ERGODIC PROPERTIES OF MULTIDIMENSIONAL BROWNIAN MOTION WITH REBIRTH†

ILIE GRIGORESCU[1] AND MIN KANG[2]

Abstract. In a bounded open region of the d dimensional space we consider a Brownian motion which is reborn at a fixed interior point as soon as it reaches the boundary. The evolution is invariant with respect to a density equal, modulo a constant, to the Green function of the Dirichlet Laplacian centered at the point of return. We determine the resolvent in closed form and prove the exponential ergodicity by Laplace transform methods using the analytic semigroup properties of the Dirichlet Laplacian. In $d = 1$ we calculate the exact spectrum of the process and note that the principal eigenvalue is equal to the second eigenvalue of the Dirichlet Laplacian, correcting an error from a previous paper. The method can be generalized to other renewal type processes.

1. Introduction

This paper generalizes to higher dimensions the results of [7]. Let $\mathcal{R}$ be a bounded open region in $\mathbb{R}^d$ with a smooth boundary (to make things precise, of class C^2) such that the origin $O \in \mathcal{R}$. For $x \in \mathbb{R}^d$, let $W_x = (w_x(t,\omega), \{\mathcal{F}_t\}_{t\geq 0})$ be a Brownian motion on $\mathbb{R}^d$ such that $P(w_x(0,\omega) = x) = 1$. On the region $\mathcal{R}$, for any $x \in \mathcal{R}$, we define a process $\{z_x(t,\omega)\}_{t\geq 0}$ with values in $\mathcal{R}$ which is identical to a standard d dimensional Brownian motion until the almost surely finite time τ when it reaches the boundary, then instantaneously returns to the origin O at τ and repeats the same evolution indefinitely. This is the multidimensional version of the problem described in [7], which may be called *Brownian motion with rebirth*, since after emulating the Brownian motion with absorbing boundary conditions (in other

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[1] Department of Mathematics, University of Miami, 1365 Memorial Drive, Coral Gables, FL 33124-4250, email: igrigore@math.miami.edu.

[2] Department of Mathematics, North Carolina State University, Harrelson Hall, Box 8205, Raleigh, North Carolina 27695 email: kang@math.ncsu.edu.

words, *killed* at the boundary) it is reborn at the origin. The state space can be shown to be compact with the topology described in (2.14) creating a shunt at the return point. As a consequence, the dynamics has an invariant measure. We identify it as the Green function for the Laplacian with pole at $\xi = 0$, modulo a normalizing factor, and prove the exponential ergodicity in Theorem 1. The key part of the proof consists in establishing that the semigroup corresponding to the rebirth process is analytic, that is, proving estimate (2.23).

In section 7.4 of [10] the average time a Brownian motion starting at x spends in the set $B \subset \mathcal{R}$ before hitting the boundary is determined as $2\int_B G(x,y)dy$. Our particle will repeat the trip from the origin to the boundary indefinitely and will stabilize in time, by ergodicity, towards the measure which gives the mean value over all configurations. The proof is analytic and uses essentially the conditions needed to carry out the contour integration in the complex plane in order to calculate the inverse Laplace transform (based on Proposition 3), which are the same as the sufficient conditions for an analytic semigroup from Theorem 7.7 in [12], applied to the Dirichlet Laplacian.

There are two venues for applications of the rebirth process.

The first originates in a variant of the Fleming-Viot branching process introduced in [1] and studied further in [8]- [9]. Assume that the singular measure δ_O giving the distribution of the rebirth location of the Brownian particle is replaced by a time-dependent deterministic measure $\mu(t,dx)$. The tagged particle process from [9] is an example in the case when $\mu(t,dx)$ is the deterministic macroscopic limit of the empirical measures of a large system of Brownian particles with branching confined to the region $\mathcal{R}$. In particular, in equilibrium, the updating measure $\mu(t,dx)$ is constant in time, being equal to $\mu(dx) = \Phi_1(x)dx$, the probability measure with density equal to the first eigenfunction of the Dirichlet Laplacian (normalized). The closed formula (2.22) captures the renewal mechanism imbedded in the process. The estimates needed for the Laplace transform inversion formula are easier to obtain in an L^2 norm than in the uniform norm. A reference in that direction is again [12]. The proofs presented in this paper are easy to modify in order to include the L^2 case. More precisely, it should be pointed out that when the measure $\mu(dx)$ has a density in $L^2(\mathcal{R})$ part of the analysis carried out for establishing (3.4)-(3.6) is simplified due to the explicit form of the resolvent of the Dirichlet Laplacian in the square norm.

The degeneracy of the update distribution at the origin gives a local (pointwise) character which increases considerably the difficulty of the problem. In that sense, one needs the analytic semigroup results from Stewart [13] and [14].

The second application is coming from mathematical finance. If $\{S(t)\}_{t\geq 0}$ denotes the asset process in a model for the derivative markets, then $\log S(t)$ is typically assumed to follow the path of a geometric Brownian motion (see [3], also [6]). The *double knock-out barrier options* have payoff equal to $S(t)$ as long as it belongs to a region $\mathcal{R}$ with the prescription that it falls back to zero as soon as the barrier or boundary is reached and starts again.

In any dimension, Theorem 1 (i) describes the spectrum $0 = \lambda_0 > \lambda_1 \geq \lambda_2 \geq \ldots$ of the generator of the process as a subset of singularities appearing in the resolvent formula (2.22), and the existence of a spectral gap is established (ii). In dimension $d = 1$ we are able to calculate the spectrum of the generator of the process exactly, and show that the spectral gap is equal to the second eigenvalue of the Dirichlet Laplacian $\lambda_1 = \lambda_2^{abs}$. This corrects an incorrect statement from [7], which is only true in one direction, namely that the spectral gap is at least equal to λ_1^{abs}, or more precisely $\lambda_1 \leq \lambda_1^{abs}$ (in $d = 1$).

2. **Results**

We shall denote by $(\Omega, \mathcal{F}, P)$ a probability space supporting the law of the family of d-dimensional coupled Brownian motions indexed by their starting points $x \in \mathcal{R}$. Let $\mathcal{A}$ be an open region in $\mathbb{R}^d$ and $x \in \mathcal{A}$. In general we shall use the notation

$$T_x(\mathcal{A}) = \inf\{t > 0 \,:\, w_x(t,\omega) \notin \mathcal{A}\}\,, \tag{2.1}$$

the *exit time* from the region $\mathcal{A}$ for the Brownian motion starting at x. Occasionally we shall suppress either x or the set $\mathcal{A}$ if they are unambiguously defined in a particular context. We shall define inductively the increasing sequence of stopping times $\{\tau_n\}_{n\geq 0}$, together with a family of adapted nondecreasing point processes $\{N_x(t,\omega)\}_{t\geq 0}$ and the process $\{z_x(t,\omega)\}_{t\geq 0}$, starting at $x \in \mathcal{R}$. Let $T_x = \tau_0 = \inf\{t \,:\, w_x(t,\omega) \notin \mathcal{R}\}$, while for $t \leq \tau_0$ we set $N_x(t,\omega) = 1_{\{\partial\mathcal{R}\}}(w_x(t,\omega))$ and $z_x(t,\omega) = w_x(t,\omega) - \int_0^t w_x(s,\omega)dN_x(s,\omega)$. We notice that $z_x(\tau_0-,\omega) = w_x(\tau_0,\omega) \in \mathcal{R}$. By induction on $n \geq 0$,

$$\tau_{n+1} = \inf\{t > \tau_n \,:\, w_x(t,\omega) - \int_0^{\tau_n} z_x(s,\omega)dN_x(s,\omega) \notin \mathcal{R}\} \tag{2.2}$$

which enables us to define, for $\tau_n < t \leq \tau_{n+1}$,

$$N_x(t,\omega) = N_x(\tau_n,\omega) + 1_{\{\partial\mathcal{R}\}}(z_x(t-,\omega))\,, \tag{2.3}$$

as well as

$$z_x(t,\omega) = w_x(t,\omega) - \int_0^t z_x(s,\omega)dN_x(s,\omega)\,. \tag{2.4}$$

We notice that $z_x(t,\omega) = 0$ for all $t = \tau_n$. The construction and the summations present in (2.2) and (2.4) are finite due to the following result.

Proposition 1. *The sequence of stopping times $\tau_0 < \tau_1 < \ldots < \tau_n < \ldots$ are finite for all n and $\lim_{n\to\infty} \tau_n = \infty$, both almost surely. Also, the integer-valued processes $N_x(t,\omega)$ defined for $t \geq 0$ have the properties (i) they are nondecreasing, piecewise constant, progressively measurable and right-continuous, and (ii) for any $x \in \mathcal{R}$, $P(N_x(t,\omega) < \infty) = 1$.*

Proof. By monotonicity, since the expected value of the first exit time from a ball centered at the origin is finite in any dimension d (for example, in [15]), we deduce that $E[T] < \infty$. As a consequence, $T = T_x < \infty$ a.s.. The time intervals between τ_n and τ_{n+1} (we include $\tau_{-1} = 0$), for any $n \geq -1$ are either T_x for the first exit time or independently identically distributed as T_0 for all the rest. Since $P(T_x = 0) = 0$ for any $x \in \mathcal{R}$ the sequence is strictly increasing. Moreover, $E[\tau_n] < \infty$, which implies $P(\tau_n < \infty) = 1$. The differences $\tau_{n+1} - \tau_n$ are i.i.d. when $n \geq 0$ which shows that $\lim_{n\to\infty} \frac{\tau_n}{n} > 0$ implying that $\tau_n \to \infty$ a.s..

The processes $N_x(t,\omega) \geq 0$ are clearly nondecreasing, integer-valued and piecewise constant. They are right-continuous by construction (2.3) preserving the same value until the next boundary hit. Progressive measurability is a consequence of the fact that the first exit times $\{\tau_n\}$ are stopping times. □

Let $A \in \mathcal{B}(\mathcal{R})$ and $p_{abs}(t,x,y)$ denote the *absorbing Brownian kernel*

$$\int_A p_{abs}(t,x,y)dy = P\Big(w_x(t,\omega) \in A\,,\, t < T_x(\mathcal{R})\Big)\,. \tag{2.5}$$

The operator Δ with Dirichlet boundary conditions on $\partial\mathcal{R}$ has a countable spectrum $\{\lambda_i^{abs}\}_{i\geq 1}$

$$0 > \lambda_1^{abs} \geq \lambda_2^{abs} \geq \ldots \tag{2.6}$$

with corresponding eigenfunctions $\{\Phi_n(x)\}$ and

$$p_{abs}(t,x,y) = \sum_{n=1}^{\infty} \exp\left(\frac{\lambda_n^{abs} t}{2}\right) \Phi_n(x)\Phi_n(y)\,. \tag{2.7}$$

The functions $\{\Phi_n(x)\}$ are smooth and form an orthonormal basis of $L^2(\mathcal{R})$ (reference [10], or [4], (6.5)). The resolvent of the absorbing Brownian motion applied to $f \in C(\mathcal{R})$ will be denoted by

$$R_\alpha^{abs} f(x) = \int_0^\infty \int_{\mathcal{R}} e^{-\alpha t} p_{abs}(t,x,y) f(y)\, dy dt\,. \tag{2.8}$$

In the following, the Laplace transform of the first exit time $T_x(\mathcal{R})$ from the domain $\mathcal{R}$ of a Brownian motion starting at x will be denoted by

$$\widehat{h^x}(\alpha) = E_x[e^{-\alpha T_x(\mathcal{R})}] = \int_0^\infty e^{-\alpha t} h^x(t)\, dt\,. \tag{2.9}$$

where $h^x(t)$ is the density function of $T_x(\mathcal{R})$. The Laplace transform (2.9) exists on the complex plane for all α with $\Re(\alpha) > \lambda_1^{abs}$ (see, in that sense, the remark following Theorem 1) and can be extended (page 211, [16]) to the resolvent set.

The law of the process $\{z_x(t,\omega)\}_{t\geq 0}$, adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ will be denoted by Q_x and the family of processes $\{Q_x\}_{x\in\mathcal{R}}$ will be denoted simply by $\{Q\}$. The construction described by equations (2.2) through (2.4) can be made deterministically for any $x \in \mathcal{R}$ and each path $w_x(\cdot) \in C([0,\infty), \mathbb{R}^d)$ resulting in a mapping preserving the progressive measurability

$$\Phi(w_x(\cdot)) = w_x(\cdot) - \int_0^{\cdot} w_x(s,\omega)\, dN_x(s,\omega)\,. \tag{2.10}$$

With this notation $\Phi : C([0,\infty), \mathbb{R}^d) \to D([0,\infty), \mathcal{R})$ and $Q_x = W_x \circ \Phi^{-1}$ is the law of the process $\{z_x(t,\omega)\}_{t\geq 0}$ with values in the region $\mathcal{R}$, a measure on the Skorohod space $D([0,\infty), \mathcal{R})$.

Let $m \in \mathbb{Z}_+$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Z}_+^d$ be a d dimensional multi-index vector and we write $|\alpha| = \sum_{i=1}^d \alpha_i$. If $\mathcal{A} \subseteq \mathbb{R}^d$ and $f : \mathcal{A} \to R$, we use the standard notation

$$\partial^{(\alpha)} f(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}}(x)$$

if the derivative exists. Naturally $C^m(\mathcal{A})$ is the set of functions for which all derivatives with multi-indices α such that $|\alpha| \leq m$ exist and are continuous. We recall that the process $\{z_x(t,\omega)\}_{t\geq 0}$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ corresponding to the underlying standard d-dimensional Brownian motion.

Proposition 2. *If $f \in C^2(\mathcal{R}) \cap C(\overline{\mathcal{R}})$, then*

$$f(z_x(t,\omega)) - f(x) - \int_0^t \frac{1}{2}\Delta f(z_x(s,\omega))ds - \int_0^t (f(0) - f(z_x(s,\omega))dN_x(s,\omega) \tag{2.11}$$

is a $\mathcal{F}_t$ - martingale with respect to Q_x.

Proof. The proof is identical with the $d = 1$ case from [7]. $\square$

Let

$$\begin{aligned} \mathcal{D} &= \Big\{ f \in C^2(\mathcal{R}) \,:\, \forall\, |\alpha| \le 2\,,\, \exists \lim\nolimits_{x\to b} \partial^{(\alpha)} f(x) \in \mathbb{R}\,,\, b \in \partial\mathcal{R} \Big\} \\ \mathcal{D}_0 &= \Big\{ f \in \mathcal{D} \,:\, \forall\, b \in \partial\mathcal{R}\,,\, \lim\nolimits_{x\to b} f(x) = f(0) \Big\}\,. \end{aligned} \tag{2.12}$$

Corollary 1. *If $f \in \mathcal{D}_0$ then*

$$f(z_x(t,\omega)) - f(x) - \int_0^t \frac{1}{2}\Delta f(z_x(s,\omega))ds \tag{2.13}$$

is a $\mathcal{F}_t$ - martingale with respect to Q_x.

The next result allows us to regard $\{z_x(t,\omega)\}_{t\ge 0}$ as a process with continuous paths on the compact state space X obtained by identifying the boundary $\partial\mathcal{R}$ and the origin O.

Let $X = \mathcal{R}$ with the topology $\mathcal{T}$ generated by the neighborhood basis

$$\begin{aligned} V_{x,r} &= \Big\{ B(x,r) \,:\, \forall r > 0 \,\text{such that}\, B(x,r) \subset \mathcal{R} \setminus \{0\} \Big\} && \text{if } x \ne 0 \\ V_{0,r} &= \Big\{ B(0,r) \cup \Big(\cup_{b\in\partial\mathcal{R}} \left(B(b,r) \cap \mathcal{R}\right) \Big) \,:\, r \in (0, \tfrac{1}{2} d(x,\partial\mathcal{R})) \Big\} && \text{if } x = 0\,. \end{aligned} \tag{2.14}$$

Remark: The space $(X, \mathcal{T})$ is compact and homeomorphic to a sphere in $\mathbb{R}^{d+1}$ with the North and South poles identified. The boundary conditions from below introduce a shunt at the origin which is responsible for the intrinsic asymmetry of the evolution.

We define the class of functions of class C^2 up to the boundary $\{0\}$ of $X \setminus \{0\}$

$$\mathcal{D}(X) = \{ f \in C^2(X \setminus \{0\}) \,:\, \exists \lim_{x\to y} \partial^{(\alpha)} f(x) \in \mathbb{R}\,,\; 0 \le |\alpha| \le 2\,,\, y \in \{0\} \cup \partial\mathcal{R} \} \tag{2.15}$$

with the notational convention that the one-sided limit $\lim_{x\to y} g(x)$ is defined as $\lim_{x\to y} g(x)$ in the topology inherited from $\mathbb{R}^d$ by the set $B(0,r) \subseteq \mathcal{R}$, $r > 0$, in the case of $y = 0$ and $X \cap B(y,r)$, if $y \in \partial\mathcal{R}$.

The inclusion mapping $\mathcal{I} : \mathcal{D}(X) \to \mathcal{D}$ is defined as $\mathcal{D}(X) \ni f \longrightarrow \mathcal{I}(f) \in \mathcal{D}$, where $\mathcal{I}(f)(x) = f(i(x))$ and $i(x) = x$ is the identification mapping from $\mathcal{R}$ to X.

Under the inclusion mapping $\mathcal{I} : \mathcal{D}(X) \to \mathcal{D}$ we define the domain

$$\mathcal{D}_0(X) = \left\{ f \in \mathcal{D}(X) \,:\, \forall b \in \partial\mathcal{R} \quad \lim_{x\to 0} f(x) = \lim_{x\to b} f(x) \right\}. \tag{2.16}$$

Corollary 2. *Let $\widehat{Q_x} = Q_x \circ i^{-1}$ be the measure induced on $C([0,\infty), X)$ by $i : \mathcal{R} \to X$. Then, $\widehat{Q_x}$ solves the martingale problem for the Markov pregenerator*

$$\mathcal{L} = (\frac{1}{2}\Delta_X, \mathcal{D}_0(X)) \tag{2.17}$$

with the convention that $\Delta_X f = \Delta\mathcal{I}(f)$ for any $f \in \mathcal{D}_0(X)$.

Remark 1. The intrinsic asymmetry of the process with respect to the direction the Brownian motion enters the origin is given by the boundary conditions on $X \setminus \{0\}$. The domain is composed of functions which are C^2 up to the boundary $\{0\}$; yet the one-sided limits on the 'south pole' neighborhood are equal, ensuring C^2 regularity on the lower sheet of the domain, while the one-sided limits on the 'north pole' (that is, the boundary inherited from $\partial\mathcal{R}$) are non necessarily equal, with the exception of the multi-index $|\alpha| = 0$ which ensures continuity.

Remark 2. We note that the domain $\mathcal{D}_0$ of the original process on $\mathcal{R}$ is not dense in $C(\overline{\mathcal{R}})$.

Proof. The argument does not change with $d > 1$ and is presented in [7]. We refer to [11] for the definition of a Markov pregenerator. The properties of $f \in \mathcal{D}_0(X)$ ensure that $\overline{\mathcal{D}_0(X)} = C(X)$. In addition, we have to show that if x is a maximum point for f, then $\Delta f(x) \leq 0$. If $x \neq 0$, this is a consequence of Taylor's formula about x. At $x = 0$ we can still apply the standard argument which shows that $\nabla f(0) = 0$ because it only depends on the ball $B(0, r)$, which is a subset of a neighborhood of the origin in $(X, \mathcal{T})$ as well, and then necessarily $\Delta_X f(0) \leq 0$. The rest is immediate from Proposition 2. □

In the following we shall use the notation $\|f\|_{C(\overline{\mathcal{R}})}$ for the supremum norm of the bounded function f and we assume that the domain $\mathcal{R}$ has boundary $\partial\mathcal{R} \in C^2$. We also recall that the Laplace transform of the first boundary hit (2.9) is analytic on the resolvent set of the Dirichlet Laplacian (2.6) due to the analyticity of the resolvent all over the resolvent set ([16]).

Given $\phi \in (\frac{\pi}{2}, \pi)$, we denote by $U_0(\phi) = U_0$ the sector of the complex plane containing the positive real axis and bounded by the two half-lines $(x, \pm\tan(\phi)x)$ for $x \leq 0$ and, for $R > 0$, we denote by $U_0(R, \phi)$ the truncated sector

$$U_0(R, \phi) = \Big\{\alpha \, : \, |arg(\alpha)| < \phi \, , \, |\alpha| > R \Big\} . \tag{2.18}$$

Theorem 1. *Let $P(t, x, dy)$ be the transition probability for the process $\{Q_x\}_{x \in \mathcal{R}}$. For any $t > 0$ the measure $P(t, x, dy)$ is absolutely continuous with respect to the Lebesgue measure on $\mathcal{R}$ and, if $N_x(t, \omega)$ is the total number of visits to the boundary up to time $t > 0$, its probability density function $p(t, x, y)$ is given by*

$$p(t, x, y) = p_{abs}(t, x, y) + \int_0^t p_{abs}(t - s, 0, y) dE[N_x(s, \omega)] \tag{2.19}$$

where

$$E[N_x(s, \omega)] = \sum_{n=1}^{\infty} P(N_x(s, \omega) \geq n) = \int_0^s \sum_{n=1}^{\infty} (h^x * (h^0)^{*\,, n-1})(r) dr \tag{2.20}$$

and satisfies the properties:

(i) for $f \in C(X)$, the contraction semigroup

$$S_t f(x) = \int_{\mathcal{R}} p(t, x, y) f(y) dy \tag{2.21}$$

maps continuous functions into continuous functions (generating a Feller process). The spectrum σ of the infinitesimal generator of (2.21) is contained among the eigenvalues of the Dirichlet Laplacian (2.6), the zeros of $1 - \widehat{h^0}(\alpha)$ and there exist $\phi \in (\frac{\pi}{2}, \pi)$, $R > 0$ and $M > 0$ such that the resolvent set $\varrho(\mathcal{L})$ includes the union of $(\lambda_1^{abs}, \infty) \setminus \{0\}$, the right half-plane $\Re(\alpha) > 0$ and the truncated sector $U_0(R, \phi)$ from (2.18).

The resolvent $R_\alpha f = \int_0^\infty e^{-\alpha t} S_t f dt$ of (2.21) is a meromorphic function on the resolvent set of the Dirichlet Laplacian with a simple pole at $\lambda_0 = 0$

$$R_\alpha f(x) = R_\alpha^{abs} f(x) + R_\alpha^{abs} f(0) \frac{\widehat{h^x}(\alpha)}{1 - \widehat{h^0}(\alpha)} \tag{2.22}$$

satisfying

$$\|R_\alpha f\|_{C(\overline{\mathcal{R}})} \leq \frac{M}{|\alpha|} \|f\|_{C(\overline{\mathcal{R}})} \quad \forall \, \alpha \in U_0(R, \phi) \, , \tag{2.23}$$

(ii) the residue at $\alpha = 0$ has kernel

$$\rho(y) = \frac{G(0, y)}{\int_{\mathcal{R}} G(0, y) dy} \tag{2.24}$$

where $G(x,y)$ is the Green function of the Laplacian with Dirichlet boundary conditions and

(iii) if α^* is the nonzero element of the spectrum σ with maximal real part, then

$$\sup_{\alpha\in\sigma\setminus\{0\}} \Re(\alpha) = \Re(\alpha^*) < 0$$

and, for any $f \in \mathcal{D}_0(X)$

$$\lim_{t\to\infty} \frac{1}{t}\log\Big(\sup_{\|f\|_{C(\overline{\mathcal{R}})}\le 1} \|S_t f(x) - \int_{\mathcal{R}} \rho(x)f(x)dx\|_{C(\overline{\mathcal{R}})}\Big) = \Re(\alpha^*)\,. \tag{2.25}$$

Corollary 3. *The process $\{Q\}$ is exponentially ergodic.*

Remark. The function defined by (2.9) has an analytic continuation on the resolvent set of the Dirichlet Laplacian, and can be re-written directly in terms of the resolvent R_α^{abs} as shown in equations (3.1)-(3.2).

3. Proof of Theorem 1

Proof. (i) The derivation of (2.19) does not depend on the dimension $d \in \mathbb{Z}_+$ hence we can refer to the proof of Theorem 1 in [7] directly.

By definition, the Laplace transform of a function $g(t)$ is equal to $\hat{g}(\alpha) = \int_0^\infty e^{-\alpha t}g(t)dt$ whenever the integral converges. From equation (2.5) $P(T_x > t) = \int_{\mathcal{R}} p_{abs}(t,x,y)dy$ we see that

$$\widehat{h^x}(\alpha) = E\Big[e^{-\alpha T_x}\Big] = -\int_0^\infty e^{-\alpha t} dP(T_x > t)\,. \tag{3.1}$$

For $\Re(\alpha) > 0$, we derive

$$\widehat{h^x}(\alpha) = -\int_{\mathcal{R}} e^{-\alpha t} p_{abs}(t,x,y)dy\Big|_0^\infty - \alpha R_\alpha^{abs}\mathbf{1}(x) = 1 - \alpha R_\alpha^{abs}\mathbf{1}(x) \tag{3.2}$$

where $\mathbf{1}(x)$ is the constant function equal to 1 and R_α^{abs} is the resolvent of the half Laplacian with Dirichlet boundary conditions (the infinitesimal generator of the absorbing Brownian motion) from (2.8). If $\Re(\alpha) > 0$ and arbitrary $x \in \mathcal{R}$ we immediately have $|\widehat{h^x}(\alpha)|_{\mathbb{C}} < 1$. Relation (2.19) is derived in [7] for $d = 1$ but the proof is identical in higher dimensions. With this in mind, for $\Re(\alpha) > 0$, we obtain from (2.19) and the formula (2.20) established in [7] that

$$\int_{\mathcal{R}} \hat{p}(\alpha,x,y)f(y)dy = \int_{\mathcal{R}} \widehat{p_{abs}}(\alpha,x,y)f(y)dy +$$

$$+\int_{\mathcal{R}} \widehat{p_{abs}}(\alpha,0,y)f(y)dy\Big(\sum_{n=1}^{\infty}\widehat{(h^x * (h^0)^{*,n-1})}\Big)(\alpha)$$

$$= \int_{\mathcal{R}} \widehat{p_{abs}}(\alpha,x,y)f(y)dy + \int_{\mathcal{R}} \widehat{p_{abs}}(\alpha,0,y)f(y)dy\Big(\sum_{n=1}^{\infty}\widehat{h^x}(\alpha)(\widehat{h^0}(\alpha))^{n-1}\Big)$$

which proves (2.22) on $\{\alpha : \Re(\alpha) > 0\}$ in the form

$$R_\alpha f(x) = R_\alpha^{abs} f(x) + R_\alpha^{abs} f(0)\,\frac{1-\alpha R_\alpha^{abs}\mathbf{1}(x)}{\alpha R_\alpha^{abs}\mathbf{1}(0)}\,. \tag{3.3}$$

For any $f \in \mathcal{D}_0(X)$ the resolvent $R_\alpha^{abs} f(x)$ is analytic on $\mathbb{C}\setminus\{\lambda_n^{abs} : n \geq 1\}$ ([16], page 211, applied to the generator of a semigroup), which implies that (3.3) can be extended as a meromorphic function outside the spectrum (2.6) of the Dirichlet Laplacian.

We shall use the results on analytic semigroups generated by strongly elliptic operators under Dirichlet boundary condition from [13] and [14]. The domain of Dirichlet Laplacian is not dense in $C(\overline{\mathcal{R}})$ in the uniform convergence norm, yet there exists a $\phi_0 \in (\frac{\pi}{2},\pi)$ for which $C(\overline{\mathcal{R}})$ belongs to the domain of the resolvent operator for any α in U_0. Moreover, there exist $R_0 > 0$ and $M^{abs} > 0$ such that the main estimate for analytic semigroups (see [12] and [16])

$$\|R_\alpha^{abs} f\|_{C(\overline{\mathcal{R}})} \leq \frac{M^{abs}}{|\alpha|}\,\|f\|_{C(\overline{\mathcal{R}})} \qquad \text{for all } \alpha \in U_0(R_0,\phi_0) \tag{3.4}$$

is valid.

We want to extend the estimate (3.4) to the resolvent (3.3) to obtain (2.23). We prove that $\alpha R_\alpha f(x)$ stays bounded for $\alpha \in U_0(R,\phi) \subseteq U_0(R_0,\phi_0)$. It is sufficient to show that there exist a radius $R' \in [R_0,\infty)$, an angle $\phi' \in (\frac{\pi}{2},\phi_0]$ and a constant $\tilde{M} > 0$ such that

$$\sup_{\alpha\in U_0(R',\phi')}\ \sup_{x\in\mathcal{R}}\Big|\alpha R_\alpha^{abs} f(0)\frac{1-\alpha R_\alpha^{abs}\mathbf{1}(x)}{\alpha R_\alpha^{abs}\mathbf{1}(0)}\Big| \leq \tilde{M}\|f\|_{C(\overline{\mathcal{R}})}\,. \tag{3.5}$$

Since $|\alpha R_\alpha^{abs} f(0)|$ can be bounded using (3.4) we only have to show that

$$\sup_{\alpha\in U_0(R',\phi')}\ \sup_{x\in\mathcal{R}}\Big|1-\alpha R_\alpha^{abs}\mathbf{1}(x)\Big| < \infty \tag{3.6}$$

and

$$\inf_{\alpha\in U_0(R',\phi')}\Big|\alpha R_\alpha^{abs}\mathbf{1}(0)\Big| > 0\,. \tag{3.7}$$

The resolvent identity applied to the constant function $\mathbf{1}$ for $\alpha, \beta \in \varrho(\mathcal{L})$ reads

$$R^{abs}_{\beta}\mathbf{1} - R^{abs}_{\alpha}\mathbf{1} = (\alpha - \beta)\, R^{abs}_{\beta}\,(R^{abs}_{\alpha}\mathbf{1})$$

and implies

$$\big(I - \big(1 - \frac{\beta}{\alpha}\big)\,\alpha R^{abs}_{\alpha}\big)\big(\beta R^{abs}_{\beta}\mathbf{1} - \mathbf{1}\big) \;=\; \alpha R^{abs}_{\alpha}\mathbf{1} - \mathbf{1}\,. \tag{3.8}$$

Let $\beta = |\alpha|$. Since we have $\|\alpha R^{abs}_{\alpha}\| \le M^{abs}$ in the operator norm from $C(\overline{\mathcal{R}})$ to $C(\overline{\mathcal{R}})$, then for all α in the truncated sector $U_0(R_0, \phi_0)$,

$$\|\big(I - \big(1 - \frac{\beta}{\alpha}\big)\,\alpha R^{abs}_{\alpha}\big)\| \le \big(1 + 2M^{abs}\big) = M_1\,.$$

Therefore,

$$\|\alpha R^{abs}_{\alpha}\mathbf{1} - 1\|_{C(\overline{\mathcal{R}})} = \|\big(I - \big(1 - \frac{\beta}{\alpha}\big)\,\alpha R^{abs}_{\alpha}\big)\big(\beta R^{abs}_{\beta}\mathbf{1} - 1\big)\,\|_{C(\overline{\mathcal{R}})} \le M_1\,\|\beta R^{abs}_{\beta}\mathbf{1} - 1\|_{C(\overline{\mathcal{R}})}\,. \tag{3.9}$$

Since $\beta > 0$ we can write $\beta R^{abs}_{\beta}\mathbf{1}(x) - 1 = -\widehat{h^x}(\beta)$ from equation (3.2). The uniform bound for (3.6) is hence equal to

$$M_1 \sup_{\alpha \in U_0(R', \phi')} \sup_{x \in \mathcal{R}} E[e^{-|\alpha| T_x}] \le M_1 \tag{3.10}$$

for any $R' \ge R_0$ and $\phi' \in (\frac{\pi}{2}, \phi]$.

Assume that (3.7) is false for any $U_0(R', \phi(R'))$ where $R' > R_0$ and $\phi(R') \in (\frac{\pi}{2}, \phi_0]$ is of the form $\phi(R') = \frac{\pi}{2} + \arcsin(\frac{1}{R'})$. Let $R_n \to \infty$. Then, there exists a subsequence $\{n_k\}_{k\ge 1}$ such that $\{\alpha_{n_k}\} \in U_0(R_{n_k}, \phi(R_{n_k}))$ violates the lower bound (3.7). The domain $U_0(R', \phi(R'))$ is closed to complex conjugation and the complex norm from (3.7) is invariant to conjugation. This shows that we can assume, without loss of generality, that $\Im(\alpha_{n_k}) > 0$. For simplicity we subindex the subsequence by n as well.

We write $\alpha_n = r_n \exp\Big(i(\frac{\pi}{2} + \epsilon_n)\Big)$. Naturally $r_n \ge R_n \to \infty$ and also $\epsilon_n < \arcsin(\frac{1}{R_n})$. On the other hand, we can select a subsequence such that $\liminf \epsilon_n = 0$. Otherwise there exists $\epsilon > 0$ such that $\epsilon_n \le -\epsilon$ for large enough n. This, together with the inequality $|\alpha_n R^{abs}_{\alpha_n}\mathbf{1}(0)| \ge 1 - |\widehat{h^0}(\alpha_n)|$ derived from (3.2) and

$$\lim_{n\to\infty} |\widehat{h^0}(\alpha_n)| \le \lim_{n\to\infty} E[e^{-r_n \cos(\frac{\pi}{2} - \epsilon) T_0}] = 0$$

would imply a contradiction with the assumption on $\{\alpha_n\}$. We have shown that $\epsilon_n \to 0$.

Equation (3.8) can be re-written in the form

$$\frac{\beta}{\alpha}\big(\beta R^{abs}_{\beta}\mathbf{1}(x) - 1\big) + \Big(1 - \frac{\beta}{\alpha}\Big)\Big(I - \alpha R^{abs}_{\alpha}\Big)\big(\beta R^{abs}_{\beta}\mathbf{1}(x) - 1\big) = \alpha R^{abs}_{\alpha}\mathbf{1}(x) - 1\,. \tag{3.11}$$

Choose

$$\beta_n = r_n \exp\Big(i(\frac{\pi}{2} - \delta_n)\Big)$$

with $\delta_n = \arcsin(\frac{1}{\sqrt{r_n}})$. Then the second term from (3.11) applied to $x = 0$

$$\Big|\Big(1 - \frac{\beta}{\alpha}\Big)\Big(I - \alpha R_\alpha^{abs}\Big)\big(\beta R_\beta^{abs}\mathbf{1}(0) - 1\big)\Big|$$

has the upper bound

$$\Big|1 - \frac{\beta}{\alpha}\Big| \|I - \alpha R_\alpha^{abs}\| \|\beta R_\beta^{abs}\mathbf{1}(0) - 1\|_{C(\overline{\mathcal{R}})} \leq \Big|1 - \frac{\beta}{\alpha}\Big|(1 + M^{abs})M_1$$

where we used (3.4) for the operator norm and (3.9)-(3.10) for the uniform norm. This term vanishes as $n \to \infty$ since $\Big|1 - \frac{\beta_n}{\alpha_n}\Big| \leq 2\Big|\sin(\frac{\delta_n + \epsilon_n}{2})\Big|$.

The first term in (3.11) at $x = 0$ satisfies the bound

$$\Big|\frac{\beta_n}{\alpha_n}\big(\beta_n R_{\beta_n}^{abs}\mathbf{1}(0) - 1\big)\Big| \leq \Big|\beta_n R_{\beta_n}^{abs}\mathbf{1}(0) - 1\Big| \leq$$

$$E\Big[e^{-\Re(\beta_n)T_0}\Big] = E\Big[e^{-(r_n \sin(\delta_n))T_0}\Big] = E\Big[e^{-\sqrt{r_n}T_0}\Big] \to 0\,.$$

These estimates show that as $n \to \infty$, the left hand side of (3.11) vanishes meanwhile the right hand side approaches -1 by the assumption made on the sequence $\{\alpha_n\}$, which is a contradiction. This concludes the proof of (3.7).

On the real axis, the function $\widehat{h^0}(\alpha)$ is the Laplace transform of the first hitting time of the boundary, equal to 1 at $\alpha = 0$ and non-increasing on $(\lambda_1^{abs}, \infty)$. The function is analytic wherever R_α^{abs} is analytic, hence $1 - \widehat{h^0}(\alpha)$ has no other zeros on a neighborhood of $(\lambda_1^{abs}, \infty)$.

Since $R_\alpha^{abs} f$ is analytic in the union of U_0 with the right half-plane $\Re(\alpha) > \lambda_1^{abs}$, the denominator $1 - \widehat{h^0}(\alpha)$ from (3.3) has only isolated zeros. We conclude that all singularities of the resolvent R_α contained in the resolvent set of the Dirichlet Laplacian are poles coinciding with the zeros of the denominator.

(ii) and (iii). We can compute the residue at $\alpha = 0$. Multiplying (2.22) by α, we get

$$\alpha R_\alpha f(x) = \alpha R_\alpha^{abs} f(x) + \frac{\alpha R_\alpha^{abs} f(0)}{\alpha R_\alpha^{abs}\mathbf{1}(0)}\left(1 - \alpha R_\alpha^{abs}\mathbf{1}(x)\right)$$

Since $R_\alpha^{abs} f$ is analytic in a neighborhood of $\alpha = 0$, it is enough to figure out the limit of $\alpha R_\alpha f(x)$ as $\alpha \to 0$ along the positive real axis. By dominated convergence, or directly from the continuity of the resolvent R_α^{abs} at $\alpha = 0$, we see that $\lim_{\alpha\to 0+} \alpha R_\alpha^{abs} f(x) = 0$, and that

$$\lim_{\alpha\to 0+} \frac{\alpha R_\alpha^{abs} f(0)}{\alpha R_\alpha^{abs}\mathbf{1}(0)} = \frac{\int_{\mathcal{R}} G(0,y)\, f(y)\, dy}{\int_{\mathcal{R}} G(0,y)\, dy} = \int_{\mathcal{R}} \rho(y)\, f(y)\, dy$$

where $\rho(y) = G(0,y)(\int_{\mathcal{R}} G(0,y)\, dy)^{-1}$.

All singularities, with the exception of zero, have negative real part. First, the singularities must be among the zeros of the denominator $\alpha R_\alpha^{abs}\mathbf{1}(0) = 1 - \widehat{h^0}(\alpha)$ since the resolvent (3.3) is a meromorphic function on U_0 and the numerator is analytic. If $\Re(\alpha) > 0$ then $|\widehat{h^0}(\alpha)| < 1$. The case $\alpha = ik$, with $k \in \mathbb{R}$ is equivalent to showing that the Fourier transform of a probability density function $f(t)$ can never attain the value one except at $k = 0$. The transform is $\int_0^\infty e^{-ikt} f(t)dt = 1$ implies that $\int_0^\infty (1 - \cos(kt)) f(t)dt = 0$, a contradiction. Let α^*, α^{**} be the nonzero elements of the spectrum with the largest two values of the real part, that is $0 > \Re(\alpha^*) > \Re(\alpha^{**})$.

Let ϵ be a positive number, chosen sufficiently small to satisfy the following construction. There exists a positive constant $R_0^* > \max(R_0, 2\Re(\alpha^{**})/\cos\phi^*)$ and an angle $\phi^* \in (\frac{\pi}{2}, \phi)$ such that we can construct a piecewise smooth, continuous non-intersecting infinite contour L_ϵ with the following properties:

(1) L_ϵ coincides with the half-lines $\{(x, \pm\tan(\phi^*)x) \,:\, x < 0\}$ for $|\alpha| > R_0^* \geq R'$, where R' is given in (3.5),

(2) $\sup_{\alpha\in L_\epsilon} \Re(\alpha) \leq \Re(\alpha^{**}) + \frac{\epsilon}{2}$,

(3) the sets $\sigma \setminus \{\alpha^*\}$ and $\{\alpha^*, 0\}$ are separated by L_ϵ, such that $\{\alpha^*, 0\}$ is contained in the same component as the positive real axis,

(4) there exists a constant $C > 0$, independent of ϵ, such that the length of the contour segment $L_\epsilon \cap B(0, R_0^*)$ is bounded above by CR_0.

The proof of part (iii) is analogous to that of part (iii) in Theorem 1 in [7]. Let $R > R_0^*$. We denote by $A_\pm = (R\cos(\phi^*), \pm R\sin(\phi^*))$ and $B_\pm = (\alpha_0, \pm R\sin(\phi^*))$.

Proposition 3 provides an inversion formula for the resolvent R_α. For $\alpha_0 > 0$

$$S_t f(x) = \frac{1}{2\pi i}\int_{\alpha_0 - i\infty}^{\alpha_0 + i\infty} e^{\alpha t} R_\alpha f(x) d\alpha\,. \tag{3.12}$$

Apply Proposition 4 to $f \to R_\alpha f$ and both poles $\zeta_0 \to 0$ and $\zeta_0 \to \alpha^*$, with $\zeta_1' = \Re(\alpha^{**}) + \frac{\epsilon}{2}$, $\zeta_1 = \Re(\alpha^{**}) + \epsilon < \Re(\alpha^*)$, $\zeta_2 = \alpha_0 - \epsilon$, and $\zeta_2' = \alpha_0 - \frac{\epsilon}{2}$, taking sectors with sufficiently large aperture to include the infinite region to the left side of L_ϵ. Notice that the bound M in the proposition is a bound in operator norm, that is $||R_\alpha f|| \leq M||f||$. At $\alpha = 0$ the resolvent has a simple pole, and let m be the (possibly greater than one) multiplicity of the pole at $\alpha=\alpha^*$. Combining formulas (3.18) and (3.19) for the multiple pole α^*, we write

$$F_*(t,f)(x)=e^{\alpha^* t}\Big(C_1 f(x)+\frac{t}{1!}C_2 f+\ldots+\frac{t^{m-1}}{(m-1)!}C_m f\Big)+\mathcal{E}_F(t,f)(x)$$

with the error term $\mathcal{E}_F(t,f)(x)$ such that $\sup_{||f||\le 1}||\mathcal{E}_F(t,f)(x)||\sim o(e^{\Re(\alpha^{**})+\epsilon t})$ and C_i, $1\le i\le m$ are bounded linear operators. Then, from the Cauchy formula,

$$|S_t f(x)-Res(0;e^{\alpha t}R_\alpha f(x))-F_*(t,f)(x)|\le \frac{1}{2\pi}\Big|\int_{A_-A_+}e^{\alpha t}R_\alpha f(x)d\alpha\Big|+\mathcal{E}(t,R)\|f\|\,. \tag{3.13}$$

The error term is the sum of the integrals over the horizontal segments A_+B_+ and A_-B_-, which approach zero for large R, plus as the error corresponding to the infinite branches $|\Im(\alpha)|>>R\sin(\phi^*)$ in the principal value (3.17). More precisely, there exists a positive constant C_0 independent of ϵ, t such that $\mathcal{E}(t,R)\le C_0R^{-1}e^{\alpha_0 t}$ for any t.

Due to properties (1) - (4), the integral over A_-A_+ from (3.13) is bounded above by

$$e^{(\epsilon+\Re(\alpha^{**}))t}\,\|f\|\,\Big(M_1(\epsilon)+M_2(\epsilon,t)\Big) \tag{3.14}$$

where $M_1(\epsilon)=(2\pi)^{-1}CR_0\Big(\sup_{\alpha\in B(0,R_0^*)\cap L_\epsilon}\|R_\alpha\|\Big)$ and

$$M_2(\epsilon,t)=\frac{(\tilde{M}+M^{abs})}{\pi R_0^*}\int_{R_0^*}^{\infty}e^{[r\cos\phi^*-(\epsilon+\Re(\alpha^{**}))]t}dr\le\frac{2(\tilde{M}+M^{abs})}{\pi R_0^*|\cos\phi^*|t}e^{[-\frac{R_0^*|\cos\phi^*|t}{2}]}\,. \tag{3.15}$$

In (3.15) we used (3.4)-(3.5) for the resolvent (3.3). We can further bound $M_2(\epsilon,t)$ by a constant $M_2(t_0)$, independent from ϵ and $t>t_0$. Finally, if $M(\epsilon,t_0)=M_1(\epsilon)+M_2(t_0)$, we have shown that

$$\Big|S_t f(x)-\int_{\mathcal{R}}\rho(y)\,f(y)\,dy-F_*(f)(x)\Big|\le e^{(\epsilon+\Re(\alpha^{**}))t}M(\epsilon,t_0)\|f\|+\mathcal{E}(t,R)\|f\|\,. \tag{3.16}$$

For large t, we have $||F_*(t,f)||\le e^{\Re(\alpha^*)t}P_m(t)$, with $P_m(t)=\sum_{i=0}^{m-1}(||C_i||/i!)t^i+\epsilon$. Writing

$$N(t)=\sup_{||f||_{C(\overline{\mathcal{R}})}\le 1}\Big|S_t f(x)-\int_{\mathcal{R}}\rho(y)\,f(y)\,dy\Big|$$

and $C(t,\epsilon,R)=e^{(\epsilon+\Re(\alpha^{**}))t}M(\epsilon,t_0)+\mathcal{E}(t,R)$, we have

$$e^{\Re(\alpha^*)t}P_m(t)-C(t,\epsilon,R)\le N(t)\le e^{\Re(\alpha^*)t}P_m(t)+C(t,\epsilon,R)\,.$$

Let $R\to\infty$, obtaining the inequalities

$$e^{\Re(\alpha^*)t}\Big(P_m(t)-e^{(\epsilon+\Re(\alpha^{**})-\Re(\alpha^*))t}M(\epsilon,t_0)\Big)\le N(t)$$

and

$$N(t) \leq e^{\Re(\alpha^*)t}\Big(P_m(t) + e^{(\epsilon+\Re(\alpha^{**})-\Re(\alpha^*))t}M(\epsilon,t_0)\Big)\,.$$

Taking the logarithm, dividing by t and letting $t \to \infty$ we conclude the proof of the theorem. □

Finally, we give the statement of the classical inversion theorems for the Laplace transform ([2]).

Proposition 3. *Let $F(t)$ be a continuous function defined for $t > 0$ such that there exists an $x_0 \in R$ with the property that*

$$\int_0^\infty e^{-x_0 t}|F(t)|dt < \infty\,.$$

Then, the Laplace transform $\hat{F}(\alpha)$ is analytic in the half-plane $Re(\alpha) > x_0$ and the following inversion formula is valid

$$F(t) = P.V\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{\alpha t}\hat{F}(\alpha)d\alpha \tag{3.17}$$

where $x \geq x_0$ is arbitrary.

We recall the definition of the sector $U_0 = U_0(\phi)$ with angle ϕ from (2.18) and denote $U_\zeta = \zeta + U_0$ the sector originating at $\zeta \in \mathbb{C}$.

Proposition 4. *Let $\zeta_1' < \zeta_1 < \zeta_0 < \zeta_2 < \zeta_2'$ and let $f(\alpha)$ be analytic in the domain $V = U_{\zeta_1'} \setminus \overline{U}_{\zeta_2'}$ with the exception of $\alpha = \zeta_0$ which is a pole of order $m \in Z_+$ with the principal part of the Laurent expansion about ζ_0 equal to*

$$\frac{c_1}{(\alpha-\zeta_0)} + \ldots + \frac{c_m}{(\alpha-\zeta_0)^m}\,. \tag{3.18}$$

Assume that there exist $R_0 > 0$ and $M > 0$ with the property $|f(\alpha)| \leq M$ if $|\alpha - \zeta_0| \geq R_0$. Then there exists a $T > 0$ such that the integral (in principal value sense)

$$F(t) = \frac{1}{2\pi i}\int_{L(\zeta_2)} e^{\alpha t} f(\alpha)d\alpha$$

is uniformly convergent for $t \geq T$ and for $t \to \infty$ we have the asymptotic expansion

$$F(t) = e^{\zeta_0 t}\Big(c_1 + \frac{c_2}{1!}t + \ldots + \frac{c_m}{(m-1)!}t^{m-1}\Big) + o(e^{\zeta_1 t})\,. \tag{3.19}$$

4. The one dimensional case

In $d = 1$, let $\mathcal{R} = (a, b)$, with $a < 0 < b$ as in [7]. Let $\lambda_k = k\pi/(b-a)$, $k = 1, 2, \ldots$, and $\sigma_{abs} = \{-\lambda_k^2/2 \, : \, k = 1, 2, \ldots\}$ be the spectrum of the half Laplacian with absorbing boundary conditions with transition kernel (2.7)

$$p_{abs}(t, x, y) = \frac{2}{b-a} \sum_{k=1}^{\infty} e^{-(\lambda_k^2/2)t} \sin \lambda_k (x-a) \sin \lambda_k (y-a)\,, \tag{4.1}$$

and resolvent kernel

$$\widehat{p_{abs}}(\alpha, x, y) = \frac{2}{b-a} \sum_{k=1}^{\infty} \frac{1}{\alpha + \lambda_k^2/2} \sin \lambda_k (x-a) \sin \lambda_k (y-a)\,. \tag{4.2}$$

The Laplace transform of the first exit time (3.1) can be written in two forms (see [7])

$$\widehat{h^x}(\alpha) = \frac{2\pi}{(b-a)^2} \sum_{k=1,\, odd} \frac{k}{\alpha + \lambda_k^2/2} \sin \lambda_k (x-a) = \frac{\cosh \sqrt{2\alpha}\Big(x - \frac{b+a}{2}\Big)}{\cosh \sqrt{2\alpha}\Big(\frac{b-a}{2}\Big)} \tag{4.3}$$

and the kernel of the resolvent (2.22) of the process given by the transition kernel (2.19) is then

$$\widehat{p}(\alpha, x, y) = \widehat{p_{abs}}(\alpha, x, y) + \widehat{p_{abs}}(\alpha, 0, y) H(\alpha, x) \tag{4.4}$$

where

$$H(\alpha, x) = \frac{\widehat{h^x}(\alpha)}{1 - \widehat{h^0}(\alpha)} = \frac{\cosh \sqrt{2\alpha}\Big(x - \frac{b+a}{2}\Big)}{\cosh \sqrt{2\alpha}\Big(\frac{b-a}{2}\Big) - \cosh \sqrt{2\alpha}\Big(\frac{b+a}{2}\Big)}\,. \tag{4.5}$$

Set $\gamma = -\lambda_1^2/2 = -\pi^2(\sqrt{2}(b-a))^{-2}$. Then we write $\sigma_{abs} = \{\gamma k^2 \, : \, k \in \mathbb{Z}_+\}$ for the spectrum of the absorbing Brownian kernel and

$$\sigma_H = \{0\} \cup \{4(1 + |a|/b)^2 \gamma k^2\,, 4(1 + b/|a|)^2 \gamma k^2 \, : \, k \in \mathbb{Z}_+\}$$

for the zeros of $\cosh \sqrt{2\alpha}\Big(\frac{b-a}{2}\Big) - \cosh \sqrt{2\alpha}\Big(\frac{b+a}{2}\Big)$. In addition, σ_{abs}^{odd} and σ_{abs}^{even} denote the subsets of σ_{abs} for k odd and k even, respectively. It is easy to see that when $b/a \notin \mathbb{Q}$, then $\sigma_H \cap \sigma_{abs} = \emptyset$. In dimension $d = 1$, we can describe the spectrum σ exactly.

Proposition 5. *The spectrum σ of the Brownian motion with return is $\sigma_{abs}^{even} \cup \sigma_H$. As a consequence, the largest nonzero point of the spectrum is $-\lambda_2^2/2 = 4\gamma$.*

Proof. Part 1. We prove that if $\beta \in \sigma_H$, then $\widehat{p_{abs}}(\alpha, 0, y)$ is analytic and nonzero at $\alpha = \beta$. For $\beta \in \sigma_H \setminus \sigma_{abs}$, we only have to show that $\widehat{p_{abs}}(\beta, 0, y) \neq 0$, which is evident since $\widehat{p_{abs}}(\beta, 0, y)$, as an element in $L^2[a,b]$, has nonzero Fourier coefficients. If $\beta \in \sigma_H \cap \sigma_{abs}$, then there exists a positive integer k such that $\beta = -k^2\gamma = -\lambda_k^2/2$, $\lambda_k = k\pi/(b-a)$ and $\cos\lambda_k(\frac{b-a}{2})(1-\cos\lambda_k a) + \sin\lambda_k(\frac{b-a}{2})\sin\lambda_k a = 0$. For k odd, $(-1)^{\frac{k-1}{2}}\sin\lambda_k a = 0$ and for k even $(-1)^{\frac{k}{2}}(1-\cos\lambda_k a) = 0$. In all cases $\sin\lambda_k a = 0$, which proves our claim.

Part 2. Since $\sigma \subseteq \sigma_{abs} \cup \sigma_H$, we divide the proof in three steps.

(i) $\sigma_H \setminus \sigma_{abs} \subseteq \sigma$. Let $p = 1, 2, \ldots$ be the multiplicity of the pole denoted by β of $H(\alpha, x)$. Then

$$(4.6) \quad \lim_{\alpha\to\beta}\Big\{(\alpha-\beta)^p \widehat{p}_{abs}(\alpha,x,y) + \widehat{p}_{abs}(\alpha,0,y)\left[(\alpha-\beta)^p H(\alpha,x)\right]\Big\} = \widehat{p}_{abs}(\beta,0,y)\bar{H}(\beta,x)$$

where $\lim_{\alpha\to\beta}(\alpha-\beta)^p H(\alpha,x) = \bar{H}(\beta,x) \neq 0$. Since $\widehat{p_{abs}}(\alpha,0,y) \neq 0$, β is a pole.

(ii) Let $\beta = -\lambda_k^2/2 \in \sigma_{abs} \cap \sigma_H$. We know that β is not a pole of $\widehat{p}_{abs}(\alpha, 0, y)$. If $p > 1$, since p_{abs} has only simple poles, the limit (4.6) is of the same type as in case (i). When $p = 1$, the limit (4.6) is

$$2(b-a)^{-1}\sin\lambda_k(x-a)\sin\lambda_k(y-a) + \widehat{p}_{abs}(-\lambda_k^2/2, 0, y)\bar{H}(-\lambda_k^2/2, x)$$

where the last factor is nonzero. As a function of y in $L^2[a,b]$ the limit is not identically zero, so β is a pole.

(iii) Let $\beta = -\lambda_k^2/2 \in \sigma_{abs} \setminus \sigma_H$. The limit (4.6) with $p = 1$ gives

$$2(b-a)^{-1}\sin\lambda_k(y-a)\Big[\sin\lambda_k(x-a) - \sin\lambda_k a\, H(-\lambda_k^2/2, x)\Big].$$

In this case, we have

$$(4.7) \qquad \lim_{\alpha\to\beta} H(\alpha,x) = \frac{\cos\lambda_k(x-a)\cos\lambda_k(\frac{b-a}{2}) + \sin\lambda_k(x-a)\sin\lambda_k(\frac{b-a}{2})}{\cos\lambda_k(\frac{b-a}{2})(1-\cos\lambda_k a) + \sin\lambda_k(\frac{b-a}{2})\sin\lambda_k a}.$$

The bracket is equal to $(\sin\frac{\lambda_k a}{2})^{-1}\cos\lambda_k(x-\frac{a}{2}) \neq 0$ for k even and vanishes for k odd, which implies that $\sigma_{abs}^{even} \subset \sigma$ and $\sigma_{abs}^{odd} \cap \sigma = \emptyset$.

Part 3. We notice that $\sup\{\beta \in \sigma_H\} < 4\gamma$, the second eigenvalue of p_{abs}, and 4γ corresponds to $k = 2$, the first even value in the spectrum. $\square$

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