PARTICLE REPRESENTATION FOR THE HEAT EQUATION WITH MASS CREATION

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Abstract. We solve the heat equation for a second-order linear operator L on the bounded domain $D \subseteq \mathbb{R}^d$ under non-classical boundary conditions (super-critical) which prescribe that the inward flux entering D at interior points with a given distribution γ be equal to $\bar{K} > 1$ times the outward flux on ∂D. A weak solution to the forward equation exists and can be represented as the the expected value of a non-conservative particle process driven by L inside D, branching at ∂D with branching constant $\bar{K}$. We provide exact estimates on its growth rate and study the associated backward equation. Uniqueness requires a mild condition on the density of the exit time from D of the diffusion driven by L. In the spacial case of $\gamma = \delta_c$, $c \in D$, we prove the existence of a strong solution. The main application and motivation is that normalized to have total mass equal to one, the solution is the hydrodynamic limit of the BSBD particle system from [19].

1. Introduction

Let $D \subseteq \mathbb{R}^d$, $d \geq 1$ a bounded domain boundary ∂D, L a second order strongly elliptic operator L defining a diffusion on D killed at the boundary and $\gamma \in M_1(D)$, a probability measure on D with $supp(\gamma) \subset D$. In this paper, we focus on a non-classical boundary problem for the heat equation determined by L where mass will be created along the support of γ and annihilated on ∂D.

The connection between Markov processes and the solutions of parabolic equations is well known in diffusion theory. If (BC) stands for certain boundary conditions on ∂D (Dirichlet, Neumann, mixed, sticky), then the process (X_t) driven by L and solving the martingale problem for test functions satisfying (BC) has a transition kernel that satisfies the corresponding heat equation. More explicitly, given a test function ϕ, the semigroup formula

$$E_x[\phi(X_t)] = \int_D p(t,x,y)\phi(y)dy\,,$$

Date: June 24, 2019.

1991 *Mathematics Subject Classification.* Primary: 60J85; Secondary: 35Q92, 35K20, 60J35, 60J70 .

Key words and phrases. Key words and phases: Branching processes, supercritical, mass creation, hydrodynamic limit, diffusion driven, accretive/dissipative heat equation, Fleming-Viot.

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where $p(t,x,y)$ the density of its transition probabilities and $E_x[\cdot]$ is the expected value of the process starting at $X_0 = x \in D$, establishes that $p(t,x,y)$ solves the backward equation $\partial_t v = Lv$ in the variables (t,x) and the forward equation $\partial_t v = L^* v$ in (t,y). This gives a representation of v, the solution of a PDE in terms of a diffusion process on D. In the Dirichlet case ($v = 0$ on ∂D), the diffusion is *killed at the boundary.* This corresponds to the dissipative character of the equation (subcritical), where the mass decreases to zero as $t \to \infty$.

We are interested in solving the heat equation with mass being created at a random point chosen with distribution γ in D and dissipated on the boundary in such a way that the total mass increases in time, leading to a super-critical regime. The equation is not symmetric. Condition (1.2) for the backward equation is straightforward, but the forward case is more difficult and the condition itself makes no sense except for a strong classical solution and special γ. To fix ideas, assuming L is the half-Laplacian and mass is created at a point $c \in D$ then the boundary condition for the forward equation expresses a flux balance (3.5), where the inward flux at c is $\bar{K}$ times the outward flux at the boundary, in addition to the usual Dirichlet boundary condition.

A strong solution may not exist except away from *supp*(γ). The delta function case can be generalized relatively easily if the support is of measure zero, for example, a finite set but also the graph of a smooth curve in D, both with a flux balance condition similar to (3.5).

Since the proportionality constant $\bar{K}$ exceeds one, v tends to infinity as $t \to \infty$. It turns out that the supercritical mass evolution required here cannot be represented by a one-particle process with state space D, as opposed to the familiar dissipative case of a one-particle killed process. Instead, it is natural to look at a measure-valued process with varying total mass. In its turn, this can be represented by an explicit multi-particle branching process on D (here called auxiliary process), where each particle is driven by L. The expected value of this process will be the analogue of a Dynkin type formula representation for $v(t,x,y)$.

1.1. **Motivation of the problem.** The motivation to study this equation comes from hydrodynamic limits. The solution v produces the natural description of the asymptotic profile $\rho(t,y)$ of the empirical measure of the *Bak-Sneppen Branching Diffusions* (BSBD), a conservative particle system studied in [19]. Since the particle system has constant mass, empirical measures are probability measures and their limit, assuming it exists, has total mass one. Let $v(t,y) = \int_D v_0(x) v(t,x,y) dx$ be the solution with mass creation starting at v_0 and $n_t = \int_D v(t,y) dy$ its total mass. This limiting profile of the particle system turns out to be exactly $\rho(t,y) = v(t,y)/n_t$, the normalization of v. Additionally, $\ln n_t$ is the limiting value of the average number of boundary hits, or equivalently, branching events, up to time $t > 0$.

Hydrodynamic limits are the result of Laws of Large Numbers whose deterministic trajectories can typically be identified as weak solutions of a PDE in the space of finite measures. To close the proof, it is essential to show uniqueness. Regularity properties of the (non-random) PDE are also important. In view of this application, and to solve a non-conservative heat equation in and of itself, we set off to study the existence and uniqueness of the weak solution; give sufficient conditions to be classical (a function), respectively strong.

We start formulating the problem. As mentioned, it is naturally obtained in *weak form*, i.e. satisfying an equation upon integration against a sufficiently smooth test function.

1.2. **Heat equation with mass creation.** The notation $M_F(D)$ designates the finite measures on $D \subseteq \mathbb{R}^d$, $M_1(D)$ the space of probability measures on D (distributions), both with the topology of convergence in distribution.

Define the time-space set of test functions smooth up to the boundary

$$(t,x) \to \phi(t,x) \in \mathcal{D} = C_b^{1,2}([0,T] \times \bar{D}; \mathbb{R})\,. \tag{1.1}$$

When $\partial D \in C^2$ this is equivalent to the set of $C_c^{1,2}(\mathbb{R} \times \mathbb{R}^d; \mathbb{R})$ of smooth functions with compact support. This space is standard in the study of martingale problems with boundary conditions, e.g. [29].

Given a test function $\phi(t,\cdot)$, with $t \geq 0$ fixed, we write $\langle \phi(t,\cdot), m\rangle = \int_D \phi(t,y) m(dy)$ for the integral against a finite measure $m(dy)$ on D, also applicable when ϕ does not depend on t. Let $\gamma \in M_1(D)$, $supp(\gamma) \subseteq D$ and a constant $\bar{K} > 1$. We note that $dist(supp(D), \partial D) > 0$.

For $\phi \in \mathcal{D}$, define the boundary condition

$$\bar{K}\langle \phi(t,\cdot), \gamma\rangle = \phi(t,y)\,, \quad y \in \partial D\,. \tag{1.2}$$

Remark. To explain the bar notation, in the construction of the auxiliary process, K is the integer valued random number of offsprings and $\bar{K}$ its mean value. In the hydrodynamic limit mentioned earlier, K is deterministic (specifically $K = 2$), but the derivation of the PDE can be done in general.

Definition 1. *We shall say that $t \to \nu_t \in C([0,\infty), M_F(D))$ is the weak solution to the heat equation for L on D with mass creation $(\gamma(dx), \bar{K})$ and initial value $\nu_0 \in M_F(D)$ if, for any test function $\phi \in \mathcal{D}$ satisfying the additional boundary condition (1.2), and any $t \geq 0$, the equality holds*

$$\langle \phi(t,\cdot), \nu_t\rangle - \langle \phi(0,\cdot), \nu_0\rangle - \int_0^t \langle \frac{\partial}{\partial s}\phi(s,\cdot) + L\phi(s,\cdot), \nu_s\rangle\, ds = 0\,. \tag{1.3}$$

Remark. Condition (1.2) sets $\phi(t, \cdot)$ constant on the boundary. Any function $\phi(t, \cdot)$ supported on a subset of $D \setminus supp(\gamma)$ satisfies the equation, showing that the problem is not vacuous.

1.3. **Idea of the proof.** Based on the *underlying* diffusion L, we shall construct a branching process $(\zeta_t)_{t \geq 0}$ defined in (4.1). The process represents the empirical measure (sum of delta measures) of a particle process with a finite number of individuals at all times. These individuals move according to L in D. Branching occurs when one of them hits ∂D. At that point, the individual is removed and K independent particles with $E[K] = \bar{K}$ are born at one random location, chosen with distribution $\gamma(dx)$ concentrated on D. We note that $supp(\gamma) \subseteq D$ and some regularity on the coefficients of L imply that the individuals have a positive lifetime and no two individuals reach the boundary at the same time, with probability one. Between boundary hits the particles follow independent diffusions driven by L. We then prove the representation of the solution to (1.3) $\nu_t = E[\zeta_t]$, introduced in (1.3) as well as its exponential growth bound.

Remark. The branching process is closely related to models in genome population dynamics. Here *mutation* is represented by the diffusive term (the Brownian term), *selection* is represented by drift (in the probabilistic, not geneticists' sense) and *recombination* is represented by the redistribution at a random point $\sim \gamma(dx)$ where the new mass is born. Genetic recombination can be seen as a repair to damaged DNA. If artificial, it is under the effect of a *catalyst*, many times simplified as contact with a portion of the boundary. The BSBD particle process mentioned earlier is exactly its normalization, or the statistical relative frequency of types.

1.4. **Summary of the paper.** The goal is to solve the above equation. Section 2 gives the conditions we assume on the underlying process driven by L. Section 3 states the main results. Theorem 1 solves the *heat equation with mass creation*, giving its representation as the expected value of the empirical measure of the *auxiliary branching process.* Theorem 2 proves some regularity properties of the solution, including being a function (having a density) away from the support of γ. Theorem 3 studies the strong solution when L is the half-Laplacian and γ is a delta function. Moreover, the weak boundary condition (1.2) - imposed on test functions - becomes the *flux balance condition* (3.5). The auxiliary branching process is defined in Section 4. The proofs of the main theorems in Section 3 are done in Section 5.

2. Conditions on the underlying process

The underlying process is the diffusion driven by a second order, strongly elliptic operator L with sufficiently smooth coefficients, given by

$$Lu(x) = \frac{1}{2} \sum_{1\le i,j\le d} a_{ij}(x)\partial^2_{x_i x_j} u(x) + \sum_{1\le j\le d} b_j(x)\partial_{x_j} u(x)\,, \qquad u \in C_c^\infty(\mathbb{R}^d)\,. \tag{2.1}$$

Due to the requirements of this section, we assume $a = (a_{ij})$ is a symmetric $d\times d$ matrix, $b = (b_j)$ a d - dimensional vector, both with $C^\infty(\mathbb{R}^d)$ components having bounded derivatives and there exists $a_0 > 0$ such that $\langle az, z\rangle \ge a_0||z||^2$, $z \in \mathbb{R}^d$ (uniform ellipticity). The diffusion is killed at τ^D, the hitting time of the boundary of a bounded domain $D \in C^2$. The killed process solves the martingale problem for $(L, \mathcal{D}(L))$, where

$$\mathcal{D}(L) = \{\phi \in C(\bar{D}) \cap C_b^2(D) \,|\, \phi(y) = 0\,,\ \ y \in \partial D\} \tag{2.2}$$

and its transition probabilities $p^D(t, x, dy)$ have densities (Dirichlet heat kernel)

$$p^D(t, x, dy) = p^D(t, x, y)dy\,, \qquad p^D(t, x, y) \in C^{1,2}((0,\infty) \times \bar{D}^2) \tag{2.3}$$

and define a *strongly continuous Feller-Dynkin semigroup* (cf. [27], Chapter III.6)

$$S_t^D\phi(x) = \int_D p^D(t, x, y)\phi(y)dy \in C(\bar{D})\,, \quad \phi \in C(\bar{D}) \ \text{(Feller property)}\,. \tag{2.4}$$

The semigroup is C_0 (strongly continuous) on $C_0(D)$, the space continuous functions vanishing at infinity on D with the supremum norm. The boundary is identified with the point at infinity.

The density exists and is continuous for $t > 0$ due to (2.3). Denote $f_D(t, x)$, $F_D(t, x)$ the density, respectively distribution functions hitting time of the boundary τ^D for initial state $x \in D$ and $t > 0$. Then

$$P_x(\tau^D > t) = 1 - F_D(t, x) = S_t^D 1(x) = \int_t^\infty f_D(s, x)\, ds\,. \tag{2.5}$$

Given $\alpha \in \mathbb{R}$, we denote the resolvent

$$R_\alpha^D\phi(x) = \int_0^\infty e^{-\alpha t} S_t^D\phi(x)dt\,, \quad R_\alpha^D(x, dy) = r_\alpha(x, y)dy\,, \qquad \phi \in C(\bar{D}) \tag{2.6}$$

provided the integral converges. The Laplace transform

$$\alpha \to \hat{f}_D(\alpha, x) = E_x[\exp(-\alpha\tau^D)] = 1 - \alpha R_\alpha 1(x)\,, \tag{2.7}$$

after an integration by parts.

Condition 1. *There exists $\alpha_- < 0$ such that*

$$\sup_{x\in\bar{D}} \hat{f}_D(\alpha_-, x) = c_D < \infty\,. \tag{2.8}$$

While (2.6) and (2.7) are always true for $\alpha > 0$, the bound c_D from (2.8) implies that the semigroup S_t^D generated by diffusion $(L, \mathcal{D}(L))$ has a spectral gap in the sense that $\sup_{x\in\bar{D}} R_{\alpha_-}1(x) < \infty$ and then $\{\alpha \in \mathbb{C} \,|\, \Re\alpha > \alpha_-\} \subseteq Res(L)$.

Additional regularity will be required for uniqueness and the existence of the strong solution, in the form of an off-diagonal bound on the Dirichlet heat kernel.

Condition 2. *For all $\beta > 0$ sufficiently small and $x, y \in \bar{D}$ with $dist(x,y) \geq \beta$, there exists $C_D(t,\beta) > 0$, depending on $t > 0$ and β only and bounded on $[0,T]$, $T > 0$ such that*

$$|p^D(t,x,y)| \leq C_D(t,\beta)\,, \quad \lim_{t\to 0} C_D(t,\beta) = 0\,. \tag{2.9}$$

The last condition is more demanding, because it assumes the density of the hitting time is smooth. It is only needed for uniqueness and the strong solution.

Condition 3. *The density $f_D \in C([0,\infty)\times\bar{D}) \cap C^{1,2}((0,\infty)\times\bar{D})$, has bounded derivatives and is a classical solution to the heat equation $\partial_t f_D = L f_D$ on $(0,\infty)\times D$ with initial value $f(0+,x) = 0$.*

2.1. **Sufficient conditions.** When L is in divergence form and uniformly elliptic with Dirichlet boundary conditions, milder assumptions on D and the coefficients than those made in this section are sufficient, e.g. Condition 1 is implied by Theorem 1.5.8. (cf.[8], p 27) where D needs not be bounded nor C^2, only strongly regular. For the bounds on the heat kernel we refer to Theorem 4.6.9 in the same [8], also discussed in [32]. On the other hand, stronger assumptions, like a smooth Dirichlet heat kernel $p^D(t,x,y)$ satisfying Gaussian off diagonal bounds would imply Condition 1 and evidently 2 (see [21], also cf. [30] for a discussion of hypoellipticity and Gaussian bounds). It is Condition 3 that needs more smoothness in ∂D and the coefficients. In the case of the half-Laplacian we prove Proposition 1 below. Other weaker conditions may be imposed, by combining regularity criteria for the Dirichlet problem ([14], Thm. 9.19) and the heat equation ([11], Thm. 6 and 7, Section 7.1).

Proposition 1. *When $L = \frac{1}{2}\Delta$ and ∂D is bounded of class C^2, then (2.3), (2.4) and Conditions 1, 2 and 3 are satisfied.*

Proof. The Dirichlet Laplacian satisfies (2.3), (2.4) and the smoothness up to the boundary is guaranteed by $\partial D \in C^2$. As mentioned in the discussion above, the spectral gap holds when D is bounded and L is in divergence form (cf.[8]). The upper bounds found in [32] for the Dirichlet heat kernel prove Condition 2. It remains to prove Condition 3.

Based on [21], p. 9, Lemma 2.5. When $L = \frac{1}{2}\Delta$, using that ∂D of class C^2, an application of Green's theorem shows that for $\psi(y) \equiv 1$

$$f_D(t,x) = -\partial_t S_t^D \psi(x) = -\frac{1}{2}\Delta_x S_t^D \psi(x) = -\frac{1}{2}\Delta_x \int_D p^D(t,x,y)\psi(y)dy$$

$$= -\int_D \frac{1}{2}\Delta_x p^D(t,x,y)\psi(y)dy = -\int_D \frac{1}{2}\Delta_y p^D(t,x,y)\psi(y)dy =$$
$$-\int_D p^D(t,x,y)\frac{1}{2}\Delta_y\psi(y)dy + \frac{1}{2}\int_{\partial D}\Big(\psi(y')\nabla p^D(t,x,y') - p^D(t,x,y')\nabla\psi(y')\Big)\cdot \mathbf{n}\, d\sigma(y') =$$
$$\frac{1}{2}\int_{\partial D}\nabla p^D(t,x,y')\cdot \mathbf{n} d\sigma(y')\,.$$

Notice that we used $\Delta_x p^D(t,x,y) = \Delta_y p^D(t,x,y)$ since $L = L^*$ and $G(x,y) = G(y,x)$ in this case.

We obtained

$$f_D(t,x) = \frac{1}{2}\int_{\partial D}\nabla p^D(t,x,y')\cdot \mathbf{n}\, d\sigma(y')\,.$$

Theorem 2.1 in [33] gives upper bounds for the gradient of the Dirichlet heat kernel in a bounded domain. Using the off-diagonal estimate we derive $\lim_{t\to 0} f_D(t,x) = 0$. □

We conclude this section with a consequence of Condition 1, that will be used throughout the paper. To simplify notation, write

$$\hat{f}_D(\alpha, m) = \langle \hat{f}_D(\alpha,x), m(dx)\rangle\,, \quad m \in M_1(D)\,. \tag{2.10}$$

Proposition 2. *Let $\gamma \in M_1(D)$. Under the assumption (2.8) there exists $-\infty < \tilde{\alpha} < 0$ such that the decreasing function $\alpha \to \hat{f}_D(\alpha,\gamma)$ satisfies*

$$\lim_{\alpha\downarrow\tilde{\alpha}} \hat{f}_D(\alpha,\gamma) = +\infty\,, \qquad \lim_{\alpha\to\infty}\hat{f}_D(\alpha,\gamma) = 0\,. \tag{2.11}$$

The Laplace transform $\alpha \to \hat{f}(\alpha,\gamma)$ can be extended to a holomorphic function on $\Re(\alpha) > \tilde{\alpha}$. For any $\bar{K} > 0$ there exists a unique $\alpha^* \in (\tilde{\alpha}, +\infty)$ such that

$$1 - \bar{K}\hat{f}_D(\alpha^*,\gamma) = 0\,. \tag{2.12}$$

Moreover, due to $\hat{f}_D(0,\gamma) = 1$, $\alpha^* > 0$ ($\alpha^* < 0$) for $\bar{K} > 1$ ($\bar{K} < 1$), with equality $\alpha^* = 0$ when $\bar{K} = 1$.

Remark. The proposition allows $\bar{K} > 0$, slightly more general than our needs for $\bar{K} > 1$. We note that α^* is necessarily positive in that case.

Proof. Consider $\gamma \in M_1(D)$ with $supp(\gamma) \subset D$. Due to (2.8) $\hat{f}_D(\alpha,\gamma) \le c_D$ for $\alpha > \alpha_-$ after integration over $\gamma(dx)$. Since $\hat{f}_D(\alpha,\gamma)$ is the Laplace transform of a non-negative random variable, $\alpha \to \hat{f}_D(\alpha,\gamma)$ is decreasing. Due to (2.8) we certainly have $\tau^D < \infty$ with probability one, which proves the limit as $\alpha \to \infty$ by dominated convergence. The existence of a finite $\tilde{\alpha}$ follows from the monotone convergence theorem, unless $\hat{f}_D(\alpha,\gamma)$ was finite for all $\alpha \to -\infty$, in which case $\tau^D = 0$ a.s., which is not allowed, again by (2.8). It follows that $\hat{f}_D(\alpha,\gamma)$ sweeps through $(0,+\infty)$ and assumes every value $\bar{K}^{-1}$. The complex extension into the half plane is standard. □

3. Main results

Theorems 1, 2 and 3 are the main results of the paper. Their proofs will be done in Section 5, based on the branching process constructed in Section 4.

Since $supp(\gamma)$ is disconnected from ∂D by hypothesis, $D(\gamma) = D \setminus supp(\gamma)$ is an open set whose boundary includes ∂D. For a finite signed measure m, we denote $|m| = m_+ + m_-$, the sum of the positive and negative parts and the total variation $||m||$.

Definition 2. *We shall say $m_\cdot \in C([0,\infty), M_F(D))$ is a regular solution if it is uniformly bounded in total variation and has a bounded density in the vicinity of the boundary, both locally uniform in time. More precisely, for any pair (t_0, T) with $0 < t_0 < T < \infty$ and any Borel set $F \subseteq D(\gamma)$, there exists a constant $C(t_0, T)$ such that*

$$\sup_{0 \le t \le T} ||m_t|| < \infty \qquad \textit{and} \qquad \sup_{t_0 \le t \le T} |m_t(F)| \le C(t_0, T)\, Leb(F)\,. \tag{3.1}$$

Remark. A regular solution has properties pertaining to the *forward equation*, especially the second inequality where we assert a weak Dirichlet boundary condition.

Theorem 1. *Assume Condition 1 holds. For any initial value $\nu_0 \in M_F(D)$, there exists a weak solution $\nu_\cdot \in C([0,\infty), M_F(D))$ of equation (1.3), where time continuity is defined in the topology of finite measures. The solution admits the representation*

$$\forall \phi \in \mathcal{D} \qquad \langle \nu_t, \phi \rangle = E_{\nu_0}[\langle \zeta_t, \phi \rangle]\,, \quad t \ge 0\,, \tag{3.2}$$

where $(\zeta_t)_{t \ge 0}$ is the auxiliary measure-valued process $(\zeta_t)_{t \ge 0}$ defined in Section 4. If, in addition, Condition 2 is satisfied, the solution is regular and when 3 holds as well, then the regular solution is unique.

From its representation (3.2), the solution satisfies an exponential growth condition.

Corollary 1. *If ν_0 is a probability measure, the total mass $n_t = \langle \nu_t, 1 \rangle$ of the solution with representation (3.2) is positive, non-decreasing and there exists a constant $c(\nu_0)$ depending only on the initial value ν_0 such that $0 < n_t \le c(\nu_0) e^{\alpha_* t}$, for any $t \ge 0$, where α^* is the constant defined in Proposition 2.*

Proof. Assuming Theorem 1, the proof is based on the representation

$$\langle \nu_t, \phi \rangle = E_{\nu_0}[\langle \zeta_t, \phi \rangle] = \int_D E_x[N_t]\, \nu_0(dx)\,, \quad t \ge 0\,,$$

applied to $\phi(t,x) \equiv 1 \in \mathcal{D}$ and Proposition 3 in Section 4. The last equality is from the construction of the branching process. Finally the exponential bound comes from (4.4), after integrating against the initial measure. The total mass n_t is non-decreasing as the expected value of a non-decreasing function. □

When $\nu_0 = \delta_x$, the solution is denoted ν_t^x and if it has a density we write $\nu_t^x(dy) = v^x(t,y)dy$. For convenience, the regularity properties of ν_t are given in the next theorem.

Theorem 2 (Properties of the density function)**.** *Assume Conditions 1 and 2. Let $D' \subset\subset D(\gamma)$, $F = D' \cup \partial D$ and $\beta = dist(supp(\gamma), F) > 0$, where $D(\gamma)$ is the same as in Definition 2.*

(i) For arbitrary times $0 < t_0 < T$, there exists a constant $C(t_0, T, \beta) > 0$ such that

$$\sup_{t\in[t_0,T],(x,y)\in D\times F} v^x(t,y) = C(t_0,T,\beta)\,, \tag{3.3}$$

implying that it satisfies (3.1) with $C(t_0,T) = C(t_0,T,\beta)$. Moreover, if $p^D(t,x,y) = 0$ when $y \in \partial D$, then the same is true for $p(t,x,y)$.

(ii) For any initial value $\nu_0 \in M_1(D)$ (probability measure, total variation one) the measure in (3.2) has a density $\nu_t(dy) = v(t,y)dy$, $y \in F$, $t > 0$ satisfying (i) with the same constant. If, in addition, $\nu_0(dy) = v_0(y)dy$, $v_0 \in C(\bar{D})$, then $v \in C([0,\infty) \times F)$.

3.1. **The strong solution.** Let $w \in C^1(\bar{D} \setminus \{c\})$). Define the *inward flux from* c the limit

$$\Phi(w,c) = \lim_{\epsilon\to 0} \int_{\partial B(c,\epsilon)} \nabla w(y) \cdot \mathbf{n}\, dS\,, \tag{3.4}$$

where $\mathbf{n}$ is the outward normal to the sphere centered at c, whenever the limit exists and is finite. We also denote the usual flux of w over the boundary $\Phi(w,\partial D) = \int_{\partial D} \nabla w(y)\cdot \mathbf{n}\, dS$, where $\mathbf{n}$ is the outward normal to ∂D.

Remark. The flux is said *inward* because it *enters* the set at c. It can be considered asymptotically equal to the opposite of the outward flux seen from the interior of the punctured set through a small ball centered at c.

Theorem 3. *Assume the same conditions as in Theorem 2. Let $L = \frac{1}{2}\Delta$, $\gamma(dx) = \delta_c(dx)$ for some $c \in D$ and $\nu_0(dy) = v_0(y)dy$, $v_0 \in C(\bar{D})$. Then the solution from Theorem 1 has a density, i.e. $\nu_t(dy) = v(t,y)dy$, integrable in the space variable for any $t \geq 0$ with $v \in C([0,\infty)\times\overline{D}\setminus\{c\}) \cap C^{1,2}((0,\infty)\times D\setminus\{c\})$ which is a solution of $\partial_t v = \frac{1}{2}\Delta v$ on $D\setminus\{c\}$ with $v(t,y)|_{\partial D} = 0$ satisfying the* flux balance condition

$$\Phi(v(t,\cdot),c) = \bar{K}\Phi(v(t,\cdot),\partial D)\,. \tag{3.5}$$

Remark. The solution $v(t,y)$ behaves like a Green function, and has a singularity of the same type. In other words $v(t,y) \sim ||y-c||^{2-d}$ as $y \to c$.

4. The auxiliary process ζ_t

Let $\tilde{D} = D \cup \{\mathfrak{o}\}$ the usual compactification of D with $D \cup \{\mathfrak{o}\}$, where $\mathfrak{o}$ is an isolated point for D. Here $\mathfrak{o}$ will be a cemetery point. We introduce a branching process constructed on a subspace of $M_F(\tilde{D})$, the space of finite measures on $\tilde{D}$. First, we construct it as a particle system Z_t having a finite but random number of particles N_t alive at any $t \geq 0$.

With the same notations as in Section 1, a single particle is placed at a random point with initial distribution $\nu_0(dx)$ at $t = 0$ and starts moving according to $(L, \mathcal{D}(L))$ until it reaches ∂D, when it goes to $\mathfrak{o}$. At that moment, instantaneously, a random integer number K of particles are born at a specific point in D distributed according to $\gamma(dx)$.

All particles start afresh, independently, and move in D until the first one dies and the branching is repeated. The procedure is continued indefinitely. The number K is always distributed according to $\pi(k)$, $k \geq 1$, where π is a probability mass function on the non-negative integers.

Two exceptional outcomes are to be taken into account, explosion and extinction, but none occurs in the setup of this paper (see below). It is shown below that the total mass is finite a.s., with an exponential bound in expected value, proving that it is not explosive. Extinction is not possible if $K \geq 1$ with probability one.

Denote by Z_t^i, $i \in \mathbb{Z}_+$ the i-th particle born in the process. If it is born at time τ, then we understand that $t \geq \tau$. Let N_t^{tot} be the total number of individuals ever born up to time $t \geq 0$. When at time τ a number $j \geq 1$ of individuals are born, their birth being simultaneous and at the same point, their ordering is not relevant. They are simply labeled $i = N_{\tau-}^{tot} + l$, $1 \leq l \leq j$. By construction, the process starts and preserves a finite number of particles during its lifetime with probability one.

We introduce the process

$$\zeta_t = \sum_{i=1}^{N_t^{tot}} \delta_{Z_t^i}\,, \quad \text{with} \quad N_t = \langle \zeta_t, \mathbf{1}_D \rangle\,, \qquad t \geq 0\,, \tag{4.1}$$

where N_t is the number of particles alive at time $t \geq 0$ and $N_t^{tot} - N_t$ is the number of killed particles, which is equal to the number of branching events up to time $t > 0$. As defined in (4.1), the process starts at a random point X with distribution $\nu_0 \in M_1(D)$.

The process (ζ_t) evolves in the space of discrete measures, more precisely

$$M_0(\tilde{D}) = \{\mu \in M_F(\tilde{D}) \mid \mu = \sum_{i \in I} \delta_{x_i}(dx),\, x_i \in \tilde{D}\,, i \in I\,, I \,\text{finite}\}\,,$$

a subset of $M_F(\tilde{D})$. We shall consider $M_0(\tilde{D})$ the space of finite configurations in $\tilde{D}$

$$M_0(\tilde{D}) = \cup_{N=1}^{\infty} \tilde{D}^{(N)} \cup \{\mathbf{0}\}$$

where $\mathbf{0}$ is the zero measure (no delta function is present) endowed with the topology of disjoint unions. Each $D^{(N)}$ is the symmetric product of D (i.e. the product space factorized by symmetries on D^N) with the product topology of D as a subset of $\mathbb{R}^d$ (see also [5], [26]). This is a Lusin space because each of the spaces in the summation is a complete separable metric (Polish) space. A net converges only if eventually all terms belong to only one element in the sum and convergence takes place in the topology of that space. In our construction, for finite configurations, we note that convergence in norm on $\tilde{D}^N$ implies weak convergence of measures.

Definition 3. *Denote (ζ_t^x) the the process with $\nu_0 = \delta_x$ as in (4.1) and if $\mu = \sum_{i\in I} \delta_{x_i}$, I finite, then $\zeta_t^\mu = \sum_{i\in I} \zeta_t^{x_i}$, where the processes $(\zeta_t^{x_i})$ are independent. We note that the points x_i are not necessarily distinct, but each individual acts as an independent copy in (ζ_t^μ).*

By construction, (ζ_t) is a pure branching process in the sense that, if $\mu_1, \mu_2 \in M_0(D)$, then

$$\mathcal{L}(\zeta_\cdot^{\mu_1+\mu_2}) = \mathcal{L}(\zeta_\cdot^{\mu_1}) * \mathcal{L}(\zeta_\cdot^{\mu_2}) \,, \tag{4.2}$$

where $\mathcal{L}(\cdot)$ denotes the probability law.

Finally, a technical clarification. We shall assume an infinite sequence of i.i.d. processes with the same law as Z_t^1 is provided, as well as an i.i.d. sequence of copies of K. At a branching event τ, we sample the current value k of the random number K and k i.i.d. copies of Z_t are sampled and the process re-started from τ as $Z_{t+\tau}$, for each particle in the union of the particles alive and the ones newly added.

The particle that was killed goes to $\mathfrak{o}$. We notice that N_t does not take into account killed particles since $\mathfrak{o} \notin D$. By construction, no branching events may occur simultaneously. The construction is thus inductive over the branching events and $N_t^{tot} < \infty$, almost surely. The process $(\zeta_t)_{t\geq 0}$ is Markov with state space $M_0(\tilde{D})$. It shall be adapted to a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where the filtration satisfies the usual conditions. We note that the law of (ζ_t) is a probability measure on the Skorokhod space $\mathbf{D}([0,\infty), M_F(\tilde{D}))$.

For the purpose of this paper, it is sufficient to have $K \geq 1$ deterministic, but in this section we let K random with $K \geq 1$ and $\bar{K} > 1$. In agreement to (2.12) we are in the supercritical case.

4.1. **Alternative construction.** Finally, we point out to an alternative approach to the construction of (ζ_t). Starting with the semigroup S_t^D of the diffusion killed at the boundary, we construct the corresponding $\tilde{S}_t^D$, a semigroup on $M_0(\tilde{D})$ governing the motion, without branching, of finite configurations of independent particles driven by S_t^D; for a finite collection of initial points, we run as many independent processes and simply follow the sum of the delta functions of their position. This semigroup is trivial in that it presents no branching and would not be useful in itself. Following [31], Section 12.5, a perturbation of $\tilde{S}_t^D$, denoted $\tilde{S}_t$, is introduced by $\Gamma(\mu, d\mu')$, a bounded kernel on $M_0(\tilde{D})$ as the solution of the integral equation

$$\mathcal{S}_t G(\mu) = \mathcal{S}_t^D G(\mu) + \int_0^t \mathcal{S}_{t-s}^D [\int_{M_0(\tilde{D})} \mathcal{S}_s G(\mu')\Gamma(\mu, d\mu')]\, ds\,. \tag{4.3}$$

In our case Γ should incorporate the branching trigger, namely the first boundary hit of the finite configuration, as well as the jump in measure when K particles are created. Let $F_D(s,\mu)$ be the distribution function of the minimum of the $N = ||\mu||$ hitting times $\tau_{x_i}^D$, provided $\mu = \sum_{i=1}^N \delta_{x_j} \equiv (x_1, \dots, x_N)$. Let

$$\Gamma(\mu, d\mu') = \mathbf{1}_{\partial D^N}(\mu) P\Big(\mu + \zeta \in d\mu'\Big) \quad \text{i.e. the distribution of } \mu + \zeta\,,$$

where $\mu \in D^N$ if at least one of its components is at the boundary (or equivalently, at $\mathfrak{o}$, the cemetery state) and ζ is an element $\zeta = (Z, Z, \dots, Z) \in D^K$ with $Z \sim \gamma$. Both the length K and the distribution of Z are random and independent. In the finite configuration formalism, addition of delta functions corresponds to concatenation

$$(x_1, \dots, x_N) + (z_1, z_2, \dots, z_k) = (x_1, \dots, x_N, z_1, \dots, z_k) \qquad \text{(modulo a permutation)}\,.$$

It is important to mention that S_t^D here is the non-defective semigroup of the absorbed motion, whereas we used the same notation for the semigroup of the killed process. This matters because at branching, particles at the boundary are not removed but will remain at $\mathfrak{o}$.

Relation (4.3) is then a reformulation of (4.18) at a more general level. For the purpose this paper, which is solving a PDE, our particle system approach is preferable, because the perturbation of the semigroup requires to adapt the soft killing in [31] with a bounded potential to the hard killing at the boundary followed by a re-proof of existence ($\mathcal{S}_t$ is only defined as a solution to an integral equation). Additionally, in this paper it is important to show the solution is smooth, and that is not practical at this generality level.

4.2. **Exponential bounds.** The next proposition shows that no explosion can occur and gives a sharp exponential upper bound of $E[N_t]$. Define the stopping times $T_m \in [0, +\infty]$ as the first time $N_t \geq m$, $m \in \mathbb{Z}_+$. The process $t \to N_t$ is rcll (cadlag) and piecewise constant, so T_m is a stopping time and T_m is nondecreasing in m. The stopping time $T_\infty = \lim_{m\to\infty} T_m$ is the *time of explosion* which is the life time of the process.

First, we prove the exact upper bound for the expected number of particles in the supercritical case, when $p_0 = 0$, i.e. $K \geq 1$ with probability one.

Proposition 3. *If $K \geq 1$, then the number of particles N_t of the process (Z_t) starting with a finite number of particles has finite expectation for any $t > 0$, assuming that $\bar{K}$ is finite.*

More precisely, there exists $C(\gamma, \bar{K}) > 0$ and $\alpha^* \geq 0$ depending only on γ and $\bar{K}$ as well, both independent of t and x, such that

$$\sup_{x \in D} E_x[N(t)] \leq C(\gamma, \bar{K}) e^{\alpha^* t} . \tag{4.4}$$

More precisely, α^* is the solution to (2.12) and depends on γ, $\bar{K}$ via the distribution of the first hitting time τ^D.

Remark. This is a simplified version of the result from [18]. Here the branching is always supercritical and extinction is not possible.

Proof. Given that $p_0 = 0$, then either $\bar{K} = E[K] > 1$ if *at least one of $p_k > 0$, $k \geq 2$*, or $K \equiv 1$ with probability one. Since in the latter case there is nothing to prove, we proceed, without loss of generality, assuming $\bar{K} > 1$. Recall T_m is the first time N_t exceeds m particles. Let N_t^x denote the number of particles at time $t \geq 0$ of the process starting with exactly one particle at $x \in D$ and $E_x[\cdot]$ the expectation with respect to this initial state. Let τ^D be the first boundary hit. Since $p_0 = 0$, the process has a non-decreasing number of particles, we have the time shift inequality holding for all ω in the sample space

$$\tau^D + T_{m-1} \circ \theta_{\tau^D}(\omega) \geq T_m(\omega) , \quad T_m(\omega) \geq T_{m-1}(\omega) , \quad m \geq 2 . \tag{4.5}$$

We remark that this is the only reason we needed a coupling in Step 1. Moreover, if $m \geq 2$ then $T_m \geq \tau^D$ implying that $\{\tau^D > t'\} = \{\tau^D > t\}$ when $t' = t \wedge T_m$.

With $t' = t \wedge T_m$, we have $t' \leq t \wedge [(T_{m-1} \circ \theta_{\tau^D}) + \tau^D]$. And on the event $\{\tau^D < t\}$,

$$t' \leq [(t - \tau^D) \wedge (T_{m-1} \circ \theta_{\tau^D})] + \tau^D . \tag{4.6}$$

We now write

$$E_x[N_{t'}] = E[N_{t'}^x] = E[\mathbf{1}_{\{t' < \tau^D\}} + \mathbf{1}_{\{t' \geq \tau^D\}} \sum_{j=1}^{K} N_{t' - \tau^D}^{Z_{\tau^D}^j}] \tag{4.7}$$

$$\leq E[\mathbf{1}_{\{t < \tau^D\}} + \mathbf{1}_{\{t \geq \tau^D\}} \sum_{j=1}^{K} N_{(t-\tau^D) \wedge (T_{m-1} \circ \theta_{\tau^D})}^{Z_{\tau^D}^j}] \tag{4.8}$$

$$= P_x(t < \tau^D) + \bar{K} \int_0^t E_\gamma[N_{(t-s) \wedge T_{m-1}}] f_D(s, x) \, ds \tag{4.9}$$

where $E_\gamma[\cdot] = \int_D E_{x'}[\cdot] \gamma(dx')$ and $f_D(t, x)$ is the density function of τ^D when the particle starts at $x \in D$. On line (4.8) we used (4.6) and on line (4.9) we used the strong Markov property and the independence of K from the past of the process.

Let $a_m(t) = E_\gamma[N_{t \wedge T_m}]$, $m \geq 0$, $t \geq 0$. By integrating $E_x[N_{t'}]$ over $\gamma(dx)$ we have

$$a_m(t) \leq \int_D P_x(t < \tau^D) \, \gamma(dx) + \bar{K} \int_0^t a_{m-1}(t-s) f_D(s, \gamma) \, ds \tag{4.10}$$

where $f_D(t,\gamma) = \int_D f_D(t,x')\gamma(dx')$. The unknown expected values satisfy the bounds $0 \le a_m(t) \le m$.

Applying Gronwall's lemma we obtain an exponential bound for $a_m(t)$ by taking advantage of $a_{m-1}(t) \le a_m(t)$. We can be more precise. The Laplace transform $\hat{g}(\alpha) = \int_0^\infty e^{-\alpha s} g(s)ds$, for an integrable function $g : [0,\infty) \to \mathbb{R}$, applied to the inequality, shows that

$$\hat{a}_m(\alpha) \le \frac{1}{\alpha}(1 - \hat{f}_D(\alpha,\gamma)) + \bar{K}\hat{a}_{m-1}(\alpha)\hat{f}_D(\alpha,\gamma) \tag{4.11}$$

A simple estimate is to bound further line (4.9) using $T_{m-1} \le T_m$ and the monotonicity of N_t to obtain $\hat{a}_{m-1}(\alpha) \le \hat{a}_m(\alpha)$.

For any $\alpha > \alpha^*$ we have

$$\hat{a}_m(\alpha) \le \frac{1}{\alpha}\Big(\frac{1 - \hat{f}_D(\alpha,\gamma)}{1 - \bar{K}\hat{f}_D(\alpha,\gamma)}\Big) \tag{4.12}$$

We note that $1 - \bar{K}\hat{f}_D(\alpha,\gamma)$ has a simple zero at α^*. Because $\alpha^* \in \mathbb{R}$, to be a multiple zero would imply that $\hat{f}'_D(\alpha^*,\gamma) = 0$. But that means $E_\gamma[\tau^D e^{-\alpha^*\tau^D}] = 0$, leading to $\tau^D = 0$ γ - almost surely, a contradiction with $supp(\gamma) \subset D$.

Notice that $t \to a_m(t)$ is non-decreasing and non-negative. The Wiener - Ikehara Tauberian theorem (cf. [24] Theorem 4.2 p. 124) proves that there exists a constant $C_1 > 0$ (obtained from the pole at α^*), dependent of γ but independent of m and t such that $\lim_{t\to\infty} e^{-\alpha^* t} a_m(t,\gamma) \le C_1$, which implies that for a constant C depending only on γ

$$a_m(t,\gamma) \le Ce^{\alpha^* t}\,. \tag{4.13}$$

Letting $m \to \infty$ and the monotonicity of $\hat{a}_m$ in m, we obtain the same inequality uniformly in m. Plugging into (4.9) we obtain

$$E_x[N_{t\wedge T_m}] \le P_x(t < \tau^D) + C\bar{K}\int_0^t e^{\alpha^*(t-s)} f_D(s,x)\,ds$$

Note that T_m goes to infinity as $m \to \infty$, so monotone convergence shows that the bound is uniform in m, which proves $E_x[N_t] < \infty$ and a fortiori the claim that N_t is finite. To estimate its growth rate we factor out e^t and then we bound the integral all by itself by letting $t \to \infty$, which will give the Laplace transform of f_D at α^*. Then

$$E_x[N_t] \le P_x(t < \tau^D) + C\bar{K}e^{\alpha^* t}\hat{f}_D(\alpha^*,x) \le 1 + C\bar{K}e^{\alpha^* t}\,. \tag{4.14}$$

This proves the claim that, uniformly in $x \in D$, there exists a constant depending only on γ and $\tilde{K}$, inequality (4.4) is true. □

4.3. **The semigroup.** In order to describe the evolution of the branching process (ζ_t) we only need to define a semigroup on $M_0(D)$. As we shall see in a moment with the introduction of (4.17), it is still important to work with general test functions on $M_F(D)$.

Define the continuous, bounded functions $F \in C_b(M_F(D))$ of the form

$$\mu \in M_F(D) \to F(\mu) = \varphi(\langle \mu, \phi_1 \rangle, \ldots, \langle \mu, \phi_l \rangle), \quad l \in \mathbb{N} \tag{4.15}$$

where $(\phi_i)_{1 \leq i \leq l}$ are test functions in $\mathcal{T}$ and $\varphi \in C_b(\mathbb{R}^d)$. A class with smooth components including $C_c^\infty(D)$ of such test functions is sufficient to determine the law of the process (4.1) as the solution to the martingale problem, see [10] p. 176. Due to Proposition 3 we can extend φ to polynomial growth functions; in fact, to functions with some positive exponential moment.

For a test function F and $\mu = \sum_{i=1}^N \delta_{x_i} \in M_0(D)$, N positive integer (with the convention that we represented the space of finite configurations as sums of delta functions) we define the semigroup on $M_0(D)$

$$\begin{aligned} \mathcal{S}_t(F)(\mu) = E_\mu[F(\zeta_t)] = E[F(\zeta_t^\mu)] &= \sum_{i=1}^N E[F(\zeta_t^{x_i})] = \sum_{i=1}^N E_{x_i}[F(\zeta_t)] \\ &= \langle E_\cdot[F(\zeta_t)], \mu \rangle = \int_D E_x[F(\zeta_t)] \mu(dx). \end{aligned} \tag{4.16}$$

The relation is a consequence of the construction of the process. Particles independent at time $s \geq 0$ remain independent forever. The only dependence is through the ancestry tree. Particles distributed deterministically at time $t = 0$ are independent.

We shall be only interested in the functionals $\mu \to F(\mu) = \langle \mu, \phi \rangle$, for some test function ϕ, in other words a *linear functional*, when $\varphi(u) = u$ and $l = 1$. In that sense we refer to the restriction of the semigroup as *the marginal transition semigroup*, formally defined in (4.17), as already mentioned in Subsection 1.4.

Now consider a mapping defined for $\phi \in C_b(D)$,

$$t \to S_t \phi(x) := E_x[\, \langle \phi, \zeta_t \rangle \,] \, (= \mathcal{S}_t F(\delta_x)). \tag{4.17}$$

Remark. The marginal semigroup will be denoted by simple upper case S_t, as opposed to $\mathcal{S}_t$ for the semigroup of the process seen on $M_0(D)$.

In the following $R_\alpha^D(\alpha) = \int_0^\infty e^{-\alpha t} S_t^D dt$ is the resolvent of the semigroup $(S_t^D)_{t\geq}$ defined in (2.4) and the same notation, without superscript D, designates the resolvent of $(S_t)_{t \geq 0}$.

Proposition 4. *The mapping (4.17) defines a continuous (but not necessarily strongly continuous) semigroup on $C(\bar{D})$ satisfying for $\phi \in C(\bar{D})$*

$$S_t \phi(x) = S_t^D \phi(x) + \bar{K} \int_0^t \int_D S_{t-s} \phi(x') \gamma(dx') dF_D(s, x), \tag{4.18}$$

with resolvent

$$R_\alpha\phi(x) = R_\alpha^D\phi(x) + \frac{\bar{K}\hat{f}_D(\alpha,x)}{1-\bar{K}\hat{f}_D(\alpha,\gamma)}\gamma R_\alpha^D\phi\,, \quad \Re(\alpha) > \alpha^*\,, \tag{4.19}$$

where α^* is defined in (2.12).

Remark.

Step 1 - Relations (4.18) and (4.19). The derivation of (4.18) follows the same reasoning as (4.7)-(4.9):

$$\begin{aligned} E_x[\phi(\zeta_t)] &= E_x[\phi(\zeta_t)\mathbf{1}_{\tau^D>t}] + E_x[\phi(\zeta_t)\mathbf{1}_{\tau^D\le t}] = S_t^D\phi(x) + E_x[E_x[\phi(\zeta_t)\mathbf{1}_{\tau^D\le t}\,|\,\mathcal{F}_{\tau^D}]] \\ &= S_t^D\phi(x) + E_x[\sum_{j=1}^{K} E_{Z^j_{\tau^D}}[\phi(\zeta_{t-\tau^D})\mathbf{1}_{T-\tau^D\ge 0}]] \\ &= S_t^D\phi(x) + \bar{K}\int_0^t\int_D S_{t-s}\phi(x')\gamma(dx')dF_D(s,x), \end{aligned}$$

where we use that the processes in the summation are iid and the number K of offsprings is independent of $\mathcal{F}_{\tau^D}$.

Since ϕ is bounded, $|S_t\phi(t,x)| \le ||\phi|| E_x[N_t]$, which is $O(e^{\alpha^* t})$ cf. (4.4). Since all functions present in (4.18) are exponentially bounded, we integrate against $e^{-\alpha t}$ to obtain the Laplace transform (in time) for $\alpha > \alpha^*$, i.e. the resolvent. The integral term in (4.18) is a convolution in the time variable, and due to (2.5), we proved that the relation between resolvents of the semigroups is

$$R_\alpha\phi(x) = R_\alpha^D\phi(x) + \bar{K}(\gamma R_\alpha\phi)\hat{f}_D(\alpha,x)\,. \tag{4.20}$$

To obtain (4.19) we integrate to the left hand side (i.e. in the variable x) the measure γ on both sides, then

$$\gamma R_\alpha\phi = \gamma R_\alpha^D\phi + \bar{K}(\gamma R_\alpha\phi)\hat{f}_D(\alpha,\gamma)\,, \tag{4.21}$$

recalling the notation (2.10). Solving

$$\gamma R_\alpha\phi = \frac{\gamma R_\alpha^D\phi}{1-\bar{K}\hat{f}_D(\alpha,\gamma)}\,, \tag{4.22}$$

and plugging back in (4.20) we establish (4.19).

The relation is valid for $\alpha > \alpha^*$. Note that

$$\hat{f}_D(\alpha,x) = 1 - \alpha R_\alpha^D\mathbf{1}(x)\,, \tag{4.23}$$

where $\mathbf{1}(x)$ is the constant function equal to one. In addition, $\{\alpha\in\mathbb{C}|\Re(\alpha) > \tilde{\alpha}\} \subseteq Res(L)$ and all functions in formula (4.19) are meromorphic on $Res(L)$. Since $\tilde{\alpha} < \alpha^*$, this shows that the denominator in formula (4.19) has a pole at α^*.

Step 2 - Semigroup property. When G is linear, i.e. $G(\mu) = \langle \phi, \mu \rangle$, $\phi \in C_b(\bar{D})$. We need to ensure that $x \to \mathcal{S}_t(\delta_x)$ is continuous for all $t > 0$. This is a consequence of (4.18). The first term is continuous by hypothesis, whereas the second is continuous by dominated convergence. Notice, as a technical point, that the Markov property was applied in the derivation of (4.18) in Part 1 to the particle process Z_t^1. At this point we can apply the Markov property to (ζ_t) to have

$$
\begin{aligned}
S_{t+s}\phi(x) &= E_x[\langle \phi, \zeta_{t+s} \rangle] = E_x[E_x[\langle \phi, \zeta_{t+s} \rangle \,|\, \zeta_s]] = E_x[\langle E_{\zeta_s}[\langle \phi, \zeta_t \rangle]] \\
&= E_x[\langle E_\cdot[\langle \phi, \zeta_t \rangle], \zeta_s \rangle] = E_x[\langle S_t\phi(\cdot), \zeta_s \rangle] \\
&= E_x[S_s(S_t\phi(\cdot))] = S_s S_t \phi(x)\,.
\end{aligned}
\tag{4.24}
$$

Step 3 - Continuity of the semigroup (but not strong continuity). We write $|S_t\phi(x) - \phi(x)| \le (I) + (II)$ where $(I) = |S_t^D\phi(x) - \phi(x)|$ and the integral term (II) has a bound based on (4.4) in Proposition 3

$$(II) \le ||\phi|| \int_0^t C(\gamma, \bar{K}) e^{\alpha^*(t-s)} dF_D(s,x) \le ||\phi|| C(\gamma, \bar{K}) e^{\alpha^* t} P_x(\tau^D \le t)\,.$$

The first term in (4.18) is strongly continuous and so (I) vanishes as $t \to 0$. The second term vanishes as well, only for each fixed $x \in D$. The convergence is not uniform, in general.
□

Based on the explicit formulas from Proposition 4, we can show the next regularity result.

Theorem 4. *Under the same conditions as in Theorem 2, the function $w(t,x) = S_t\phi(x)$ belongs to $C([0,\infty) \times \bar{D}) \cap C^{1,2}((0,\infty) \times D)$, is bounded for $t \in [0,T]$, $T > 0$ and satisfies the heat equation $\partial_t w = Lw$ with boundary condition (1.2) and initial value $w(0,x) = \phi(x)$.*

Proof. According to Proposition 4, for a fixed $\phi \in \mathcal{D}$, $t \to S_t\phi(x)$ is continuous. We re-apply (4.18) we obtain the function is of class C^1 for $t > 0$. To see that, the first term satisfies the condition by hypothesis, and the second term is a time integral of a continuous function. The derivatives in x appear only from $S_t^D f(x)$ or $f(t,x)$, so the regularity in x inherited from the heat kernel $p^D(t,x,y)$ assumed in (2.3) is preserved.

The heat equation is linear; since the first term of (4.18) satisfies it, we have to prove that

$$U(t,x) = \int_0^t a(t-s) f_D(s,x) ds\,, \qquad a(t) = \int_D S_t\phi(x')\gamma(dx')$$

satisfies the heat equation as well. All functions involved are C^1 in t and C^2 in x. First we write $U(t,x) = \int_0^t a(s) f_D(t-s,x) ds$ and differentiate

$$\partial_t U(t,x) = a(t) f_D(0+,x) + \int_0^t a(s) \partial_t f_D(t-s,x) ds = \int_0^t a(s) L f_D(t-s,x) ds = LU(t,x)$$

due to Condition 3. Using the formula for the semigroup (4.18) we can check that $(t,x) \to S_t\phi(t,x) \in \mathcal{D}$ and satisfies the boundary condition (1.2) for all $t > 0$. Since the semigroup is continuous cf. Proposition 4, w satisfies the initial value by letting $t \to 0$. □

Corollary 2. *Under the same conditions as Theorem 1, for any $t_0 > 0$, the function $w^{t_0}(t,x) = w(t_0 + t, x)$ belongs to $\mathcal{D}$ and satisfies the boundary conditions (1.2).*

Proof. The $C^{1,1}$ regularity away from $t = 0$ was proved in Theorem 4. On $[t_0, T] \times D$, the function $w(t,x)$ is the sum of $S_t^D \phi(x)$ and the integral term in (4.18). By assumption, the density $f_D(t,x) \in C_b^{1,2}([t_0,\infty) \times D$. This concludes the proof. □

5. Proofs of the main theorems

The proof is organized in two steps. The first step is to show that the (possibly) particular solution $(\nu_t)_{t\geq 0}$, obtained as expected value of the measure-valued process $(\zeta_t)_{t\geq 0}$ by the representation (3.2), has all the properties of the solution in Theorems 1 and 2, except uniqueness, which will be proven at the end, in step two.

Without loss of generality we can assume that $\nu_0 \in M_1(D)$ (is a probability distribution). For a test function ϕ, we put

$$\langle \phi(t,\cdot), \nu_t \rangle := E_{\nu_0}[\langle \zeta_t, \phi(t,\cdot)\rangle] = E_{\nu_0}[\sum_{j=1}^{N_t} \phi(t, Z_t^j)] \,. \tag{5.1}$$

Based on the estimate on N_t from Proposition 3, ν_t has finite total mass. It belongs to $M_F(D)$ simply because it is the expected value of a finite random measure.

5.1. **Proof of Theorem 1: Existence.** *Part 1 - existence of a finite measure-valued solution.*

Let $\phi \in \mathcal{D}$ such that $\phi(t,\cdot)$ satisfies (1.2) for $t \geq 0$. We need to check that $(\langle \zeta_t, \phi(t,\cdot)\rangle)_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Calculating the jump at a boundary hit τ by particle i, we obtain

$$\langle \zeta_\tau, \phi(\tau,\cdot)\rangle - \langle \zeta_{\tau-}, \phi(\tau-,\cdot)\rangle = \phi(\tau, Z') + \phi(\tau, Z'') - \phi(\tau, Z_i(\tau-)) \,,$$

where Z', Z'' are independent random variables with respect to $\mathcal{F}_{\tau-}$ with distribution γ. Note that $\phi(t-,y) = \phi(t,y)$ so the left-hand side limit in time was dropped. The conditional expectation of this jump, given $\mathcal{F}_{\tau-}$, vanishes, i.e.

$$\overline{K} \int_D \phi(\tau, y)\gamma(dy) - \phi(\tau, Z_i(\tau-)) = 0 \,, \qquad Z_i(\tau-) \in \partial D$$

as long as ϕ satisfies the boundary condition (1.2). The test function is bounded with smooth bounded derivatives, implying by a standard application of the optional sampling theorem that the functional is a continuous martingale between boundary hits. Taking the expected value obtains (1.3), since ν_t is defined as the expected value of ζ_t by (5.1).

Part 2: Time continuity. Using (4.17) we see that, as a function of time, the deterministic process ν_t, $t \geq 0$, is in the Skorokhod space of right continuous with left limit paths. From (4.18), applied to $\phi(x) \to \phi(t,x)$, we have that $\langle \phi(t,\cdot), \nu_t \rangle = S_t\phi(t,x)$ is the sum of a known continuous part given by S_t^D and a time integral of a bounded function. This shows that $t \to \langle \phi(t,\cdot), \nu_t \rangle$ is continuous. We note that this was shown for a fixed ϕ. If the mapping $t \to \langle \nu_t, \phi \rangle$ is continuous, then $t \to \nu_t$ is continuous in the topology of finite measures.

5.2. **Proof of Theorem 2.** *Claim (i).* Since $D' \subset\subset D(\gamma)$ we see that $dist(supp(\gamma), F) = \beta > 0$. Consider ϕ non-negative in formula (4.18) with $supp(\phi) \subseteq F$. Integrate over γ in the variable x to obtain

$$u(t,\phi) = u^D(t,\phi) + \bar{K} \int_0^t u(t-s,\phi) f_D(s,\gamma)\, ds \tag{5.2}$$

with $u(t,\phi)$ and $u^D(t,\phi)$ denoting $\int_D S_t\phi(x)\gamma(dx)$, respectively the same integral for the semigroup S_t^D; at the same time, we simplified notation for $f(t,\gamma) = \int_D f_D(t,x)\gamma(dx)$. Notice that $\sup_{s\in[0,\infty]} f(s,\gamma) = C(f_D,\gamma) < \infty$. Gronwall's inequality proves that

$$u(t,\phi) \leq u^D(t,\phi) e^{\bar{K}C(f_D,\gamma)t} = \ .$$

The inequality is true for any ϕ. The right-hand side in the equation above integrates an absolutely continuous function $y \to p^D(t,x,y)$ against ϕ. By re-inserting into (5.2) and then removing ϕ, we obtain that S_t is absolutely continuous in y with density satisfying the bound

$$0 \leq \int_D p(t,x,y)\gamma(dx) \leq e^{\bar{K}C(f,\gamma)t} \int_D p^D(t,x,y)\gamma(dx)\,. \tag{5.3}$$

This proves that the original equation (4.18), the semigroup kernel has a density in the variable y

$$\begin{aligned} p(t,x,y) &= p^D(t,x,y) + \bar{K} \int_0^t \int_D p(t-s,x',y)\gamma(dx') f_D(s,x)\, ds \\ &\leq p^D(t,x,y) + \bar{K} \int_0^t e^{\bar{K}C(f_D,\gamma)(t-s)} \int_D p^D(t-s,x',y)\gamma(dx') f_D(s,x)\, ds \\ &\leq p^D(t,x,y) + \bar{K} e^{\bar{K}C(f_D,\gamma)t} \int_0^t \int_D p^D(t-s,x',y)\gamma(dx') f_D(s,x)\, ds \\ &\leq p^D(t,x,y) + \bar{K} e^{\bar{K}C(f_D,\gamma)t} \int_0^t C_D(t-s,\beta) f_D(s,x) ds \\ &\leq p^D(t,x,y) + \bar{K} e^{\bar{K}C(f_D,\gamma)t} \Big(\sup_{t\in[0,T]} C_D(t,\beta) \Big) P_x(\tau_D < t)\,. \end{aligned} \tag{5.4}$$

For the last two lines we recall that $\beta = dist(supp(\gamma), \partial F)$ and used the off-diagonal bound $p^D(t-s,x,y) \leq C_D(t-s,\beta)$ postulated in Condition 2, eq. (2.9). Set

$$C(t_0,T,\beta) := \sup_{t\in[t_0,T],x,y\in\bar{D}} p^D(t,x,y) + \bar{K}e^{\bar{K}C(f_D,\gamma)t}\Big(\sup_{t\in[0,T]} C_D(t,\beta)\Big)\,. \tag{5.5}$$

concluding the first part of claim (i), on the density bound.

To prove the claim about the density vanishing at the boundary ∂D, we first note that (5.3) is valid for $y \in F$. Dominated convergence in the time integral from the third line of (5.4) implies the claim. Dominated convergence is guaranteed by the off-diagonal bound (2.9), i.e. $p^D(t-s,x,y) \leq C_D(t-s,\beta)$, independent of y.

Claim (ii). Integrating (5.4) against the initial distribution $\nu_0(x)$ we obtain $v(t,y)$. The bound is not larger than $||\nu_0||C(t_0,T,\beta)$, where $||\nu_0||$ is the total variation. Continuity for $p^D(t,x,y)$, the first term in (5.4), is given by assumption (2.3) for $t>0$. Integrating against the continuous initial density v_0 makes this term continuous at $t=0$. Another factor that becomes a continuous is $f_D(s,v_0) = \int_D f_D(s,x)v_0(x)dx$. Time continuity of the second term, a time integral, is guaranteed because the integrand is bounded (again we use the off-diagonal bound). To prove continuity in y we have to re-do the argument based on Gronwall's lemma from the proof of claim (i). Fix $y_0 \in F$. Integrate (5.4) against γ in the variable x and subtract the values at y and y_0. With the absolute value we obtain the inequality (5.2) where $u(t,\phi)$ is replaced by $u(t,y,y_0) = |\int_D |p(t,x,y) - p(t,x,y_0)|\,\gamma(dx)$ and $u^D(t,y,y_0)$ the analogue quantity for the kernel $p^D(t,x,y)$. Following the steps from the proof of claim (i) we obtain an inequality corresponding to line three of (5.4); after integrating against the initial density v_0 this reads

$$\begin{aligned}0 \leq |v(t,y)-v(t,y_0)| = |p(t,v_0,y)-p(t,v_0,y_0)| &\leq |p^D(t,v_0,y)-p^D(t,v_0,y_0)| \\ &+\bar{K}e^{\bar{K}C(f_D,\gamma)t}\int_0^t |p^D(t-s,\gamma,y)-p^D(t-s,\gamma,y_0)|f_D(s,\gamma)\,ds\,.\end{aligned}$$

Once again, dominated convergence and the off-diagonal bound (2.9) from Condition 2 show that $y \to y_0$ implies the claim (ii).

5.3. **Proof of Theorem 1: Regularity.** The proof of Theorem 2 only requires Conditions **??** and 2. Under these conditions we prove the solution is regular, as stated in Theorem 1.

Recall the constant obtained in (5.5). The bounds involved are obtained for $D(\gamma) = D \setminus supp(\gamma)$. By setting $C(t_0,T) = C(t_0,T,\beta)$ we see that ν_t^x is a regular solution in the sense of (3.1).

5.4. **Proof of Theorem 1: Uniqueness.** Recall that when $\nu_0 = \delta_x$, the solution is denoted $v^x(t,y)$ for $t>0$. We shall use the time reversal in the semigroup, or equivalently the backward equation.

Fix $T>0$ and $g\in C_b(\bar D)$. Using $\nu_t^x(dy)$ from (5.1) we set

$$w(t,x)=\langle \nu_t^x, g\rangle=\int_D g(y)\nu_t^x(dy)=S_t g(x)\,.$$

We point out that this is the point where Condition 3 is used. According to Theorem 4 this function satisfies $\partial_t w=Lw$. The function w also verifies the boundary conditions (1.2). It is also continuous as $t\to 0$ with value $\nu_0(dy)=g(y)dy$. Define $\phi(t,x):=w(T-t,x)$, $t\in[0,T]$. For any sufficiently small $\epsilon>0$, ϕ is a test function in $\mathcal{D}$ on the time interval $[0,T-\epsilon]$ and satisfies the boundary conditions (1.2).

Let $m_t(dx)$ be a weak solution satisfying (1.3). Then

$$\langle \phi(t,\cdot),m_t\rangle=\langle\phi(0,\cdot),m_0\rangle\,,\qquad 0\le t\le T-\epsilon\,. \tag{5.6}$$

This implies

$$\begin{aligned}\langle\phi(T-\epsilon,\cdot),m_{T-\epsilon}\rangle &=\langle\phi(0,\cdot),m_0\rangle\\ &=\langle w(T,\cdot),m_0\rangle=\langle\langle g(\cdot),\nu_T\rangle,m_0\rangle=\int_D\int_D \nu_T^x(dy)g(y)m_0(dx)\\ &=\int_D g(y)\int_D \nu_T^x(dy)m_0(dx)=\int_D g(y)\nu_T^{m_0}(dy)=\langle g,\nu_T^{m_0}\rangle\,.\end{aligned}\tag{5.7}$$

The left hand side is

$$\begin{aligned}\langle\phi(T-\epsilon,\cdot),m_{T-\epsilon}\rangle&=\langle\int_D g(y)\nu_\epsilon^x(dy),m_{T-\epsilon}\rangle=\langle S_\epsilon g,m_{T-\epsilon}\rangle\\ &=\langle S_\epsilon g-g,m_{T-\epsilon}\rangle+\langle g,m_{T-\epsilon}\rangle\,.\end{aligned}$$

Assume we prove that the first term vanishes as $\epsilon\to 0$. Then, since the second term approaches $\langle g,m_T\rangle$, we proved that $m_T=\nu_T^{m_0}$. This will be true for arbitrary $T>0$, concluding the proof.

It remains to analyze the first term. The solution $m_\cdot$ is a regular solution in the sense of (3.1), its total variation $||m_t||$ is bounded uniformly for $t\in[0,T]$. Write $S_tg=S_t^Dg+(S_t-S_t^D)g$ in view of applying (4.18). Using the *strong continuity of the semigroup* S_t^D we obtain

$$|\langle S_\epsilon^D g-g,m_{T-\epsilon}\rangle|\le ||S_\epsilon^D g-g||\sup_{t\in[0,T]}||m_t||\quad\to\quad 0\,.$$

Again, from (4.18), the second part of $S_\epsilon g$ satisfies

$$|(S_t-S_t^D)g(x)|\le\bar K\int_0^t\int_D|S_{t-s}g(x')\gamma(dx')|dF_D(s,x)$$

$$\le\bar K||g||\int_0^t C(\gamma,\bar K)e^{\alpha^*(t-s)}dF_D(s,x)\le\bar K||g||C(\gamma,\bar K)e^{\alpha^* t}P_x(\tau^D\le t)$$

We only need to show that $\langle P_x(\tau^D\le\epsilon),m_{T-\epsilon}(dx)\rangle\to 0$ as $\epsilon\to 0$. Pick $F\subset D\setminus D(\gamma)$ $F=\{x'\in D\,|\,dist(x',\partial D)<\delta\}$, for arbitrary but sufficiently small $\delta>0$ independent of ϵ.

From the properties of the diffusion $(L, \mathcal{D}(L))$ we have that $x \to P_x(\tau^D \leq \epsilon)$ converges to zero as $\epsilon \to 0$, uniformly in $x \in D \setminus F$.

Fix $t_0 > 0$ such that for all ϵ, $T - \epsilon > t_0$. In integral notation

$$|\langle P_x(\tau^D \leq \epsilon), m_{T-\epsilon}(dx)\rangle| = |\int_D P_x(\tau^D \leq \epsilon)\, m_{T-\epsilon}(dx)| \leq (I) + (II)\,.$$

For the first term

$$(I) = |\int_{D\setminus F} P_x(\tau^D \leq \epsilon)\, m_{T-\epsilon}(dx)| \leq \sup_{x\in D\setminus F} P_x(\tau^D \leq \epsilon) \sup_{t\in[t_0,T]} ||m_t||$$

due to the remarks above. For the second term

$$(II) \leq |\int_F P_x(\tau^D \leq \epsilon)\, m_{T-\epsilon}(dx)| \leq Leb(F) C(t_0, T)\,,$$

because $m_{\cdot}$ satisfies (3.1). Recall that D is bounded and C^2, meaning that $Leb(F) = O(\delta)$ and consequently $\limsup_{\epsilon\to 0}[(I) + (II)] \sim O(\delta)$, δ independent of ϵ. Since δ can be arbitrarily small, we proved the claim.

5.5. **Proof of Theorem 3.** A function $\phi(t, \cdot)$ with compact support in $D \setminus \{c\} = D(\gamma)$ trivially satisfies the boundary conditions (1.2) if

$$\phi(t, c) = \bar{K}\phi(t, x') = 0\,, \qquad x' \in \partial D\,. \tag{5.8}$$

At this point we apply the parabolic Weyl lemma (for the heat equation) (cf. [30] and [22]) noticing that all $\phi(t, \cdot) \in C_c^\infty(D \setminus \{c\})$ are admissible as test functions. By uniqueness, the weak solution v is given by (5.4) according to Theorem 2 (i). In part (ii) of the same theorem, we proved continuity up to the boundary ∂D. Moreover, the last assertion in part (i) states that if $y \to p^D(t, x, y)$ vanishes on ∂D, so does $y \to p(t, x, y)$. Consequently, the same is true for the integral in the variable x, namely $y \to v(t, y)$. We have shown that the density $v(t, y)$ is smooth for $t > 0$ and $y \in D \setminus c$ and continuous for $t > 0$, $y \in \bar{D} \setminus \{c\}$ and satisfies the strong forward equation (note that $L = L^*$ here)

$$\partial_t v = \frac{1}{2}\Delta v \quad \text{on } D \setminus \{c\}\,, \qquad v(t, \cdot)|_{\partial D} = 0\,. \tag{5.9}$$

Due to $\partial D \in C^2$, regularity theorems (cf. [11], Ch. 7, Theorem 7 and [14] Ch. 6.3 Theorem 6.14) imply that $v \in C^2(D(\gamma) \cup \partial D)$. The regularity at the boundary, here C^2 (in general, it must be at least C^1) will be needed when we apply the classical Green's theorem below.

In the following, $D_\epsilon = \{x \in D | dist(x, \partial D) > \epsilon\}$. Due to $\partial D \in C^2$, for ϵ sufficiently small $B(c, 3\epsilon)$ and $D \setminus D_{3\epsilon}$ are away from each other.

For sufficiently small ϵ, there exists a smooth function $y \to h_\epsilon(y)$ on $\bar{D}$ having the properties:

(i) $h_\epsilon(y) = 1$ on $D \setminus D_\epsilon$ and $B(c, \epsilon)$

(ii) $supp\, h_\epsilon \subseteq D \setminus D_{3\epsilon} \cup B(c, 3\epsilon)$

(iii) $\nabla h_\epsilon(y) = 0$ on $\partial D_{3\epsilon}$, ∂D_ϵ, $\partial B(c,\epsilon)$ and $\partial B(c,3\epsilon)$.

Denote

$$D^1_\epsilon = D_\epsilon \setminus \overline{D_{3\epsilon}}\,, \quad D^2_\epsilon = B(c,3\epsilon) \setminus \overline{B(c,\epsilon)}\,.$$

When ϕ satisfies the boundary conditions (1.2), the functions $\phi_1 = h_\epsilon \phi$ and $\phi_2 = (1-h_\epsilon)\phi$ satisfy (1.2) and (5.8), respectively. Let $\phi(c) = B$. Then $\phi(S) = \bar{K}B$, S on the boundary. Together with the properties of h_ϵ, we have the estimates

$$supp\, \phi_1 \subseteq D^1_\epsilon \cup D^2_\epsilon \tag{5.10}$$

$$\phi_1 = 2B + o(\epsilon) \quad \text{on} \quad \overline{B(c,\epsilon)} \quad \phi_1 = o(\epsilon) \quad \text{on} \quad \partial B(c,3\epsilon) \tag{5.11}$$

$$\phi_1 = B + o(\epsilon) \quad \text{on} \quad D \setminus D_\epsilon \quad \phi_1 = o(\epsilon) \quad \text{on} \quad \partial D_{3\epsilon}\,. \tag{5.12}$$

$$\nabla \phi_1 = o(\epsilon) \quad \text{on} \quad \partial(D^1_\epsilon \cup D^2_\epsilon) \tag{5.13}$$

We know that $v(t,y)$ satisfies (1.3) in simplified form (no dependence on t in ϕ). This translates into

$$Eq(1) + Eq(2) = 0$$

where

$$Eq(i) = \langle v(t,\cdot), \phi_i\rangle - \langle v(0,\cdot), \phi_i\rangle - \int_0^t \langle v(s,\cdot), \frac{1}{2}\Delta\phi_i\rangle ds\,, \qquad i = 1,2$$

after splitting the equation into the ϕ_1 and the ϕ_2 parts. The ϕ_2 part satisfies (5.8) and is zero in a neighborhood of ∂D and c, implying that $Eq(2) = 0$. It follows that $Eq(1) = 0$. We now write the integrand (multiplied by two for simplicity)

$$\langle v(s,\cdot), \Delta\phi_1\rangle = \int_D v(s,y)\Delta\phi_1(y)dy = \int_{D^1_\epsilon \cup D^2_\epsilon} v(s,y)\Delta\phi_1(y)dy$$

where we see that $supp\,\Delta\phi_1 \subseteq D^1_\epsilon \cup D^2_\epsilon$ by eliminating the neighborhood of the boundary ∂D and of c where $h_\epsilon \equiv 1$ (so its derivatives are zero) as well as where $h_\epsilon \equiv 0$. Using the second Green formula, the integral on the right hand side splits into $I_1 + I_2 + I_3$ with

$$I_i = -\int_{\partial D^i_\epsilon} \phi_1(S)\nabla v(s,S)\cdot \mathbf{n}\, dS + \int_{\partial D^i_\epsilon} v(s,S)\nabla\phi_1(S)\cdot\mathbf{n}\, dS\,, \qquad i = 1,2$$

and

$$I_3 = -\int_{D^1_\epsilon \cup D^2_\epsilon} \Delta v(s,y)\phi_1(y)dy\,.$$

Due to the assumptions on h_ϵ

$$\begin{aligned} I_1 + I_2 &= -\phi_1(c)\Phi(v(s,\cdot),c) - \phi_1(S)\Phi(v(s,\cdot),\partial D) + o(\epsilon) = \\ &- B\Big[\Phi(v(s,\cdot),c) - \bar{K}\Phi(v(s,\cdot),\partial D)\Big] + o(\epsilon)\,, \qquad \bar{K} = 2\,. \end{aligned} \tag{5.14}$$

The parenthesis multiplied by B isolates the boundary conditions. Suppose we can show that all remaining terms in $Eq(1)$ are $o(\epsilon)$. Since B is arbitrary, letting $\epsilon \to 0$, we would obtain the time integral of the bracket is zero, that is

$$\int_0^t \Big[\Phi(v(s,\cdot),c) - 2\Phi(v(s,\cdot),\partial D)\Big]\,ds = 0\,.$$

Since the integrand is continuous in s, the derivative is zero for all $s > 0$, which proves the claim.

Now we have to prove that the other terms are $o(\epsilon)$. We can break down $Eq(1)$ in two terms. One, which includes I_3

$$\int_{D^1_\epsilon \cup D^2_\epsilon} \Big[v(t,y) - v(0,y) - \int_0^t \frac{1}{2}\Delta v(s,y)\Big]\,\phi_1(y)\,dy$$

and another one on the complement set

$$\int_{B(c,\epsilon)\cup D\setminus D_\epsilon} \Big[v(t,y) - v(0,y)\Big]\,\phi_1(y)\,dy = o(\epsilon)\,.$$

Note that the time integral part in the second term is zero because of $h_\epsilon = const$ on the set of integration and its derivatives vanish. In the first term, the parenthesis is identically equal to zero since the integration set is away from the boundary and from c and we have shown that v solves the heat equation in the interior of $D \setminus \{c\}$. The second term is vanishing as $\epsilon \to 0$ due to the fact that $p(t,x,y)$ has integral equal to $n^x_t < \infty$ and thus $v(t,\cdot)$ is always integrable. Due to the definition of h_ϵ, the argument is concluded by dominated convergence.

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