (2.3), using the notation $\sigma_s \in M_F(D)$ for the value at time $s \in [0,\infty)$, define the functional $\Phi : \mathbb{D}([0,\infty), M_F(D)) \to \mathbb{R}$

$$\Phi(\sigma_\cdot) := \sup_{t \in [0,T]} \left| \langle \sigma_t, \phi(t,\cdot)\rangle - \langle \sigma_0, \phi(0,\cdot)\rangle - \int_0^t \langle \sigma_s, \frac{\partial}{\partial s}\phi(s,\cdot) + L\phi(s,\cdot)\rangle \, ds \right| . \tag{6.8}$$

For arbitrary $M > 0$ define $\Phi_M(\sigma_\cdot) = \Phi(\sigma_\cdot) \wedge M$. Now Φ_M is bounded and continuous. We practically follow steps 2-4 of Proposition 2 in [9].

We are now ready to apply Portmanteau theorem. Equation (6.2) established in Proposition 5 shows that

$$\lim_{N \to \infty} E[\Phi_M(\nu_\cdot^N)] = 0\,. \tag{6.9}$$

Let $(\nu_\cdot^\circ, n_\cdot^\circ)$ be a limit point of the tight pair of transformed processes. To simplify notation, we use the same index N for the subsequence converging to the limit point. Since $(\nu_\cdot^N, n_\cdot^N) \Rightarrow (\nu_\cdot^\circ, n_\cdot^\circ)$ and Φ_M is continuous and bounded, we obtain that $E[\Phi_M(\nu_\cdot^\circ)] = 0$. The monotone convergence theorem for $M \to \infty$ proves

$$E[\Phi(\nu_\cdot^\circ)] = 0 \quad \text{and then} \quad \Phi(\nu_\cdot^\circ) = 0 \quad a.s. \tag{6.10}$$

It is sufficient to remark that, being C - tight, the limit is continuous in time. It follows that we can pick a set of measure zero, common to all $T > 0$.

It is important to recall that ϕ was chosen to satisfy the boundary condition (2.3). As a consequence, ν_t° solves the heat equation with mass creation (2.2). Moreover, the proof of tightness and specifically (5.3) in Lemma 1 show that $\nu_\cdot^\circ$ is regular. By uniqueness (Theorem 1), $\nu_\cdot^\circ = \nu_\cdot$ almost surely.

We move on to $n_\cdot^N$. According to the first part of this proof, the tightness is inherited from Theorem 4 which proves that any limit point, and thus $\nu_\cdot$, is an element of $M_F(D)$, i.e. has all its mass concentrated on D (more is true, as we said, since $\nu_\cdot$ is regular). This and the fact D is bounded, allows us to use an arbitrary test function $\phi \in C(\bar{D})$ to write the bounded continuous functional on $\mathbb{D}([0,\infty), M_F(D))$

$$\Psi(\sigma_\cdot) := \sup_{t\in[0,T]} \Big| \langle \sigma_t, \phi(\cdot)\rangle - \langle \nu_t, \phi(\cdot)\rangle \Big| \,, \tag{6.11}$$

where we note that $\nu_\cdot$ is the deterministic solution and $\sigma_\cdot$ is the variable. Write $\psi_M = \Psi \wedge M$, $M > 0$ and repeating the reasoning on Φ_M we obtain that

$$\lim_{N\to\infty} E[\Psi_M(\nu_\cdot^N)] = E[\Psi_M(\nu_\cdot)] = 0\,.$$

Adopting $\phi(x) = 1$ we obtain the same for $n_\cdot^N = \exp A_\cdot^N$ and $n_t = \exp A_t$. Then $\Psi_M(n_\cdot^\circ) = 0$ and letting $M \to \infty$ once again $n_\cdot^\circ = n_\cdot$ almost surely. This concludes the identification of the limit.

We still need to prove convergence as in (3.5), Definition 3. Notice that $\Psi(\nu_\cdot^N) \leq [e^{A_T^N} + e^{A_T}]\|\phi\|$, using the supremum norm of ϕ.

$$E[\Psi(\nu_\cdot^N)] \leq E[\Psi(\nu_\cdot^N), A_T^N \leq L] + E[e^{A_T^N} + e^{A_T}]\|\phi\|, A_T^N > L]\,.$$

The second term is bounded above by

$$e^{-L}\|\phi\| e^{A_T} E[e^{2A_T^N}] \leq e^{-L}\|\phi\| e^{A_T} M(2,T)\,.$$

based on (5.23) with $\beta = 2$. Adopting $M = [e^L + e^{A_T}]\|\phi\|$, the first term is bounded by $E[\Psi_M(\nu_\cdot^N)]$. Letting $N \to \infty$ and then $L \to \infty$ we obtained $\lim_{N\to\infty} E[\Psi(\nu_\cdot^N)] = 0$, which immediately implies weak convergence in probability (3.5) for $\nu_\cdot^N$. Putting $\phi \equiv 1$, the same is shown for $A_\cdot^N$, completing the proof. $\square$

6.1. **Proof of Theorem 2.** At this point we have to reverse the transformation from Proposition 6. We notice that trivially both $n_t^N \geq 1$ and $n_t \geq 1$ and such have a lower bound away from zero. Setting $Y_t^N = A^N = \ln n_t^N$, with its limit $\ln n_t = A_t$, since $x \to \ln x$ is uniformly Lipschitz on $x \in [1, \infty)$, weak convergence in probability (3.5) for $A_\cdot^N$ is implied by the same for $n_\cdot^N$.

On the other hand we see that if $H(a,b) = e^{-a}b$ then the mean value theorem gives

$$|\langle \mu_t^N, \phi\rangle - \langle \mu_t, \phi\rangle| = |H(A_t^N, \langle \nu_t^N, \phi\rangle) - H(A_t, \langle \nu_t, \phi\rangle| \leq |A_t^N - A_t| + |\langle \nu_t^N, \phi\rangle - \langle \nu_t, \phi\rangle|\,,$$

which implies (3.5) is satisfied for $Y_t^N = \langle \mu_t^N, \phi\rangle$ and its limit $Y_t = \langle \mu_t, \phi\rangle$. □

7. Sketch of the tagged particle limit

The material proved in Sections 5 and 6 allows to develop the scaling limit of the tagged particle. We do not prove the result here, leaving it to an upcoming paper. However, we formally identify the limit in Subsection 7.2. The technical steps are outlined in a result we obtained in [11].

Fix the particle tag i and consider $N \geq i$ or simply take $i = 1$. We are interested in proving

$$X_\cdot^N \Rightarrow X. \tag{7.1}$$

and identifying the limit X_t as a stochastic process indexed by $t \geq 0$. These results require both convergence in distribution of $\mu_0^N \Rightarrow \mu_0$ and $X_0^{N,1} \Rightarrow X_0^1$.

7.1. **Tightness.** To prove the tightness *of each individual particle's number of visits to the absorbing boundary* $(A_t^{N,i})$, which is well defined for $N \geq i$, but of course is not continuous, even in the limit, we turn to the tightness criterion for processes in the Skorokhod space.

Proposition 7. *Let $i \in \mathbb{N}$ fixed and assume $X_0^{N,i} \Rightarrow X_0^i$ with $P(X_0^i \in dx) \in M_1(D)$. Then, $(A_t^{N,i})$ is tight, verifying, for any $T > 0$*

$$\sup_{N\geq 1} E[A_T^{N,i}] < +\infty \tag{7.2}$$

$$\lim_{\eta\to 0}\limsup_{N\to\infty}\sup_{t\in[0,T]} P(w'_{A^{N,i}}(\eta) > \epsilon) = 0\,. \tag{7.3}$$

Proof. Being a counting process, it follows that the only way the modulus of continuity $w'_{A^{N,i}}(\eta)$ in the Skorokhod J_1 - topology would exceed $\epsilon > 0$ is that it is at least one. More precisely, the union of the hitting times and the initial $t = 0$ must contain at least two elements within distance η. Otherwise, we can always optimize the partition of mesh η as to include that times and in that case $w'_{A^{N,i}}(\eta) = 0$. Then, either

(i) *there are at least two hits to the boundary* in $[t, t+\eta]$, or

(ii) *there is exactly one hit, but within η from $t = 0$.*

In case (i), the particle is redistributed, meaning that $\{\tau_X^D \leq \eta\}$ is a sub-event, fitting the exact conditions of Lemma 2 with $q(dx) = \gamma(dx)$.

In case (ii) $\{\tau_X^D \leq \eta\}$ is a sub-event as well. We split

$$P(\tau_X^D \leq \eta) = E[P_{X_0^N}(\tau_X^D \leq \eta\,,\, dist(X_0^{N,i}, \partial D) \geq \delta)]$$

$$+E[P_{X_0^N}(\tau_X^D \leq \eta\,,\, dist(X_0^{N,i}, \partial D) \geq \delta)] \leq c(\delta)\eta + P(dist(X_0^{N,i}, \partial D) < \delta)\,,$$

because the first term in the upper bound fits the exact conditions of Lemma 2 as, for example, it was applied in (5.19), while the second term will be vanishing due to the continuity theorem and the assumption that the initial point converges in distribution to a value that does not charge ∂D. □

Using Propositions 3 and 7 we write the differential equation for the test function corresponding to the tagged particle, i.e. of the form $F(X) = \phi(X_1)$, $\phi \in C^2(\bar{D})$. All integrands are bounded, and the integrators in time are either the Lebesgue measure dt or one of the counting measures A_t^N or $A_t^{N,1}$. It follows that $(X^{N,1})_{N>1}$ is tight. Moreover, it satisfies the following martingale problem, defining a Markov process which is time inhomogeneous.

We know from Theorem 4 that $A_\cdot^N$ converges in probability to the deterministic, continuous, increasing function $A_\cdot$. From Theorem 1 we know that the total mass defines an absolutely continuous measure $dA_t = a_t dt$. This measure induces a non-homogeneous Poisson measure $\alpha(t)$ with jumps at times $A^{-1}(\theta)$, where θ are the jumps of a Poisson process of intensity one. By construction, this process can be independent from a countable sequence of mutually independent diffusions $(L, \mathcal{D}(L))$, which will serve as building blocks between jumps.

7.2. **The law of the tagged particle.** The tagged particle process $(X_t^1)_{t\geq 0}$ starts at X_0^1. It moves according to the diffusion $(L, \mathcal{D}(L))$ until the minimum of either the first arrival in $\alpha(t)$ or the first hitting time of the absorbing boundary. At such times, it instantaneously redistributes to a random point with distribution $\gamma(dx)$ and continue until the next jump time, dictated by the minimum described above. The process is well defined because no two jumps are simultaneous, and visits to the boundary are sufficiently far apart due to the tightness argument on $A^{N,1}_{\cdot} \Rightarrow A^1_{\cdot}$. The limit $A^1_{\cdot}$ is the number of visits to the absorbing boundary by the tagged particle $(X^1_t)_{t\geq 0}$, but its total average number of jumps is $\alpha(t) + A^1(t)$.

8. Appendix

Theorem 2, the main result of this paper, uses a partial differential equations result summarized in Theorem 1, which is proven in detail in [12].

The existence of the weak solution $(\nu_t)_{t\geq 0}$ of Definition 1, eq. (2.2)-(2.3) is based on the construction of the auxiliary branching process (ζ_t) and relation (8.2). In short, when $\nu_0 = \delta_x$, $x \in D$, the branching process is well defined and equals the expected value of the measure valued process starting with one particle $\nu^x_t = E_x[\zeta_t]$, $t \geq 0$.

8.1. **The auxiliary processes Z_t and ζ_t.** In this section, we outline the construction of a particle system Z_t having a random total number of particles N_t. This is a counting process resulting from branching upon reaching the boundary of the domain. In that sense, our dynamics, including the conservative process (X^N_t) given in (3.1), is intimately related to super-critical branching. See the comments in Subsection 3.3. This states that the expected value of the empirical measure, seen as finite measure-valued random trajectory, is the solution to (2.2)-(2.3). The formal construction, definition, and proof of the regularity properties of this process, as well as related questions to its evolution semigroup, are done in [12].

At $t = 0$, a single particle is placed at a random point with distribution $m_0(dx) \in M_1(D)$. The particle, starts moving according to $(L, \mathcal{D}(L))$, until it reaches ∂D, when it dies. Instantaneously, two particles are born at the same random point in D chosen with distribution γ. All particles start afresh and continue an independent motion in D until the first one dies and the branching is repeated. We note that particles depend on each other only through ancestry, and not through their motion.

We shall make the convention that a particle hitting the absorbing boundary jumps, instead of being killed upon contact, which makes particle labelling easier. Then each particle has a Markovian motion once it is born, namely the *Brownian motion with rebirth* introduced in [7], also studied in [8, 4]. Under (1.4), the particle system is well defined, having a constant number of particles between branchings. The branching times for a strictly increasing sequence, since they never coincide; all with probability one. We assume it is defined on a filtered probability space, and built constructively, up to the limit of the strictly increasing sequence of branching times, denoted by τ^*, a stopping time in $[0, +\infty]$.

The model can be easily generalized to have a random number K of offspring created at the recombination point, including a smaller number than one, leading to the possibility of dissipation of mass (e.g. K may be Poisson distributed), but we shall only consider a number of exactly $K = 2$ for our purpose of representing the solution of (2.2)-(2.3).

The first particle is denoted Z_t^1, the second Z_t^2, and so on. Let the number of particles at time t be denoted N_t, which, only in this special case, coincides with the number of branchings - a feature that while convenient, is not essential to the construction.

In principle, τ^* could be finite with positive probability, in which case the system is said *explosive*. In [12] it is shown that this is not the case. Corollary 1 to Theorem 1 in [12] (here Theorem 1 (i)) gives the exact bound $||\nu_t|| \leq ||\nu_0|| e^{\alpha_* t}$ for the total variation of the solution present in the regularity condition (2.6), where $\alpha = \alpha_* > 0$ solves (3.12). This implies that N_t has exponential moments up to the critical value $\alpha_* > 0$.

Denote the empirical measure

$$\zeta_t = \sum_{i=1}^{N_t} \delta_{Z_t}. \tag{8.1}$$

This is a finite measure-valued Markov process, i.e. living on $\mathbb{D}([0, \infty), M_F(D))$. In the technical construction, the state space is not the whole $M_F(D)$, but a strict subset denoted $M_0(D)$, the space of discrete measures on D. This aspect is not important in the present paper. For a detailed construction we point to [12].

Based on the exponential estimate on N_t we define the expected value $\nu_t^x(dx)$ of the empirical measure of the process $(Z_t)_{t \geq 0}$ starting with one particle at x. Technically, we should denote this initial point by the non-random delta measure δ_x, for consistency with the measure valued setup. We can see that $x \to \nu_t^x(dx)$ is continuous in the topology of weak convergence and then the second integral in (8.2) is well defined.

For a bounded test function ϕ and a probability measure $\nu_0(dx) = v_0(x)dx \in M_1(D)$, we put

$$\langle \nu_t^x, \phi(t,\cdot)\rangle := E_x[\sum_{j=1}^{N_t} \phi(t, Z_t^j)], \qquad \nu_t^{v_0} = \int_D v_0(x)\nu_t^x dx\,. \tag{8.2}$$

Then, by uniqueness, the function $\nu_t^{v_0}$ is the stochastic representation of the weak solution ν_t of the heat equation with particle creation at $\gamma(dx)$ satisfying (2.2)-(2.3). The solution has the regularity properties of Theorem 1. Moreover, if the total mass is $n_t = \langle \nu_t, 1\rangle$, then $n_t > 0$ for all $t \geq 0$ and if $\nu_0 \in M_1(D)$ then $n_t > 1$ for $t > 0$, as well as differentiable with continuous derivative. This justifies the definition $\ln n_t = \int_0^t a_s ds$.

8.2. **The strong solution.** In terms of analytic properties, a strong solution exists only outside the support of γ. In that case we can formulate the boundary conditions for the density, i.e. the forward equation, in terms of the flux balance. The second main result in this paper, Theorem 3, is based on the following result.

Theorem 5 (Theorem 3 in [12])**.** *Let $L = \frac{1}{2}\Delta$, $\gamma(dx) = \delta_c(dx)$ for some $c \in D$ and $\nu_0(dy) = v_0(y)dy$, $v_0 \in C(\bar{D})$. Then the solution from Theorem 1 has a density, i.e. $\nu_t(dy) = v(t,y)dy$, integrable in the space variable for any $t \geq 0$ with $v \in C([0,\infty) \times \overline{D} \setminus \{c\}) \cap C^{1,2}((0,\infty) \times D \setminus \{c\})$ which is a solution of $\partial_t v = \frac{1}{2}\Delta v$ on $D \setminus \{c\}$ with $v(t,y)|_{\partial D} = 0$ satisfying the* flux balance condition *(3.9).*

We conclude with a simple example in the one dimensional case. Let $D = (0,1)$, $\partial D = \{0,1\}$, $\gamma = \delta_c$, $c \in (0,1)$ and $L = \frac{1}{2}\frac{d^2}{dy^2}$ with $\nu_0(dx) = v_0(x)dx$. Then $L = L^*$, $\nu_t(dy) = v(t,y)dy$ with $v(0+,\cdot) = v_0(\cdot)$ and v has continuous time derivative. In addition, one can verify directly that for any $t > 0$, v is smooth in $(0,c) \cup (c,1)$ and satisfies the boundary conditions

$$\begin{aligned} \forall t > 0 \qquad v(t,c-) = v(t,c+)\,, \quad v(t,0) = v(t,1) = 0 \\ (v'(t,c+) - v'(t,c-)) + 2v'(0) = 0\,. \end{aligned} \tag{8.3}$$

A similar case (with reflection at $x = 1$) is studied in [18] with some additional considerations on the quasi-invariant measure.

References

[1] Asselah, Amine; Ferrari, Pablo A.; Groisman, Pablo; Jonckheere, Matthieu; *Fleming-Viot selects the minimal quasi-stationary distribution: The Galton-Watson case.* Ann. Inst. Henri Poincare Probab. Stat. 52 (2016), no. 2, 647-668.

[2] P. Bak and K. Sneppen, *Punctuated equilibrium and criticality in a simple model of evolution*, Physical Review Letters 71 (1993) (24): 4083-4086.

[3] P. Bak, *How Nature Works: the Science of Self-organized Criticality*, Copernicus; 1 edition, April 23, 1999.

[4] Ben-Ari, Iddo; Pinsky, Ross G. *Spectral analysis of a family of second-order elliptic operators with nonlocal boundary condition indexed by a probability measure.* J. Funct. Anal. 251 (2007), no. 1, 122-140.

[5] Burdzy, K.; Holyst, R.; March, P. *A Fleming-Viot particle representation of the Dirichlet Laplacian.* (2000) Comm. Math. Phys. 214, no. 3.

[6] Ferrari, P. A.; Maric, N. *Quasi stationary distributions and Fleming-Viot processes in countable spaces.* (2007) Electron. J. Probab. 12, no. 24, 684–702

[7] Grigorescu, Ilie; Kang, Min *Brownian motion on the figure eight.* J. Theoret. Probab. 15 (2002), no. 3, 817-844.

[8] Grigorescu, Ilie; Kang, Min *Ergodic properties of multidimensional Brownian motion with rebirth.* Electron. J. Probab. 12 (2007), no. 48, 1299-1322.

[9] Grigorescu, Ilie; Kang, Min. *Hydrodynamic limit for a Fleming-Viot type system* Stochastic Process. Appl. 110 (2004), no. 1, 111-143.

[10] Grigorescu, Ilie; Kang, Min *Markov processes with redistribution.* Markov Processes and Related Fields (2013), Volume 19, Issue 3, pages 497-520. (2013)

[11] Grigorescu, Ilie; Kang, Min *Tagged particle limit for a Fleming-Viot type system.* Electron. J. Probab. 11 (2006), no. 12, 311-331.

[12] Grigorescu, I; Song, Y. *Particle representation for the heat equation with mass creation* (2019) Preprint.

[13] Grigorescu, Ilie; Kang, Min *Quasi-stationarity for a non-conservative evolution semigroup*, (2019) Preprint.

[14] J. Jacod and A. Shiryaev, *Limit Theorems for Stochastic Processes*, 2nd ed., Springer, Berlin Heidelberg, 2003.

[15] Meleard, Sylvie; Villemonais, Denis Quasi-stationary distributions and population processes. Probab. Surv. 9 (2012), 340-410.

[16] L.C.G. Rogers and D. Williams, *Diffusions, Markov Processes and Martingales*, Vol. 1, 2nd ed.,Cambridge University Press 2000.

[17] Villemonais, Denis *Interacting particle systems and Yaglom limit approximation of diffusions with unbounded drift.* Electron. J. Probab. 16 (2011), no. 61, 1663-1692.

[18] Song, Yishu. *Hydrodynamic Limit of Bak-Sneppen Branching Diffusions*(2016) Ph.D. Thesis. Open Access Dissertations. Paper 1702. http://scholarlyrepository.miami.edu/oa_dissertations/1702