QUASI-STATIONARITY FOR A NON-CONSERVATIVE EVOLUTION SEMIGROUP

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Abstract. We investigate a non-conservative semigroup $S_t = E[\zeta_t]$ determined by a branching process (ζ_t) evolving in on an open domain $D \subseteq \mathbb{R}^d$. Branching occurs upon exiting D. Between branching the particles are driven by the a Dirichlet transition kernel $p^D(t,x,y)$. When a particle is killed at the boundary, a random number K is born at a point in D with distribution γ. We determine the exact exponential rate of the total mass $n_t = S_t\mathbf{1} \sim \exp(\alpha^* t)$ as a function of $\bar{K} = E[K]$ in all regimes - super/sub critical according to the sign of $\bar{K} - 1$. We prove the Yaglom limit $S_t/n_t \to \nu$. The qsd ν is proportional to the resolvent of the Dirichlet kernel at a point α^* in bijection to $\bar{K}$, spanning the real part of the resolvent set. The problem is motivated by the Bak-Sneppen branching diffusions (BSBD) [15] and the Fleming-Viot particle models. The semigroup, divided by its total mass, gives the hydrodynamic limit of the BSBD with branching intensity $\bar{K}$. Since ν is the asymptotic profile under equilibrium, the family of qsd ν, indexed by $\bar{K}$, provides an explicit example of self-organizing equilibrium. In addition, an extension of the Wiener-Ikehara Tauberian Theorem for a class of non-increasing functions is proved in the Appendix.

1. Introduction

This paper is the third in a sequence, motivated by the study of the limiting profile of the Bak-Sneppen branching diffusions (BSBD), a particle system introduced in [15]. The second paper [16] constructs an auxiliary super-critical branching process (ζ_t) whose normalized semigroup $S_t = E[\zeta_t]$, i.e. divided by its total mass, gives the hydrodynamic

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limit of the BSBD. Here, we aim to extend the construction from [16] to the subcritical regime and analyze the behavior of the semigroup as $t \to \infty$, including its quasi-stationary distributions, Yaglom limits and the relation to the BSBD.

Let $p^D(t,x,y)$ be the Dirichlet kernel of a diffusion on the open domain $D \subseteq \mathbb{R}^d$ with smooth boundary, a process we shall call the *underlying diffusion*. Let $\gamma(dx)$ be a probability measure on D that does not charge the boundary (i.e. $\gamma(D) = 1$).

The process $(\zeta_t)_{t\geq 0}$ lives on the space of finite configurations on D. An element of the state space can be seen as a sum of delta measures at the locations of a finite number of particles distributed randomly in D. In time, the particles evolve independently according to $p^D(t,x,y)$, until the first one hits the boundary ∂D. Instantaneously, that particle is killed and a random integer number $K \geq 0$ of new particles, independent of the evolution up to that moment, is born at a point chosen with distribution γ. The new finite set of particles starts a new episode until the next boundary hit. We note that, almost surely, no simultaneous boundary visits occur. Under mild conditions on the semigroup, the duration of an episode is positive, and the branching events can be ordered in strictly increasing order with probability one. The motion continues up to either extinction at $T_{ext} > 0$ or explosion at $T_\infty > 0$, both possibly infinite. In fact, we shall prove that the number of particles is almost surely finite at any $t > 0$ and thus $T_\infty = +\infty$.

For a test function ϕ, we write $S_t\phi(x) = E_x[\langle \zeta_t, \phi\rangle]$ and $n_t^x = S_t\mathbf{1}(x)$ for the total mass, with bracket notation for the integration over D and $\mathbf{1}(x)$ the constant function equal to one. The main results are Theorems 1, 2 and 3. Theorem 1 determines the exact asymptotic rate α^*, a number with the same sign as $\bar{K} - 1$ (determining criticality) such that $n_t^x \sim e^{\alpha^* t}$. Theorem 2 provides explicit formulas for the semigroup and resolvents. The precise correspondence between α^* and $\bar{K}$ is given in Proposition 1.

Theorem 3 proves the limit

$$\lim_{t\to\infty} \frac{S_t\phi(x)}{S_t 1(x)} = \int_D \phi \, d\nu \qquad \text{(Yaglom limit)}\,,$$

and identifies the unique quasi-stationary distribution (qsd) ν as the probability measure having density, modulo a normalizing constant, equal to $\langle \gamma, r^D_{\alpha^*}\rangle$. Here the resolvent kernel r^D_α of p^D is integrated over γ and evaluated at the value α^* which depends on $\bar{K}$. On the technical side, we establish a Tauberian result in Lemma 1, a generalization of the Wiener-Ikehara Theorem when the functions involved are not necessarily increasing.

1.1. **Connection to particle systems and self-organizing equilibrium.** In the BSBD, $N \geq K \geq 1$ are positive integers, N fixed, deterministic and K possibly random. Denote $\mu^N(t, dx)$ the empirical measure of N particles that evolve as follows. Inside D they move independently according to $p^D(t, x, y)$. When a particle reaches ∂D, it is killed *together with* $K - 1$ *particles among the remaining* $N - 1$. All the K particles killed are re-born at a random point inside D with distribution $\gamma(dx)$ and restart the motion until the next boundary hit. The total mass is preserved and $\mu^N(t, dx)$ is a probability measure. We emphasize that, as opposed to (ζ_t), the BSBD is conservative.

In [15], the number $K = 2$ is non-random, but it can equally be, for example, binomial with arbitrary mean value. The dynamics combines the Fleming-Viot random redistribution and the mean-field Bak-Sneppen evolutionary fitness model [4]. In such a framework, two limits are of interest for $\mu^N(t, dx)$.

One, as $N \to \infty$, is a Law of Law Numbers (hydrodynamic limit) $\mu^N(t, dx) \to u(t, x)dx$ which is a deterministic time-indexed process, equal to the solution of a parabolic PDE associated to the generator L of the underlying diffusion. However, the limit $u(t, x)$ cannot be described by the heat equation with classical boundary conditions. It turns out that it can be represented as the normalization (mass one) of the semigroup $S_t = E[\zeta_t]$ discussed in this paper. This was our original motivation to study the semigroup. More precisely, setting $v(t, x) = S_t\phi(x)$, $\phi \in C(\bar{D})$, solves the *heat equation with mass creation* $\partial_t v - Lv = 0$ with $v(t, x) = \bar{K}\langle\gamma, v\rangle$ when $x \in \partial D$, a PDE introduced in [16]. Given the total mass $n_t^x = S_t\mathbf{1}(x)$, the hydrodynamic limit of the BSBD is $u(t, x) = v(t, x)/n_t^x$, $u(0+, x) = \phi(x)$ (Theorem 2 from [15]).

The representation of the limiting profile of a particle system with constant mass via the normalization of a non-conservative semigroup is well known for the Fleming-Viot branching system [8, 18, 24, 12, 2]. In that case, $K = 1$ and the redistribution measure is not constant, depending on the configuration. Denote S_t^D the semigroup generated by $p^D(t, x, y)$. In perfect analogy with the normalization for S_t, here $v(t, x) = S_t^D\phi(x)$ and $u(t, x) = v(t, x)/n_t^x$.

As soon as $\bar{K} > 1$, the profile cannot be a-priori described as the transition kernel of a one particle Markov semigroup and we need to move up to a measure valued process. In short, we need more than one particle to represent a super-critical process with mass creation.

The other limit is as $t \to \infty$. In principle, $\mu^N(t,dx) \to \mu^N_{eq}(dx)$ leads to the empirical measure under equilibrium. While this limit is not explicit, heuristically, letting $N \to \infty$ (after $t \to \infty$) leads to the qsd associated to the semigroup S_t. This becomes natural when we may commute the limits, as would be the case, for instance, under a uniform rate of convergence to equilibrium (say, a spectral gap, or a uniformly decaying variance in N as $t \to \infty$). This justifies the interest in the Yaglom limits $\lim_{t\to\infty} S_t\phi(x)/n^x_t = \nu$, for all $\bar{K}$, as the semigroup represents the limit after $N \to \infty$ in non-equilibrium. In essence, we show there exists a bijection between the interaction intensity $\bar{K}$ and a family of quasi-stationary distributions ν, which are macroscopic particle profiles of the BSBD in stationary state, an example of an explicit representation of self-organizing equilibrium [3].

1.2. **Conditions on the underlying diffusion.** With usual notations, $M_F(D)$ designates the finite measures on $D \subset \mathbb{R}^d$ and $M_1(D)$ the space of probability measures on D (distributions), both with the topology of convergence in distribution (weak* topology). Same notations hold for a more general set than D.

For the time-space functions $\phi \in C([0,\infty) \times \bar{D}; \mathbb{R})$, $\langle m, \psi(t,\cdot)\rangle$ for $t \geq 0$ and $\langle m, \psi(\cdot)\rangle$, or sometimes simply $\langle m, \psi\rangle$ when there is no time variable, shall denote the integral against a signed measure $m(dx)$ on D.

Let $p(t,x,y)$, $t > 0$, $x, y \in \mathbb{R}^d$ be the transition kernel for a diffusion $(X_t)_{t\geq 0}$ generated by a second-order strongly elliptic operator L with smooth coefficients with bounded derivatives. Assume D is a C^2 bounded domain in $\mathbb{R}^d$. Put $\tau^D = \inf\{t > 0 \,|\, X_t \in D^c\}$ for the exit time from D of the diffusion. Under such conditions, $\tau^D < \infty$ a.s. and let $p^D(t,x,y)$, $t > 0$ be the Dirichlet kernel defined by

$$p^D(t,x,y)dy = P_x(X_t \in dy, \tau^D > t)\,. \tag{1.1}$$

The associated semigroup is a *strongly continuous Feller-Dynkin semigroup* in the sense of [25], Chapter III.6, i.e. defines a C_0 (i.e. strongly continuous) semigroup on $C_0(D)$, the space continuous functions vanishing on ∂D with the supremum norm. In addition, it has the strong Feller property. For $\phi \in B(D)$ (set of bounded functions)

$$S^D_t\phi(x) = \int_D p^D(t,x,y)\phi(y) \in C_0(D) \subseteq C(\bar{D})\,. \tag{1.2}$$

The boundary ∂D can be assimilated to the cemetery state and the killed process $X_t^D = X_t \mathbf{1}(\tau^D > t)$ has transition semigroup S_t^D and uniquely solves the martingale problem $(L, \mathcal{D}(L))$ with $\mathcal{D}(L) = \{\phi \in C^2(\bar{D}) \,|\, \phi(x) = 0\,, x \in \partial D\}\,.$

The density, respectively distribution functions $f_D(t,x)$, $F_D(t,x)$ of τ^D for the initial state $x \in D$ at $t > 0$ satisfy

$$(1.3) \qquad P_x(\tau^D > t) = 1 - F_D(t,x) = \int_t^\infty f_D(s,x)\, ds\,.$$

We assume that

$$(1.4) \qquad f_D\,,\, \partial_t f_D \in C((0,\infty)\times \bar{D};\mathbb{R}) \quad \text{and} \qquad f_D(0+,x) = 0\,.$$

The Laplace transform $\alpha \to \hat{f}_D(\alpha,x) = E_x[\exp(-\alpha\tau^D)]$ is defined for all real α where the integral is finite and thus for $\{\alpha \in \mathbb{C} \,|\, \Re(\alpha) \geq 0\}$. In particular we denote $\hat{f}_D(\alpha,\rho) = \langle \rho(dx), \hat{f}_D(\alpha,x)\rangle$ for $\rho \in M_1(D)$, a probability measure on D.

Because D is bounded and regular $\tau^D > 0$ a.s. and $\hat{f}_D(+\infty,x) = 0$ by monotone convergence. It will be assumed that there exists a negative exponential moment $\alpha_- < 0$ such that for all $x \in D$

$$(1.5) \qquad \sup_{x\in D} \hat{f}_D(\alpha_-,x) = c_D < +\infty\,.$$

On $\Re(\alpha) > 0$ and, by extension, wherever the integral converges, we define the resolvent of the semigroup

$$(1.6) \qquad R_\alpha^D \phi(x) = \int_0^\infty e^{-\alpha t} S_t^D dt\,.$$

Due to the relation

$$(1.7) \qquad \frac{1}{\alpha}(1 - \hat{f}_D(\alpha,x)) = R_\alpha^D \mathbf{1}(x)\,,$$

for $\mathbf{1}(x) \equiv 1$ it follows that the resolvent can be extended on the half-plane $\Re(\alpha) > \alpha_-$, equivalent to $(\alpha_-, +\infty) \subseteq Res(L)$.

We start with a basic result proven in [16].

Proposition 1 (Proposition 2 in [16]). *Let $\gamma \in M_1(D)$. Under the assumption (1.5) there exists $-\infty < \tilde{\alpha} < 0$ such that the decreasing function $\alpha \to \hat{f}_D(\alpha,\gamma)$ satisfies*

$$(1.8) \qquad \lim_{\alpha\downarrow\tilde{\alpha}} \hat{f}_D(\alpha,\gamma) = +\infty\,, \qquad \lim_{\alpha\to\infty} \hat{f}_D(\alpha,\gamma) = 0\,.$$

The Laplace transform $\alpha \to \hat{f}(\alpha, \gamma)$ can be extended to a holomorphic function on $\Re(\alpha) > \tilde{\alpha}$. For any $\bar{K} > 0$ there exists a unique $\alpha^* \in (\tilde{\alpha}, +\infty)$ solving

$$1 - \bar{K}\hat{f}_D(\alpha^*, \gamma) = 0\,. \tag{1.9}$$

Moreover, due to $\hat{f}_D(0, \gamma) = 1$, $\alpha^* > 0$ ($\alpha^* < 0$) for $\bar{K} > 1$ ($\bar{K} < 1$), with equality $\alpha^* = 0$ when $\bar{K} = 1$.

Remark. Theorem 1 will show that α^* is the exponential growth rate of the total mass of the branching process $E[\langle \zeta_t, 1\rangle] \leq Const \cdot e^{\alpha^* t}$.

1.3. **A Tauberian condition.** A non-negative function $f \in C([0,\infty);\mathbb{R})$ is said to have *exponential monotonicity* if there exists $\alpha_+ \geq 0$ such that $e^{\alpha_+ t} f(t)$ is non-decreasing. When f is differentiable, it is sufficient that $f'(t) + \alpha_+ f(t) \geq 0$ for all $t > 0$. When, in addition, $f(t) > 0$ for all $t \geq 0$, the condition is equivalent to having a finite lower bound for the the logarithmic derivative.

Condition 1 (Tauberian)**.** *For every $x \in D$, the density $f_D(t, x)$ has exponential monotonicity.*

Such an assumption will be necessary in the application of a variant of the Wiener-Ikehara Tauberian Theorem (see Section 6.2).

Condition 1 is non-trivial. In general, it is easier to verify for $t \to +\infty$, where series expansions of the heat kernel are available, e.g. for Bessel processes in Proposition 2, and, more generally, from the theory of symmetric compact operators, when applicable. In the case of Brownian motion and regular D, a Bessel-Fourier series shows that $f_D'(t)/f_D(t) \to \lambda_1$, where $\lambda_1 = \tilde{\alpha} < 0$ is the first eigenvalue. It is worth menioning a connection to the well-known Li-Yau estimate [23], Theorem 1.2, which gives another proof of the lower bound at $t \to \infty$.

At zero, it is enough that the distribution function $P_x(\tau^D \leq t)$ be convex on some small interval. A random variable, here on $[0, \infty)$, is *unimodal* if its distribution function is first convex and then concave, i.e. the density, if it exists, is first increasing, then decreasing with only one maximum point. Thus, a sufficient condition near zero is that τ^D be unimodal. However, this is not sufficient as $t \to \infty$.

It is known that one-dimensional diffusions have unimodal exit times [26]. This implies that Bessel processes have the property, and thus, Brownian motion in a ball centered at the origin, in all dimensions, has it as well. Likewise, all processes with coefficients and sets D amenable (via symmetries or other transformations) to the one-dimensional case.

Joining the observations, we obtain a sufficient condition, proven in the Appendix.

Proposition 2. *For any starting point $x \in D$, the density of the exit time of d-dimensional Brownian motion from $D = B(0, r)$, $r > 0$, the ball centered at zero, has exponential monotonicity.*

1.4. **Plan of the proof.** In Section 2, the auxiliary branching process (ζ_t) is constructed. Section 3 extends the upper bound on the number of particles from the case $K \geq 1$ (Proposition 3) to arbitrary K (Proposition 4) by coupling. Theorem 1 is the main result giving the exact exponential growth rates. Section 4, with main result Proposition 6, analyzes the probability of extinction and other aspects of the sub-critical case. Section 5 proves exact formulas (Theorem 2) for the semigroup S_t and its resolvent. Theorem 3, together with Corollaries 3 and 4 prove the Yaglom limits and give explicit formulas for the qsd as a function of $\bar{K}$. Asymptotic limits are given in strong form (5.8) under Condition 1 and Cesaró form (5.14) under no assumptions on the monotonicity of $f_D(t, x)$. We also note that S_t also provides a natural example of a continuous semigroup of bounded operators which is not of class C_0.

Finally, the Appendix (Section 6) has two parts. The first part proves Lemma 1 that allows the use of Tauberian Theorem s (e.g. Wiener-Ikehara 4 for strong limits and Karamata's Theorem for the Cesaró limits) applied to functions that are not necessarily non-decreasing. The second part provides a general setup, definitions and basic results (Theorem 5) for the concepts of Yaglom limits and qsd's in the context of a non-conservative semigroup such as the one under consideration.

2. The branching processes Z_t and ζ_t

This section follows closely the construction from [16], Section 4.

Let $\tilde{D} = D \cup \{\mathfrak{o}\}$ be the usual compactification of D with $D \cup \{\mathfrak{o}\}$, where $\mathfrak{o}$ is an isolated point for D, that will be the cemetery point.

Recall $M_F(\tilde{D})$ denotes the space of finite measures on $\tilde{D}$. Ultimately, we shall construct a process (ζ_t) on the space of discrete measures $M_0(D)$, a proper subspace of $M_F(\tilde{D}) \subseteq M_F(\tilde{D})$. In order to do that, we consider an infinite supply of i.i.d. processes evolving in $\tilde{D}$, denoted generically by Z_t^i, with law given by the transition kernel $p^D(t, x, y)$.

We introduce a particle system having a finite but random number of particles N_t alive at $t \geq 0$. With the same notations as in Section 1, the number $K \sim \pi$ where π is a probability mass function on the non-negative integers with $P(K = k) = p_k$, $k \geq 0$. Throughout the paper $\bar{K} = E[K] < +\infty$.

A single particle Z_t^1 is placed at a random point with initial distribution $\nu_0(dx)$ at $t = 0$ and starts moving driven by $p^D(t, x, y)$ until it reaches ∂D, when it goes to $\mathfrak{o}$. At that moment, instantaneously, a random integer number K of particles are born at a specific point in D distributed according to $\gamma(dx)$. All particles start afresh, independently, and move in D until the first one reaches ∂D and the branching is repeated. The procedure is continued until one of two exceptional stopping times, explosion or extinction, terminate the process. Explosion cannot occur in our setup as will be proven in Theorem 1, but extinction may occur when $p_0 > 0$.

Denote by Z_t^i, $i \in \mathbb{Z}_+$ the i-th particle born in the process. If it is born at time τ, then we understand that $t \geq \tau$. Let N_t^{tot} be the total number of individuals ever born up to time $t \geq 0$. When at time τ a number $j \geq 1$ of individuals are born, their birth being simultaneous and at the same point, their ordering is not relevant. They are simply labeled $i = N_{\tau-}^{tot} + l$, $1 \leq l \leq j$. By construction, the process starts and preserves a finite number of particles during its lifetime with probability one.

Definition 1. *The process of particles alive at time $t \geq 0$ is*

$$\zeta_t = \sum_{i, Z_t^i \notin \mathfrak{o}} \delta_{Z_t^i}, \quad \textit{with} \quad N_t = \langle \zeta_t, \mathbf{1}_D \rangle, \qquad t \geq 0. \tag{2.1}$$

Here N_t is the number of particles alive at time $t \geq 0$ and $N_t^{tot} - N_t$ is the number of killed particles, also equal to the number of branching events up to time $t > 0$. As defined in (2.1), $\zeta_0 = \delta_X$, $X \sim \nu_0(dx)$. In this way, we denote $E_{\nu_0}[\cdot] = \int_D E_x[\cdot] \nu_0(dx)$ and $E_x[\cdot]$ corresponds to $\nu_0 = \delta_x$.

The process (ζ_t) evolves in the space of discrete measures inside D, more precisely

$$M_0(D) = \{\mu \in M_F(D) \mid \mu = \sum_{i \in I} \delta_{x_i}(dx), \, x_i \in D, i \in I, I \text{ finite}\},$$

a subset of $M_F(D)$. We shall see $M_0(D)$ as

$$M_0(D) = \cup_{N=1}^{\infty} D^{(N)} \cup \{\mathbf{0}\}$$

where $\mathbf{0}$ is the zero measure (no delta function is present) endowed with the topology of disjoint unions. Each $D^{(N)}$ is the symmetric product of D (i.e. the product space factorized by symmetries on D^N) with the product topology of D as a subset of $\mathbb{R}^d$ (see also [6], [19]) and the general construction of branching processes in [10] and the recent monograph [22]. This is a Lusin space because each of the spaces in the summation is a complete separable metric (Polish) space. A net converges only if eventually all terms belong to only one element in the sum and convergence takes place in the topology of that space. In our construction, for finite configurations, we note that convergence in norm on $\tilde{D}^N$ implies weak convergence of measures. We notice that (2.1) disregards, in fact, any labelling and the original labels of Z_t^i were important only in the construction. The same is true for the cemetery state that is not present in the final form of (ζ_t).

In the construction, we assume the existence of an infinite sequence of i.i.d. processes with the same law as Z_t^1, as well as an i.i.d. sequence of copies of K. At a branching event τ, we sample the current value k of the random number K and k i.i.d. copies of Z_t are sampled and the process re-started from τ as $Z_{t+\tau}$, for each particle in the union of the particles alive and the ones newly added.

A few final remarks. The particle that was killed goes to $\mathfrak{o}$. We notice that, by definition, N_t does not take into account killed particles since $\mathfrak{o} \notin D$. By construction, no branching events may occur simultaneously. The construction is thus inductive over the branching events $\tau_1 < \tau_2 < \ldots < \tau_m < \ldots$ that can be ordered, and $N_t^{tot} < \infty$, both happening almost surely until T^*. The process $(\zeta_t)_{t\geq 0}$ is Markov with state space $M_0(D)$. It shall be adapted to a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where the filtration satisfies the usual conditions. We note that the law of (ζ_t) is a probability measure on the Skorokhod space $\mathbf{D}([0,\infty), M_F(D))$.

Definition 2. *Denote (ζ_t^x) the the process with $\nu_0 = \delta_x$ as in (2.1) and if $\mu = \sum_{i\in I} \delta_{x_i}$, I finite, then $\zeta_t^\mu = \sum_{i\in I} \zeta_t^{x_i}$, where the processes $(\zeta_t^{x_i})$ are independent. We note that the points x_i are not necessarily distinct, but each individual acts as an independent copy in (ζ_t^μ).*

By construction, (ζ_t) is a pure branching process in the sense that, if $\mu_1, \mu_2 \in M_0(D)$, then

$$\mathcal{L}(\zeta_\cdot^{\mu_1+\mu_2}) = \mathcal{L}(\zeta_\cdot^{\mu_1}) * \mathcal{L}(\zeta_\cdot^{\mu_2})\,, \tag{2.2}$$

where $\mathcal{L}(\cdot)$ denotes the probability law.

3. Asymptotics for the total number of particles

The next proposition shows that no explosion can occur and gives a sharp exponential upper bound of $E_x[N_t]$. Define the stopping times $T_m \in [0, +\infty]$ as the first time $N_t \geq m$, $m \in \mathbb{Z}_+$. The process $t \to N_t$ is rcll (cadlag) and piecewise constant, so T_m is a stopping time and T_m is nondecreasing in m. The stopping time $T_\infty = \lim_{m\to\infty} T_m$ is the *time of explosion*. It will be shown that $T_\infty = +\infty$ almost surely.

First, state a result proven in [16] giving the exact upper bound for the expected number of particles in the trivially supercritical case when $K \geq 1$.

Proposition 3 (Proposition 3 in [16])**.** *Let K be such that $P(K=0) = p_0 = 0$ with finite $\bar{K}$ and $\alpha^* \geq 0$ be the solution to (1.9), which depends on γ but not on t and x. Then, the number of particles N_t of the process (Z_t) has finite expectation for any $t > 0$. More precisely, there exists $C(\gamma, \bar{K}) > 0$ such that*

$$\sup_{x\in D} E_x[N(t)] \leq C(\gamma, \bar{K}) e^{\alpha^* t}\,. \tag{3.1}$$

To extend the proof of the upper bound to arbitrary K, we proceed by coupling with a process that, path by path, will always have a larger number of particles and satisfies the conditions of Proposition 3.

Proposition 4. *For an arbitrary random integer $K \geq 0$ with finite $\bar{K}$, let $\tilde{K}$ be a random integer with distribution defined by $P(\tilde{K} = k) = p_k$, if $k \geq 2$ and $P(\tilde{K} = 0) = 0$, respectively $P(\tilde{K} = 1) = p_0 + p_1$. We note that $\tilde{K}$ satisfies the conditions of Proposition 3. Let $\tilde{\alpha}^*$ be the solution to eq. (1.9) and $\tilde{C}$ denoting the constant C in (3.1), corresponding to $\tilde{K}$. Then*

$$N_t \leq \tilde{N}_t \quad a.s. \qquad and \qquad \sup_{x\in D} E_x[N(t)] \leq \tilde{C} e^{\tilde{\alpha}^* t}\,. \tag{3.2}$$

Remark. Theorem 1 will prove that the correct exponential growth rate is α^*, corresponding to the value $\bar{K}$. Proposition 4 is necessary to ensure that the total mass is finite at all times $t \geq 0$. Naturally $\alpha^* \leq \tilde{\alpha}^*$, so the current bound is generally not optimal.

Proof. We first remark that in the case when $p_k = 0$ for all $k \geq 2$, then $0 \leq N_t \leq 1$ almost surely, proving that $E_x[N_t] \leq 1$ is trivially bounded. In case at least one of these probabilities is non-zero, then we proceed to Step 1.

Step 1. First we couple the process with a new process $\tilde{Z}_t$ having the same evolution mechanism as Z_t with the exception that the number of particles born at a boundary hit is $\tilde{K}$. The processes are identical up to the first boundary hit, starting with the same number of particles at the same locations and following the same paths driven by p^D. At the first boundary hit, if Z_t draws a sample K of the number of particles to be born and $K \geq 1$, then the two processes continue to be identical until the first time K equals zero. At that moment, $\tilde{Z}_t$ will continue with an additional particle born at location chosen with the same distribution $\gamma(dx)$. The offspring of this particle follows the dynamics using the distribution of $\tilde{K}$ for the numbers of births upon each boundary hit and will be independent forever of Z_t. The rest of the process continues its evolution. This coupling will follow the same paths for the original particles or the particles born when $K \neq 0$, while the other particles of $\tilde{Z}_t$ not belonging to Z_t follow independent paths from an infinite supply of diffusive paths driven by p^D on D. It is important that, path-by-path, $N_t \leq \tilde{N}_t$ and $\tilde{N}_t$ is non decreasing in $t \geq 0$. We notice that $E[\tilde{K}] = p_0 + E[K]$. We know that at least one of p_k, $k \geq 2$ is positive, and then $E[\tilde{K}] > 1$.

Let $\tilde{\alpha}^*$ be the solution to (1.9) corresponding to $\tilde{K}$. Also, write $\tilde{C}$ for the constant C in (3.1) for $\tilde{N}_t$. Then (3.1) holds for $\tilde{N}_t$, thus it holds for N_t, which is exactly (3.2). $\square$

Knowing that $E_x[N_t] < \infty$ for all $t > 0$, we can use the martingale theory to prove the next proposition. The exact behavior of the extinction probabilities is given in Proposition 6.

Proposition 5. *The process $t \to N_t$ is a sub - (super-), (simple) martingale with respect to the filtration $\mathcal{F}_t$, according to $\bar{K} > 1$, (< 1), $(= 1)$, respectively. When $\bar{K} \leq 1$ then $\lim_{t\to\infty} N_t \geq 0$ exists a.s. and $n_t^x \leq 1$, $t \geq 1$.*

Proof. We write the counting process $N_t = \sum_s (N_s - N_{s-})$ and see that due to the strong Markov property the conditional expectation of the increment is exactly $\bar{K}-1$, which proves the claim. $\square$

Recall that τ^D is the hitting time of the absorbing boundary for the underlying diffusion. Proposition 4 showed that for any $\rho \in M_1(D)$, $n_\rho(t) = \langle \rho, n_t^x \rangle = E_\rho[N_t] \leq \tilde{C} e^{\alpha^* t} < \infty$, implying non-explosion $N_t^\rho < \infty$ almost surely. When $\rho(dz) = \delta_x(dz)$, we write $n_\rho(y) = n_x(t)$.

Theorem 1 is the main result in this section.

Theorem 1. *The expected value of the total number of particles at time $t \geq 0$ has continuous derivatives on $t \in [0, \infty)$ and satisfies*

$$n_x(t) = P_x(\tau^D > t) + \bar{K} \int_0^t n_\gamma(t-s)\, dF_D(s,x)\,. \tag{3.3}$$

Its Laplace transform

$$\hat{n}_x(\alpha) = \frac{1}{\alpha}\Big[1 + (\bar{K}-1)\frac{\hat{f}_D(\alpha,x)}{1-\bar{K}\hat{f}_D(\alpha,\gamma)}\Big] \tag{3.4}$$

is meromorphic on the half-plane $\Re(\alpha) > \tilde{\alpha}$ with exactly one simple pole at α^* with value denoted $B(x,\bar{K})$, where α^* is the solution of (1.9). When $\bar{K}=1$, then $B(x,\bar{K})=1$ and $n_x(t) \equiv 1$, $\hat{n}_x(\alpha) = 1/\alpha$. When $\bar{K} \neq 1$, then

$$B(x,\bar{K}) = (1-\frac{1}{\bar{K}})\frac{\hat{f}_D(\alpha^*,x)}{\alpha^* \hat{f}'_D(\alpha^*,\gamma)} > 0\,. \tag{3.5}$$

Under either of the assumptions $\bar{K} \geq 1$ or Condition 1, the asymptotic growth rate at $t \to \infty$ is α^*,

$$\lim_{t\to\infty} e^{-\alpha^* t} n_x(t) = B(x,\bar{K})\,. \tag{3.6}$$

Remarks. We recall that $\tilde{\alpha} < 0$ is the infimum of the real part of the spectrum for the underlying diffusion. Due to (1.7), there is no actual pole at $\alpha = 0$. As we have seen in Proposition 1, In the subcritical $\bar{K} < 1$, critical $\bar{K} = 1$ and critical $\bar{K} > 1$ we have $\alpha^* < 0$, $\alpha^* = 0$ and $\alpha^* > 0$, respectively.

Proof. Step 1. Proof of (3.3). Since $T_\infty = +\infty$ a.s., we can write the renewal equation, also almost surely

$$N_t^x = \mathbf{1}_{\{t<\tau^D\}} + \mathbf{1}_{\{t\geq\tau^D, K\geq 1\}} \sum_{j=1}^{K} N_{t-\tau^D}^{Z^j_{\tau^D}} \quad . \tag{3.7}$$

Here (Z_t^j) are i.i.d. processes started at a random point with distribution γ. For all $j \geq 1$

$$E[\mathbf{1}_{\{t\geq\tau^D\}} N_{t-\tau^D}^{Z^j_{\tau^D}} \,|\, \tau^D = s] = n_\gamma(t-s)$$

which obtains

$$\begin{aligned} n_x(t) &= P_x(\tau^D < t) + \sum_{k=1}^{\infty} p_k E[\mathbf{1}_{\{t\geq\tau^D\}} \sum_{j=1}^{k} N_{t-\tau^D}^{Z^j_{\tau^D}}] \\ &= P_x(\tau^D < t) + \Big(\sum_{k=1}^{\infty} k p_k\Big) \int_0^t n_\gamma(t-s) f_D(s,x)\, ds\,. \end{aligned}$$

$$= P_x(\tau^D < t) + \bar{K}\int_0^t n_\gamma(t-s) f_D(s,x)\, ds\,,$$

proving (3.3). Since $n_x(t)$, respectively $n_\gamma(t)$ are bounded for $t \in [0,T]$, $T > 0$, we obtain that $t \to n_x(t)$ is Lipschitz, so certainly continuous and non-negative. We know that $t \to n_\gamma(t)$ si continuous and $t \to f_D(t,x)$ is $C^1([0,\infty);\mathbb{R})$. Simple Calculus shows that $n'_x(t)$ exists and is continuous.

Step 2. Proof of formula (3.4.) We can treat the case $\bar{K} = 1$ directly. The branching times have continuous distributions so the sequence of successive branchings is totally ordered with no simultaneous branchings. Between branchings, the number of particles is constant and at time zero it is one. At any such branching, the expected jump in number of particles is zero. It follows that $n_x(t) \equiv 1$. The Laplace formula is then trivial and matches (3.4).

Now let $\bar{K} \neq 1$. Using the general exponential bound (3.2), we now know that the Laplace transform of (3.3) is well defined on a set including $\Re(\alpha) > \tilde{\alpha}^* \geq 0$. On this set we obtain

$$\int_0^\infty e^{-\alpha t} n_x(t) dt = \hat{n}_x(\alpha) = \frac{1}{\alpha}(1 - \hat{f}_D(\alpha,x)) + \bar{K}\hat{n}_\gamma(\alpha)\hat{f}_D(\alpha,x)\,. \tag{3.8}$$

Integrating over γ and doing the algebra we obtain the ratio of two holomorphic functions

$$\hat{n}_\gamma(\alpha) = \frac{\langle \gamma, R^D_\alpha \mathbf{1}\rangle}{1 - \bar{K}\hat{f}_D(\alpha,\gamma)}\,. \tag{3.9}$$

We can plug back into the first equation to conclude the proof of the formula (3.4).

The Laplace transform (3.4) was proven to exist, due to Proposition 4, for real $\alpha > \tilde{\alpha}^*$. At this point, it can be uniquely extended as far as the functions $\hat{f}_D(\alpha,x)$ and $\hat{f}_D(\alpha,\gamma)$ are holomorphic, which is the half-plane $\Re(\alpha) > \tilde{\alpha}$. We show below it is a meromorphic function with pole at $\alpha^* < \tilde{\alpha}^*$. Because of (1.7), there is no singularity at zero.

Recall that α^* and $\tilde{\alpha}^*$ solve (1.9) for $\bar{K}$, corresponding to the unmodified K, respectively for $\tilde{K}$, corresponding to the coupling with the dominant $\tilde{K}$. The inequality $\tilde{\alpha}^* \geq \alpha^*$ follows from $E[\tilde{K}] \geq E[K]$. We know that $\alpha^* > \tilde{\alpha}$. At this point it is clear that α^* is a pole for $\hat{n}_x(\alpha)$.

Step 3. Proof of (3.5). The formula is trivial if $\bar{K} = 1$. When $\bar{K} \neq 1$ we notice that $(\alpha - \alpha^*)\hat{n}_x(\alpha)$ has a first term in $1/\alpha$ that vanishes as $\alpha \to \alpha^*$ and the function $\alpha \to \hat{f}_D(\alpha,x)$

is continuous at α^* and has nonzero value. It remains to verify that

$$\lim_{\alpha\to\alpha^*}\frac{\frac{1}{\bar K}-\hat f_D(\alpha,\gamma)}{\alpha-\alpha^*}=-\hat f'_D(\alpha^*,\gamma)=E_\gamma[\tau^D e^{-\alpha^*\tau^D}]>0\,.$$

We note that $(1-1/\bar K)\alpha^*>0$.

Step 4. Proof of the asymptotics. Again, (3.6) is trivial when $\bar K=1$ with $n^x_t\equiv 1$. Otherwise, for $t\to\infty$, we have two cases. The case when we assume $\bar K\geq 1$, Proposition 5 shows that $n_x(t)$ is non-decreasing. As such, the classical Wiener-Ikehara Tauberian Theorem (here Theorem 4) applies.

In the case when Condition 1 is satisfied, no restriction on p_0 is needed. We split (3.3) in two. First, let $\delta>0$ such that $\tilde\alpha+\delta<\alpha^*\wedge 0$. Then, by Markov's inequality

$$e^{-\alpha^* t}P_x(\tau^D>t)\leq \hat f_D(\tilde\alpha+\delta,x)e^{-(\alpha^*-\tilde\alpha-\delta)t}\to 0\qquad t\to+\infty\,. \tag{3.10}$$

Second, we calculate the integral term using our Lemma 1 together with Proposition 8. Finally, we notice that in (3.3), the first term is holomorphic at $\alpha=\alpha^*$. It follows that the pole of $\hat n_x(\alpha)$ is the same as the pole issued from the convolution part. □

Corollary 1. *With no assumption on the monotonicity of $t\to f_D(t,x)$, if $\bar K>1$ then*

$$\lim_{t\to\infty}e^{-\alpha^* t}\int_0^t n^x_s ds\to B(x,\bar K)/\alpha^*$$

and if $\bar K\leq 1$, then

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t e^{-\alpha^* s}n^x_s ds\to B(x,\bar K)\,.$$

Proof. The two limits are immediate consequence of Lemma 1, parts (ii) and (iii). □

4. The subcritical case

For any $\rho\in M_1(D)$ we write

$$u_\rho(t)=E[e^{-\lambda N^\rho_t}\mathbf{1}_{[0,\infty)}(N^\rho_t)]\quad\text{and}\quad v_\rho(t):=P_\rho(T_{ext}\leq t)\,, \tag{4.1}$$

where u_ρ is finite for all $\lambda\geq 0$ and v_ρ is the distribution function of the time of extinction. We already know that $N^\rho_t<\infty$ a.s., so the indicator function in the formula is redundant, but we use the formula for consistency. See the discussion in the Remarks subsection. When $\rho(dz)=\delta_x(dz)$ $u_\rho(t)=u_x(t)$, $v_\rho(t)=v_x(t)$, noticing that $u_\rho(t)=\langle\rho,u_\cdot(t)\rangle$ and $v_\rho(t)=\langle\rho,v_\cdot(t)\rangle$. In addition, recall $\Theta_K(s)=\sum_{k=0}^\infty p_k s^k$, $s\leq 1$, the generating function of K.

Proposition 6. *The convolution formulas hold*

$$u_x(t) = e^{-\lambda} P_x(\tau^D > t) + \int_0^t \Theta_K(u_\gamma(t-s)) f_D(s,x)\, ds\,, \tag{4.2}$$

$$v_x(t) = \int_0^t \Theta_K(v_\gamma(t-s)) f_D(s,x)\, ds\,, \qquad v_x(t) = P(N_t^x = 0)\,, \tag{4.3}$$

where (4.3) can be obtained from (4.2) by letting $\lambda \to \infty$*.* The probability of extinction $v_{ext} := P_x(T_{ext} < \infty)$ *is the unique fixed point* $0 \le v_{ext} \le 1$ *of* Θ_K

$$v_{ext} = \Theta_K(v_{ext})\,, \qquad v_{ext} = \lim_{t\to\infty} v_x(t) \tag{4.4}$$

deriving that $v_{ext} = 0$ *if* $p_0 = 0$*,* $v_{ext} \in (0,1)$ *if* $p_0 > 0$ *and* $\bar{K} > 1$ *and* $v_{ext} = 1$ *when* $p_0 > 0$ *and* $\bar{K} \le 1$*.*

Proof. We derive (4.2) in analogous fashion to the proof of formula (3.3). The generating function appears because at the time $\tau^D = s$ (under conditioning) in the integral, the process will instantaneously have K independent copies of itself.

To prove (4.3), we can calculate

$$\begin{aligned} P_x(T_{ext} > t) &= P_x(\tau^D > t) + \int_0^t (1 - \Theta_K(P_\gamma(T_{ext} \le t-s))) f_D(s,x) ds \\ &= 1 - \int_0^t \Theta_K(P_\gamma(T_{ext} \le t-s)) f_D(s,x) ds \end{aligned} \tag{4.5}$$

implying

$$P_x(T_{ext} \le t) = \int_0^t \Theta_K(P_\gamma(T_{ext} \le t-s)) f_D(s,x) ds\,, \tag{4.6}$$

which proves the claim.

The limit $\lambda \to \infty$ follows by dominated convergence, after noticing that $0 \le u_\gamma(t-s) \le 1$ and $f_D(s)$ is integrable.

To check (4.4) we see that $\Phi_K(s) - s$ equals p_0 at $s = 0$ and 0 at $s = 1$. It is also convex. When $\bar{K} = \Phi'_K(1) > 1$ the function has a nontrivial fixed point, while when $\bar{K} \le 1$ the function is non-increasing so $s = 1$ is the only fixed point. In both cases it is unique. □

4.1. **Discussion and remarks.** 1) Results like (4.4) are well known in the classical theory of branching processes. From the point of view of survival, only the branching number K and intensity $\bar{K}$ matter. This is due to the fact that $E[\tau^D]$ is bounded above and below away from zero under γ. However, the semigroup of the branching process ζ_t is not trivial, as we see in Theorem 3.

2) We note that (4.1) is a moment generating function (in λ) of $\omega \to N_t^x(\omega)$, where ω is the random element, for t fixed, whereas (3.4) is the Laplace transform (in α) of the expected value $t \to n_t^x$, a deterministic function.

3) When γ is a quasi stationary distribution for the underlying process L, the process N_t^γ is Markovian, equal to a birth and death chain in continuous time.

4) The process N_t^γ is not Markovian in general, since the holding times between branchings are not exponential. However, this is a branching process where each particle branches a after i.i.d. times τ^D (starting from a point $Z \sim \gamma$).

When $\lambda = 0$ in (4.2) the trivial solution gives that $N_t^x = +\infty$ has zero probability. The probability of extinction $v_x(t) = P(N_t^x = 0)$ has solution equal to zero if $p_0 = 0$. An immediate consequence of the equation (4.3) after integrating over $\gamma(dx)$ is given below.

Corollary 2. *If $\bar{K} \leq 1$, the distribution function of the time of extinction under γ is the fixed point of*

$$v_\gamma = \Theta_K(v_\gamma(\cdot)) \star f_D(\cdot, \gamma) \,. \tag{4.7}$$

4.2. **The Bernoulli case.** In general, as soon as K is supported on $k > 1$, Equation (4.7) is not solvable in closed form, even in simple cases like $\Theta_K(s) = (1-p) + ps^2$, $0 < p < 1$, including for $p = \frac{1}{2}$.

When $K \in \{0, 1\}$, $p_0 = 1 - p$, $p_1 = p$, the generating function $\Theta_K(s) = (1-p) + ps$ and the process is always sub-critical unless trivial. The sub-case $p = 0$ is the simple killed process and $p = 1$ is the *Brownian motion with rebirth* from [17, 5]. These are the only cases when the particle process can be cast as a one-particle Markov process with state space D. For $p \in (0, 1)$, $p = \bar{K} < 1$ and extinction occurs in finite time almost surely and

$$\hat{v}_\gamma(\alpha) = \frac{1-p}{\alpha} \frac{\hat{f}_D(\alpha, \gamma)}{1 - p\hat{f}_D(\alpha, \gamma)} \tag{4.8}$$

and the Laplace transform of the tail $P_\gamma(T_{ext} > t) = 1 - v_\gamma(t)$

$$P_\gamma(\widehat{T_{ext} >} \cdot)(\alpha) = \frac{1}{\alpha} - \hat{v}_\gamma(\alpha) = \frac{1}{\alpha} \frac{1 - \hat{f}_D(\alpha, \gamma)}{1 - p\hat{f}_D(\alpha, \gamma)} \,, \tag{4.9}$$

with a pole at α^*. In this case $\alpha^* < 0$ In the same way as in Theorem 1 for (3.6), we apply Lemma 1 showing that

$$P_\gamma(T_{ext} > t) \sim e^{\alpha^* t} \,. \tag{4.10}$$

If, in addition, γ is chosen among the quasi-stationary distributions, we have $\tau^D \sim exp(\theta)$, then $T_{ext} \sim exp((1-p)\theta)$, the thinned exponential. In the Brownian motion case $\theta = |\tilde{\alpha}|$, $\tilde{\alpha}$ the first eigenvalue and we can obtain $\alpha^* = (1-p)|\tilde{\alpha}|$.

5. The branching process (ζ_t) and its semigroup

The process (ζ_t) in (2.1) is now well defined for any distribution of the branching number K. When $E[K] < \infty$, it is not explosive, with a precise exponential bound on the expected number of particles. In the case $p_0 > 0$, the probability of extinction is positive, and if in addition $\bar{K} \leq 1$, then it is equal to one. We are now interested in the semigroup properties induced by the Markov property structure of the process. In the first subsection we summarize the construction given in [16].

5.1. **The Semigroup.** The process (ζ_t) in (2.1) is constructed on the state space of finite configurations D, denoted by $M_0(D)$. This space is embedded in the Polish space $M_F(D)$. Define the space of test functions as continuous, bounded functions $F \in C_b(M_F(D))$ of the form

$$\mu \in M_F(D) \to F(\mu) = \varphi(\langle \mu, \phi_1 \rangle, \dots, \langle \mu, \phi_l \rangle)\,, \quad l \in \mathbb{N} \tag{5.1}$$

where $(\phi_i)_{1 \leq i \leq l} \in C_b(D)$ and $\varphi \in C_b(\mathbb{R}^d)$. A class with smooth components ϕ_i, e.g. belonging to $C_c^\infty(D)$ of such test functions is sufficient to determine the law of the process (2.1) as the solution to the martingale problem, see [11].

Due to Proposition 3 we can extend φ to polynomial growth functions (and more). In fact, we shall be only interested in the functionals $\mu \to F(\mu) = \langle \mu, \phi \rangle$, for some test function ϕ, in other words a *linear functional*, when $\varphi(u) = u$ and $l = 1$. In that sense we refer to the restriction of the semigroup as *the marginal transition semigroup*, formally defined in (5.3), as already mentioned in Subsection 1.4.

For a test function F and $\mu = \sum_{i=1}^N \delta_{x_i} \in M_0(D)$, N positive integer (with the convention that we represented the space of finite configurations as sums of delta functions) we define the semigroup on $M_0(D)$

$$\begin{aligned}\mu \to \mathcal{S}_t(F)(\mu) = E_\mu[F(\zeta_t)] = \sum_{i=1}^N E[F(\zeta_t^{x_i})] = \sum_{i=1}^N E_{x_i}[F(\zeta_t)] \\ = \langle \mu, E_\cdot[F(\zeta_t)] \rangle = \int_D E_x[F(\zeta_t)] \mu(dx)\,.\end{aligned} \tag{5.2}$$

The relation is a consequence of the construction of the process. Particles independent at time $s \geq 0$ remain independent forever. The only dependence is through the ancestry tree. Particles distributed deterministically at time $t = 0$ are independent.

Definition 3. *For $\phi \in C_b(D)$, we define $S_0\phi = \phi$ and for $t > 0$*

$$S_t\phi(x) := E_x[\langle \zeta_t, \phi \rangle]\,, \quad x \in D\,. \tag{5.3}$$

We now show that S_t is a continuous semigroup in the sense of (6.2) in the Appendix. Theorem 2 is an extension of Proposition 4 in [16]. Recall that S_t^D is the semigroup with transition kernel $p^D(t,x,y)$ defined in (1.2).

Theorem 2. *The mapping (5.3) defines a continuous (but not strongly continuous, in general) semigroup on $C_b(D)$ with $S_t\phi \in C(\bar{D}) \subseteq C_b(D)$, $t > 0$ satisfying*

$$S_t\phi(x) = S_t^D\phi(x) + \bar{K}\int_0^t \langle \gamma, S_{t-s}\phi \rangle \, dF_D(s,x)\,, \qquad \lim_{x \to \partial D} S_t\phi(x) = \bar{K}\langle \gamma, S_t\phi \rangle\,. \tag{5.4}$$

The semigroup has resolvent $R_\alpha\phi(x) = \int_0^\infty e^{-\alpha t} S_t\phi(x)\, dt$

$$R_\alpha\phi(x) = R_\alpha^D\phi(x) + \frac{\bar{K}\hat{f}_D(\alpha,x)}{1 - \bar{K}\hat{f}_D(\alpha,\gamma)}\langle \gamma, R_\alpha^D\phi \rangle\,, \quad \Re(\alpha) > \alpha^*\,, \tag{5.5}$$

where α^* is defined in (1.9).

Remarks.

1) Boundedness. The result is based on a renewal equation essentially the same as (3.3), which is the special case of (5.3) when $\phi(x) \equiv \mathbf{1}(x)$. In this case we write $S_t\mathbf{1}(x) = E_x[\,\langle\, \zeta_t, \mathbf{1}\,\rangle]$. The proof is done in Proposition 4, [16]. In short

$$|S_t\phi(x)| = |E_x[\,\langle\, \zeta_t, \phi\,\rangle| \leq ||\phi|| S_t\mathbf{1}(x) = ||\phi|| \sup_{x \in \bar{D}} E[N_t^x] = ||\phi|| \sup_{x \in \bar{D}} n_t^x \leq \tilde{C}e^{\tilde{\alpha}^* t}\,, \tag{5.6}$$

showing that S_t is a bounded operator.

2) S_t is point-wise continuous but not a C_0 semigroup. The semigroup S_t is not strongly continuous, because the limit $\lim_{t\to 0}|S_t\phi(x) - \phi(x)| = 0$ is not uniform in $x \in \bar{D}$. As $x \to \partial D$, the limit approaches $\bar{K}\langle \gamma, \phi \rangle \neq 0$, in general.

3) Boundary conditions. In [16], the forward and backward equations satisfied by the kernel of the semigroup are investigated from an analytic point of view (solution of a parabolic PDE). The second relation from (5.4) is a boundary condition (BC) that defines the solution $(t,x) \to v(t,x)$, $v(0+,x) = \phi(x)$ of $\partial_t v = Lv$, where L is the same operator as the generator of the semigroup (S_t^D), only with new domain given by (BC).

4) The resolvent. Formula (5.5) is immediately obtained by integration against $e^{-\alpha t}$ over time. Its extension to $\Re(\alpha) > \alpha^*$ is valid because all functions involved are holomorphic on the half-plane since $\alpha^* > \tilde{\alpha}$ by construction.

Proof. Semigroup property. The semigroup property is simply the Chapman-Kolmogorov relation for (ζ_t), which is standard in the theory of Markov processes. The proof follows closely Proposition 4, [16].

When G is linear, i.e. $G(\mu) = \langle \phi, \mu \rangle$, $\phi \in C_b(D)$. We can apply the Markov property to (ζ_t) to have

$$
\begin{aligned}
S_{t+s}\phi(x) &= E_x[\langle \zeta_{t+s}, \phi \rangle] = E_x[E_x[\langle \zeta_{t+s}, \phi \rangle \mid \zeta_s]] = E_x[\langle E_{\zeta_s}[\langle \zeta_t, \phi \rangle]] \\
&= E_x[\langle \zeta_s, E_\cdot[\langle \zeta_t, \phi \rangle] \rangle] = E_x[\langle \zeta_s, S_t\phi(\cdot) \rangle] \\
&= E_x[S_s(S_t\phi(\cdot))] = S_s S_t \phi(x)\,.
\end{aligned}
\tag{5.7}
$$

Continuity at the boundary of $x \to S_t\phi(x)$. When $x \in D$, the strong Feller property of S_t^D and the regularity of $x \to f_D(s,x)$ (1.4) show that $x \to S_t\phi(x)$ is continuous on D. For the next step, it is convenient to write $dF_D(s,x) = f_D(s,x)ds$ in the integrand of the convolution formula. For $x \to \partial D$ we use convergence in distribution for $F_D(s,x)$ since we know that τ^D has distribution delta at time zero. The value of the limit is equal to $\bar{K}\langle \gamma, S_t\phi \rangle$.

The resolvent. The resolvent formula is proven in Proposition 4, [16] and is almost identical to Step 2 of Theorem 1 in this paper. □

5.2. **Quasi-stationarity.** Quasi stationarity and Yaglom limits are defined in the Appendix, eq. (6.5), generalizing in a natural form the classical concepts for a dissipative semigroup S_t. Instead of $S_t\mathbf{1} < 1$, we now consider a non-conservative semigroup where $S_t\mathbf{1} \neq 1$, including both the super-critical $S_t\mathbf{1} \to \infty$ and sub-critical evolution $S_t\mathbf{1} \to 0$.

Theorem 3. *Assume Condition 1 is satisfied. Then, for any $x \in D$, $\phi \in C_b(D)$, the strong Yaglom limit (6.7) exists*

$$
\lim_{t\to\infty} \frac{S_t\phi(x)}{S_t\mathbf{1}(x)} = \langle \nu, \phi \rangle
\tag{5.8}
$$

and $\nu \in M_1(D)$ is equal to

$$\nu(dx) = C(\alpha^*) \int_D \gamma(dx') R^D_{\alpha^*}(x', dx)\,, \qquad C(\alpha^*)^{-1} = \langle \gamma, R^D_{\alpha^*} \mathbf{1} \rangle \tag{5.9}$$

and $\alpha^* > \tilde{\alpha}$ solves (3.6). When $\bar{K} \neq 1$, we can write $C(\alpha^*) = \alpha^*/(1 - \frac{1}{\bar{K}})$ and when $\bar{K} = 1$ $\alpha^* = 0$ and ν is proportional to the Green function.

Proof. Step 1. The pole at α^*. Formula (5.5) for the resolvent of the semigroup S_t can be extended to a meromorphic function on $\Re(\alpha) > \tilde{\alpha}$ with one simple pole at $\alpha = \alpha^*$. First, we calculate the value of the pole

$$\lim_{\alpha \to \alpha^*} (\alpha - \alpha^*) R_\alpha \phi(x) = \lim_{\alpha \downarrow \alpha^*} (\alpha - \alpha^*) R^D_\alpha \phi(x) \tag{5.10}$$

$$+ \bar{K} \lim_{\alpha \downarrow \alpha^*} \left[\frac{(\alpha - \alpha^*)}{1 - \bar{K} \hat{f}_D(\alpha, \gamma)} \right] \lim_{\alpha \downarrow \alpha^*} \left[\hat{f}_D(\alpha, x) \langle \gamma, R^D_\alpha \phi \rangle \right] \tag{5.11}$$

$$= B(x, \bar{K}) \left[\frac{\langle \gamma, R^D_{\alpha^*} \phi \rangle}{\langle \gamma, R^D_{\alpha^*} \mathbf{1} \rangle} \right] . \tag{5.12}$$

Here $(\alpha - \alpha^*) R^D_\alpha \phi(x) \to 0$ as $\alpha \to \alpha^*$. The last line is due to the identity (1.7) $1 - \alpha \langle \gamma, R^D_\alpha \mathbf{1} \rangle = \hat{f}_D(\alpha, \gamma)$ and the fact that $1 - \bar{K} \hat{f}_D(\alpha^*, \gamma) = 0$.

Step 2. The limit of $e^{-\alpha^* t} S_t \phi(x)$. The proof follows nearly identically Step 4 of the proof of Theorem 1. First, we show directly that $\lim_{t \to \infty} e^{-\alpha^* t} S^D_t \phi(x) = 0$. The bound $|e^{-\alpha^* t} S^D_t \phi(x)| \leq ||\phi|| e^{-\alpha^* t} P_x(\tau^D > t)$ proves the limit because of (3.10).

A Tauberian theorem is needed, in the form of Lemma 1 followed by Proposition 8. First, without loss of generality we assume $\phi \geq 0$, otherwise $\phi = \phi_+ - \phi_-$ by decomposing in positive and negative parts. The limit is linear in ϕ via the factor $\langle \gamma, R^D_{\alpha^*} \phi \rangle$ so we obtain the exact limit as a difference of the two functions $\phi_\pm$.

We want to ensure that $t \to \int_D S_t \phi(x') \gamma(dx')$ in the convolution from (5.4) is non-negative (which we explained) and that there exists $\alpha_+ \geq 0$ such that $t \to e^{\alpha_+ t} f_D(t, x)$ is non-negative and non-decreasing. The second condition is in the hypothesis (Condition 1). These prove the limit equals the value of the pole at α^*

$$\lim_{t \to \infty} e^{-\alpha^* t} S_t \phi(x) = B(x, \bar{K}) \left[\frac{\langle \gamma, R^D_{\alpha^*} \phi \rangle}{\langle \gamma, R^D_{\alpha^*} \mathbf{1} \rangle} \right] . \tag{5.13}$$

We already know that $e^{-\alpha^* t} S_t \mathbf{1}(x) \to B(x, \bar{K}) > 0$ from (3.6) in Theorem 1. Since $B(x, \bar{K}) > 0$ we divide the two and obtain (6.7). $\square$

In the absence of Condition 1, a slightly weaker result is available. This is the usual Cesaró limit Tauberian theorem. It is important to mention that no assumption except $\bar{K} < \infty$ is necessary.

Corollary 3. *Let ν be the qsd from (5.9). With no assumption on the monotonicity of $t \to f_D(t,x)$, for any $C_b(D)$*

$$
\text{(5.14)} \qquad \begin{aligned} &\textit{if} \quad \bar{K} > 1 \quad \lim_{t\to\infty} \frac{\int_0^t S_s\phi(x)ds}{\int_0^t n_s^x ds} = \langle \nu, \phi \rangle \quad \textit{and} \\ &\textit{if} \quad \bar{K} \le 1 \quad \lim_{t\to\infty} \frac{\int_0^t e^{-\alpha^* s} S_s\phi(x)ds}{\int_0^t e^{-\alpha^* s} n_s^x ds} = \langle \nu, \phi \rangle \,. \end{aligned}
$$

Proof. We split, as in the proof of Theorem 3, $\phi = \phi_+ - \phi_-$, and apply Lemma 1 (ii) for the first limit, respectively (iii), for the second limit. Corollary 1 calculates the same limit for n_t^x. The ratio of the two limits proves (5.14). $\square$

The last result is more general as shown in Theorem 5, Part 2). We include the direct calculations as well as the identification of the eigenvalue.

Corollary 4. *Under the same conditions as in Theorem 2, the probability measure ν from (5.9) is a left-side eigenfunction of the resolvent R_α with eigenvalue $(\alpha - \alpha^*)^{-1}$. If, in addition, the Yaglom limit (5.8) holds, then ν is the left-side eigenfunction of S_t, $t > 0$ with eigenvalue $e^{\alpha^* t}$.*

Proof. We start with the resolvent. Recall that $C(\alpha^*)^{-1} = \langle \gamma, R^D_{\alpha^*}\mathbf{1}\rangle$. We want to show $\nu R_\alpha = (\alpha - \alpha^*)^{-1}\nu$, and more explicitly

$$
\text{(5.15)} \qquad \langle \gamma, R^D_{\alpha^*} R_\alpha \phi\rangle = (\alpha - \alpha^*)^{-1} \langle \gamma, R^D_{\alpha^*}\phi\rangle \,, \quad \alpha > \alpha^* \,.
$$

To see this, first we calculate, in view of (5.5) and (1.7)

(5.16)

$$\langle\gamma, R^D_{\alpha^*}\hat f_D(\alpha,\cdot)\rangle = \langle\gamma, R^D_{\alpha^*}\mathbf{1}\rangle - \frac{\alpha}{\alpha-\alpha^*}\langle\gamma,(R^D_{\alpha^*}\mathbf{1}-R^D_{\alpha}\mathbf{1})\rangle \qquad \text{(resolvent identity)}$$

$$= \langle\gamma, \frac{1}{\alpha^*}(1-\hat f_D(\alpha^*,\cdot))\rangle - \frac{\alpha}{\alpha-\alpha^*}\langle\gamma, \Big(\frac{1}{\alpha^*}(1-\hat f_D(\alpha^*,\cdot)) - \frac{1}{\alpha}(1-\hat f_D(\alpha,\cdot))\Big)\rangle \tag{5.17}$$

$$= C(\alpha^*)^{-1} - \frac{1}{\alpha-\alpha^*}\Big(\alpha C(\alpha^*)^{-1} - (1-\hat f_D(\alpha,\gamma))\Big) \tag{5.18}$$

$$= \frac{1}{\alpha-\alpha^*}\Big(-(1-\frac{1}{\bar K}) + (1-\hat f_D(\alpha,\gamma))\Big) \tag{5.19}$$

$$= \frac{1}{\bar K(\alpha-\alpha^*)}\Big(1-\bar K\hat f_D(\alpha,\gamma)\Big)\,. \tag{5.20}$$

Using the resolvent identity once more for the first term of (5.5), after simplification, we proved (5.15).

Choosing $\phi = \mathbf{1}$ we obtain $\nu R_\alpha\mathbf{1} = (\alpha-\alpha^*)^{-1}$ so $\lambda = \nu S_t\mathbf{1} = e^{\alpha^* t}$. Since any strong Yaglom limit is a qsd (Theorem 5), we obtained that λ is the exact eigenvalue for S_t. $\square$

6. Appendix

In this section we present some technical results and give some background on quasi-stationary distributions and Yaglom limits.

6.1. Exponential monotonicity.

The first section explores when exponential monotonicity - Condition 1 - is satisfied.

Proposition 7. *Let $f \in C([0,\infty);\mathbb{R})$ be positive with continuous derivative on $(0,\infty)$ and non-decreasing on some interval $(0,t_0]$, $t_0>0$. If $\liminf_{t\to\infty} f'(t)/f(t) > -\infty$, then f has exponential monotonicity.*

Proof. At $t\to\infty$, there exist $M_1 > -\infty$ and $t_1>0$ such that if $t\ge t_1$, then $\frac{f'(t)}{f(t)} \ge -M_1$. If $t_1>t_0$ we take M_0 the maximum between zero and $\sup_{t\in[t_0,t_1]}\left|\frac{f'(t)}{f(t)}\right|$. Finally, let $\alpha_+ = M_0\vee M_1 > -\infty$, then $\forall t>0$, $\frac{f'(t)}{f(t)} \ge -\alpha_+$ and $e^{\alpha_+ t}f(t)$ is non-decreasing. $\square$

Proof of Proposition 2. For the distribution of τ^D, being unimodal means that the density $f_D(t,x)$ has a non-negative derivative on some $(0,t_0]$. It is known that one-dimensional general diffusions have unimodal exit times ([26] - Rösler 1980, Annals of Prob), also more recently [20]. This takes care of the condition at $t\to 0$.

For $t \to \infty$, we use the formulas in Serafin [27]. More generally, this proof can be extended to other operators in higher dimensions using the classical theory of compact operators. The Dirichlet Laplacian on the unit ball gives a Bessel-Fourier series expansion for $F_D(t)$ with eigenvalues $\lambda_n = -\frac{j^2_{\mu,n}}{2}$ where $j^2_{\mu,n}$ in decreasing order, with $0 > \lambda_1 > \lambda_2$, where λ_1 is simple.

The values λ_n are given by the n-th nonzero roots of J_μ, $\mu = \frac{d}{2} - 1$, J_μ the Bessel function of first type. The exact form of the series implies $\lim_{t\to\infty} e^{-\lambda_1 t} f_D(t,x)$ is positive as well as $\lim_{t\to\infty} e^{-\lambda_1 t}\partial_t f_D(t,x)$ is finite negative and thus we obtain

$$\lim_{t\to\infty} \frac{\partial_t f_D(t,x)}{f_D(t,x)} = \lambda_1 < 0\,.$$

This implies the conditions of Proposition 7 are satisfied and Condition 1 is proven. $\square$

6.2. **Conditions for the Tauberian Theorem.** We shall use the Wiener-Ikehara Theorem for the density $a(t)$ of a Lebesgue-Stieltjes measure $A(t) = \int_0^t a(s)ds$ on the positive axis ([7], Theorem I-W, p.5). For a more general version, see [21] , Theorem 4.2 p. 124.

Theorem 4 (Wiener-Ikehara). *Suppose that the function F has the following properties:*

(i) For $\Re(\alpha) > 1$, $F(\alpha) = \int_0^\infty e^{-\alpha t} a(t)dt$, where $a(t)$ is a non-decreasing function with $a(0) \geq 0$.

(ii) For $\Re(\alpha) > 1$, $\alpha \neq 1$, $F(\alpha) = G(\alpha) + \frac{1}{\alpha - 1}$, where $G(\alpha)$ is continuous on the half-plane $\Re(\alpha) \geq 1$.

Then $e^{-t}a(t) \to 1$ as $t \to \infty$.

The next lemma will explain how we apply the Wiener-Ikehara theorem *for the density $a(t)$ when it is not non-decreasing.* The key ingredient is the exponential monotonicity property from Condition 1. When $a(t)$ is just non-negative, the Karamata Tauberian Theorem provides a limit at $t \to \infty$ in the Cesaró limit sense. This is used in the next lemma, part (iii).

Lemma 1. *Let $\alpha_1 < \alpha^*$ be two real numbers and $g : [0,\infty) \to [0,\infty)$ a continuous function with $g(0) \geq 0$ such that its Laplace transform $\hat{g}(\alpha) = \int_0^\infty e^{-\alpha t} g(t)dt$ is holomorphic on the half-plane $\{\alpha | \Re(\alpha) > \alpha_1,\, \alpha \neq \alpha^*\}$ and has a pole at α^* equal to q. Denote $G(t) = \int_0^t g(s)ds$.*

(i) If there exists $\alpha_+ > 0$ such that $t \to e^{\alpha_+ t} g(t)$ is non-decreasing (g is exponentially monotone), then $\lim_{t\to\infty} e^{-\alpha^* t} g(t) = q$.

(ii) If $\alpha^* > 0$, then $\lim_{t\to\infty} e^{-\alpha^* t} \int_0^t g(s)ds = q/\alpha^*$ and

(iii) If $\alpha^* \le 0$, then $\lim_{t\to\infty} t^{-1}\int_0^t e^{-\alpha^* s} g(s)ds = q$.

Remark. Independently from our work, a more general variant of part (i) of the Lemma, appears in a recent preprint ([14], Theorem 2). We note that (ii) is not true when $\alpha^* \le 0$. On the other hand, if $\alpha^* > 0$, the limit in (ii) implies the limit in (iii), so the statements cannot be improved with the same assumptions.

Proof. Part (i). For A, B real and $B > 0$, denote

$$g_{A,B}(t) = \frac{1}{B} e^{\frac{A}{B}t} g(\frac{t}{B}) \qquad \text{with Laplace transform} \quad \hat{g}_{A,B}(\alpha) = \hat{g}(B\alpha - A)\,.$$

We choose $A > \max\{-\alpha_1, \alpha_+\}$ and then $B = \alpha^* + A > 0$ such that

$$1 = \frac{\alpha^* + A}{B} > \frac{\alpha_1 + A}{B} > 0\,.$$

Since $A \ge \alpha_+$, $g_{A,B}(t)$ is non-decreasing and $\hat{g}_{A,B}(\alpha)$ is holomorphic except the pole at $\alpha = 1$ in the half-space $\Re(\alpha) > 1 - (\alpha^* - \alpha_1)/B$ which includes $\Re(\alpha) = 1$. Calculating that $\tilde{q} = \frac{q}{B}$ is the pole of $\hat{g}_{A,B}(\alpha)$ at $\alpha = 1$, we verified that the new function $g_{A,B}(t)$ satisfies the conditions of Wiener-Ikehara Theorem 4.

Then $\lim_{t\to\infty} e^{-t} g_{A,B}(t) = \tilde{q}$, which is equivalent to the limit in Part (i).

Part (ii). Unless $g \equiv 0$, the Laplace transform of $\int_0^t g(s)ds$ will have a pole at $\alpha = 0$. Adopt $\alpha_1 := 0$, α^* remaining the same in Part (i), this time applied to the integral, which is non-decreasing. This obtains the limit.

Part (iii). Part (i) cannot be applied even after a shift as done in the proof of Part (i). The function $t \to e^{-\alpha^* t} g(t)$ is non-negative and its Laplace transform exists for real $\alpha > 0$. At $\alpha \to 0^+$, the transform has a limit. The Karamata Tauberian Theorem [21], Theorem 8.1) implies the exact limit at (iii). $\square$

Proposition 8. *Let $h, f : [0,\infty) \to \mathbb{R}$ be continuous non-negative and there exists $\alpha_+ \ge 0$ such that $t \to e^{\alpha_+ t} f(t)$ is non-decreasing. Then*

$$t \to e^{\alpha_+ t}(h \star f)(t) = e^{\alpha_+ t}\int_0^t h(s) f(t-s)ds \tag{6.1}$$

is non-decreasing.

Proof. Step 1. We prove that the result is true for $\alpha_+ = 0$. Pick $t' > t \ge 0$. Then

$$(h \star f)(t') - (h \star f)(t) = \int_t^{t'} h(s) f(t'-s)ds + \int_0^t h(s)[f(t'-s) - f(t-s)]ds \ge 0\,.$$

Step 2. It is sufficient to write (6.1)

$$e^{\alpha_+ t}(h \star f)(t) = e^{\alpha_+ t}\int_0^t h(s)f(t-s)ds = \int_0^t [e^{\alpha_+ s}h(s)][e^{\alpha_+(t-s)}f(t-s)]ds$$

and then apply Step 1 to $e^{\alpha_+ t}h(t)$ and $e^{\alpha_+ t}f(t)$. $\square$

6.3. **Yaglom limits and quasi-invariant distributions.** For $D \subseteq \mathbb{R}^d$ a bounded domain, we shall consider $(S_t)_{t\geq 0}$ will be a continuous semigroup of bounded operators on the space $C_b(D)$ with the supremum norm, denoted by $||\cdot||$, i.e. $\forall \phi \in C_b(D)$

$$(6.2) \qquad \begin{aligned} &(i) \quad \forall\, t \geq 0\,, \qquad S_t\phi \in C_b(D) \quad \text{and} \quad S_0\phi = \phi \\ &(ii) \quad \forall\, t, t' \geq 0\,, \qquad S_{t+t'}\phi = S_t S_{t'}\phi \\ &(iii) \quad \forall x \in D\,, \quad t \to S_t\phi(x) \quad \text{is continuous}\,. \end{aligned}$$

Define the resolvent

$$(6.3) \qquad \alpha \to R_\alpha\phi(x) = \int_0^\infty e^{-\alpha t} S_t\phi(x)\, dt$$

for $\alpha \in \mathbb{C}$ such that the integral exists. We shall assume that there exists $\alpha_1 < 0$ such that

$$(6.4) \qquad \forall \alpha > \alpha_1 \qquad \sup_{x\in D}\int_0^\infty e^{-\alpha t} S_t 1(x)\, dt < +\infty\,.$$

A stronger condition is that there exist $\alpha' \geq 0$, $C > 0$ such that $||S_t|| \leq Ce^{\alpha' t}$. This is the case of the semigroup (5.3) in this paper with $\alpha' := \alpha^* \vee 0$.

In view of (6.4), the resolvent is holomorphic on $\Re(\alpha) > \alpha_1$.

When $\nu \in M_1(D) \subset M_F(D)$ is positive, then $\langle \nu, S_t 1\rangle > 0$. A probability measure $\nu \in M_1(D)$ is said a *quasi-stationary distribution* (qsd) for the semigroup S_t if

$$(6.5) \qquad \frac{\langle \nu, S_t\phi\rangle}{\langle \nu, S_t 1\rangle} = \langle \nu, \phi\rangle\,, \qquad \forall\, t \geq 0\,.$$

In the context of this paper, the process we study the the semigroup $t \to E_x[\,\langle\, \zeta_t, \phi\,\rangle\,] = S_t\phi(x)$ formally defined in (5.3). Theorem 3 proves it satisfies the definition (5.2).

Then, the definition (6.5) of a qsd ν reads explicitly as

$$(6.6) \qquad E_\nu[\sum_{i=1}^{N_t} \phi(Z_t^i)] = E_\nu[N_t]\cdot\langle \nu, \phi\rangle\,, \qquad \forall \quad t \geq 0\,.$$

A probability measure $\nu \in M_1(D)D$ is said a *Yaglom limit* for the semigroup S_t if there exists a probability measure ν' such that, for all $\phi \in C_b(D)$

$$\lim_{t\to\infty} \frac{\langle \nu', S_t\phi\rangle}{\langle \nu', S_t 1\rangle} = \langle \nu, \phi\rangle\,. \tag{6.7}$$

In that case we say ν' is in the *domain of attraction* of ν. If a Yaglom limit has domain of attraction all delta functions, or equivalently, any probability measure ν' on D, it is said a *strong Yaglom limit*.

Theorem 5 is straightforward and well known for dissipative semigroups. See the monograph [9] and [13] for the relation between qsd and eigenfunctions.

Theorem 5. *Assume (S_t) has the properties (6.2). Then*

1) If ν is a qsd, then the expected value of the total mass $\langle \nu, S_t 1\rangle$ is exponential;

2) The qsd ν is a left side eigenfunction of the semigroup, as well as of the resolvents;

3) Any strong Yaglom limit ν is a qsd and is in its own domain of attraction. A strong Yaglom limit, if it exists, is unique.

Proof. 1) Using $\phi = S_s\psi$ and $\psi(x) = \mathbf{1}(x)$ we obtain that $t \to \nu S_t \mathbf{1} = n_t^\nu$ is continuous and satisfies $n_{t+t'}^\nu = n_t^\nu n_{t'}^\nu$, hence is an exponential function. In case the semigroup is dissipative $||S_t|| \le 1$, the time to extinction is exponentially distributed.

2) For the semigroup, it is due to the definition, and for the resolvent we obtain the relation directly by integration.

3) Let t, t' positive. Then, applying the definition (6.7) with $S_{t'}\phi$ in place of ϕ,

$$\lim_{t\to\infty} \frac{\langle \nu', S_t S_{t'}\phi\rangle}{\langle \nu', S_t 1\rangle} = \langle \nu, S_{t'}\phi\rangle\,. \tag{6.8}$$

$$\frac{\langle \nu', S_t S_{t'}\phi\rangle}{\langle \nu', S_t 1\rangle} = \frac{\langle \nu', S_{t+t'}\phi\rangle}{\langle \nu', S_{t+t'} 1\rangle} \cdot \frac{\langle \nu', S_t S_{t'} 1\rangle}{\langle \nu', S_t 1\rangle}\,. \tag{6.9}$$

Let $t \to \infty$. The first factor converges to $\langle \nu, \phi\rangle$ as $t + t' \to \infty$ and the second factor uses (6.7) with $S_{t'}1$ in place of ϕ, to converge to $\langle \nu, S_{t'}1\rangle$. The equality of the two limits shows that ν is a qsd. $\square$

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