SECOND ORDER APPROXIMATION OF A FLUID LIMIT NEAR EQUILIBRIUM

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Abstract. We analyze the fluctuations near a stationary point of the deterministic fluid limit of a random genetic model established in [4]. Near such a point, the vector field is small, but on the Central Limit Theorem scale, the departures from the equilibrium point converge in distribution, as a time dependent process, to a linear diffusion with drift intensity proportional to the derivative of the vector field. When the equilibrium is stable the constant is negative and the limit is the classical Ornstein-Uhlenbeck process. The result is under *natural scaling* in the sense that we do not add noise to a deterministic dynamical system, instead we study the second order approximation of a random process that scales to a deterministic ordinary differential equation.

1. Introduction and mathematical model

The genetic model for the fixation mechanism in a chromosome introduced in [10] and generalized in [6] follows the evolution of words

$$\mathbf{Z} = (Z^1, Z^2, \ldots, Z^L) \in S := \{0, 1, \ldots, N-1\}^L$$

of length L taking values in the first $N-1$ positive integers together with zero, which is singled out as a special character. A certain preferred configuration, here $\mathbf{0} = (0, \ldots, 0)$, is attained by random mutations. Once a zero component is reached, the probability to switch back to a non-zero character is $g(\mathbf{Z}) \in [0,1]$, where $g(\cdot)$ is a function depending on the current word (configuration). The case $g = 0$ corresponds to the *Muller ratchet effect*, and $g \in (0,1)$ to a degree of *stickiness* of the preferred configuration.

Let's denote by U the number of non-zero components of $\mathbf{Z}$. We assume there exists $\gamma \in C^1([0,1])$, $0 \leq \gamma(u) \leq 1$ such that

$$g(\mathbf{Z}) = \gamma(\frac{U}{L}), \qquad Z \in S, \quad 0 \leq U \leq L. \tag{1.1}$$

Date: December 27, 2018.

1991 *Mathematics Subject Classification.* Primary: 60F05; Secondary: 37H10, 37H20, 92D15.

Key words and phrases. Key words and phases: solw-fast system, perturbation of a Dynamical System, fluctuation limit, Ornstein-Uhlenbeck, Genetic model of evolution.

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The process U_t, $t \geq 0$, otherwise a function of Z_t, is in itself a Markov chain on the space $\{0, 1, ..., L\}$. Following the discrete time model from [6], the generating function of the transition probabilities associated to U_{t+1}, at update time t is given by

$$E\left(s^{U_{t+1}} | U_t = U\right) = \left(\frac{1}{N} + \left(1 - \frac{1}{N}\right) s\right)^U \left(1 - \gamma(\frac{U}{L}) + \gamma(\frac{U}{L}) s\right)^{L-U}, \qquad s > 0 . \tag{1.2}$$

In other words, after jump, U_{t+1} is the sum of two independent binomials, one with $U = U_t$ trials and probability of success (to remain non-zero) $1 - \frac{1}{N}$ and one with $L - U$ trials corresponding to the zero components, with probability of success (i.e. to convert into a non-zero character) equal to $\gamma(\frac{U}{L})$. If $\gamma(0) = 0$, the state $U = 0$ is absorbing (the cemetery state of the Markov process) and extinction occurs with probability one.

In this paper we consider the continuous time, pure jump version version of this dynamics. The infinitesimal generator is, for $f \in C([0, 1])$

$$\mathcal{A}^L f(u) = L\left[\frac{u}{N}\left(f(u - \frac{1}{L}) - f(u)\right) u + \gamma(u)(1 - u)\left(f(u + \frac{1}{L}) - f(u)\right)\right]. \tag{1.3}$$

We note that time is sped up by a factor of L and that when $\gamma(0) = 0$ the process will never leave the state 0.

One can see the process as the evolution of the empirical measure (relative frequency)

$$u_t^L = \frac{U_t}{L} = \frac{1}{L}\sum_{j=1}^{L} \mathbf{1}_{\{0\}}(Z_t^j) . \tag{1.4}$$

This and $\gamma = \gamma(u^L)$ point out to a *mean-field* dependence, leading to the natural scaling of a Law of Large Numbers. When in time-dependent setup and established for dependent particles, such a scaling limit is known as a *fluid limit*. The empirical measure of the zero states, here simply U_t/L, converges in probability to the deterministic solution of an ODE (1.6). This article aims at a second approximation of the solution, on a CLT scale, near equilibrium points.

Theorem 1 (from [4]) establishes the fluid limit. It is the continuous time analogue of Theorem 3 in [6]. In that paper, the scaled process is a deterministic discrete time dynamical system. With $\gamma(u)$ defined in (1.1), let

$$H(u) := \left(-\frac{1}{N}\right) u + \gamma(u)(1 - u) , \qquad 0 \leq u \leq 1 . \tag{1.5}$$

Theorem 1 (Theorem 1 from [4])**.** *Assume that u_0^L converges in probability, as $L \to \infty$, to the deterministic state $\bar{u} \in [0, 1]$ and $\gamma = \gamma(u)$, $0 \leq u \leq 1$ from (1.1). Then, as $L \to \infty$, the Markov process $(u_t^L)_{t\geq 0}$ converges in distribution to the deterministic process $(u_t)_{t\geq 0}$ on*

$[0,1]$, *equal to the unique solution of*

$$\frac{du}{dt} = H(u)\,, \qquad u(0) = \bar{u}\,. \tag{1.6}$$

2. Fluctuation Near Equilibrium

In this paper we will analyze behavior near equilibrium points. We will show that if we use a different scaling near equilibria, given the appropriate initial values, the second approximation to the process u_t behaves as a diffusion.

To describe fluctuations around an equilibrium point u' of the function H, we introduce a new process (y_t^L) of the form

$$y_t^L = \sqrt{L}\Big(u_t^L - u'\Big)\,, \qquad H(u') = 0 \quad \text{and} \quad u_t^L = \frac{U_t^L}{L}\,. \tag{2.1}$$

This amounts to studying the fluctuation process (2.1) for (1.4) since $u_t^L = U_t^L/L$ and so y_t^L is exactly the diffusive or Central Limit Theorem scaling. Throughout we assume the initial condition at time zero

$$\sqrt{L}(u_0^L - u') \xrightarrow{p} y_0 \in \mathbb{R}\,, \tag{2.2}$$

i.e. y_0^L converges in probability to a real value y_0 when we start from values $\sim O(L^{-1/2})$.

In the main result, Theorem 2, we prove that the process (y_t^L) converges in distribution to (y_t), an Ornstein-Uhlenbeck (O-U) process with drift parameter $r = H(u')$. We note the convergence is not only for marginals at given time t, but as a process, which is seen as a random variable on the Skorokhod space of rcll paths. In a dynamical systems context, a random perturbation of an ODE induces a similar behavior after linearization - see [2], Ch. 1 and references on slow-fast systems.

It is remarkable that, at the microscopic level of the U_t process, the only equilibrium is zero. In [3] the relation between quasi-stationary distributions and equilibrium points of the scaled system is investigated in more detail.

Theorem 2 (Perturbation near Equilibrium)**.** *Let H be defined as in (1.5), with u' an equilibrium point of H. Provided the initial condition (2.2) then, the process (y_t^L) from (2.1) converges in distribution, as $L \to \infty$, to the one-dimensional O-U process with generator*

$$\mathcal{U}g(y) = -ryg'(y) + \frac{1}{2}\sigma^2 g''(y)\,, \qquad r = -H'(u')\,, \quad \sigma^2 = \frac{2u'}{N} \tag{2.3}$$

starting at $y_0 \in \mathbb{R}$.

The arguments given in this paper prove a slightly stronger result, in that H needs to have a continuous derivative only in a neighborhood of an equilibrium point $u' \in (0,1)$. This is due to the fact that under the scaling considered in (2.1) only a neighborhood $u' \pm O(\frac{1}{\sqrt{L}})$

ever appears in calculations involving H' directly. We did not make this distinction because it complicates the exposition and most applications use a smooth γ. Moreover, Theorem 2 improves the result of Theorem 3.1 in [3], by removing the condition that H' be Lipschitz.

In [4] function γ was a power law $\gamma(u) = cu^a$ in all applications. Here we require that $\gamma \in C^1([0,1])$, which would be satisfied if $a \geq 1$. Moreover, in a region of the parameters (a, c, N) with $a > 1$ and cN sufficiently large, the ODE has at least two stationary solutions (equlilibrium points), besides zero, one stable and one unstable (cf. Proposition 1, [4]). Evidently, at those points the question of a scaled version of the second approximation is meaningful.

Stability and recurrence are put in direct correspndence, mirrored in the sign of the parameter $r \in \mathbb{R}$. Classical results imply that when $r = H'(u') > 0$ (u' stable) the O-U process is recurrent and when $r < 0$ (u' unstable) it is transient. We note that Theorem 2 holds when $r = 0$ (non-hyperbolic equilibrium [7]). Since the fluctuation limit is a one-dimensional Brownian motion, it is null-recurrent. Recalling that $y = 0$ means $u = u'$ (2.1) on the larger scale $\sqrt{L}$, $L \to \infty$, the initial point is, macroscopically, within $1/\sqrt{L}$ from u' and points on $[0,1]$ at macroscopic distance are sent to infinity. However, the microscopic behavior can be studied in detail, e.g. the hitting time τ_0 of $y = 0$ (i.e. u') can be explicitly calculated. For this, fine properties and explicit formulas for τ_0 we refer the reader to [1].

The diffusion coefficient $\sigma^2 = 2u'/N$ reflects the particular interaction model (1.2) and (1.5) under consideration. The factor $1/N$ is proportional to the intensity of *mutation*, while u' shows that a larger value of the equilibrium is more volatile. If $u' = 0$, the process is a nonrandom trajectory solving the ODE (2.3). Finally, if $u' = H(u') = 0$, which can only occur if $\gamma(0) = 0$, $y_t \equiv y_0$.

3. The differential formula and martingales

The pure jump Markov process (u_t) with generator $\mathcal{A}^L$ (1.3) evolves in the state space $[0,1]$. We assume it is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, where the filtration satisfies the usual conditions.

More generally, for any $f \in C_c(\mathbb{R})$ (including the space $C([0,1])$ by canonical extension)

$$M_t^{f,L} = f(u_t^L) - f(u_0^L) - \int_0^t \mathcal{A}^L f(u_s^L)\, ds \tag{3.1}$$

is an $(\mathcal{F}_t)$ - martingale with predictable quadratic varation

$$\langle M^{f,L} \rangle_t = \int_0^t \mathcal{A}^L f^2(u_s^L) - 2f(u_s^L)\mathcal{A}^L f(u_s^L)\, ds\,. \tag{3.2}$$

This can be written explicitly

(3.3)

$$\langle M^{f,L}\rangle_t = \int_0^t L\left[\frac{u_s^L}{N}\left(f(u^L-\frac{1}{L})-f(u^L)\right)^2+\gamma(u_s^L)(1-u_s^L)\left(f(u_s^L+\frac{1}{L})-f(u_s^L)\right)^2\right]ds\,.$$

If the state space $\mathbb{X}$ is Polish, i.e. separable, complete metric space, Feller processes on $\mathbb{X}$ can be canonically constructed (Ch. III [8]) on the Skorokhod space $\mathcal{D}([0,\infty),\mathbb{X})$ of right-continuous with left-limit paths (rcll), a space endowed with the J_1 metric topology. The law of such a process is a probability measure on $\mathcal{D}([0,\infty),\mathbb{X})$. Tightness is the notion of pre-compactness of probability laws defined by Prokhorov's theorem. We shall use the stronger notion of $C-tightness$, which guarantees that any limit point of a precompact set is supported on the subset of contiunous paths $C([0,\infty),\mathbb{X})$.

Definition 1. *A sequence of processes $(Y_\cdot^L)_{L>0}$ on a Polish space $(\mathbb{X},||\cdot||)$ with right-continuous with left limits paths (in the Skorokhod space) is* C-tight, *if for any $T\geq 0$*

$$\text{(3.4)}\qquad (i)\qquad \lim_{M\to\infty}\limsup_{L\to\infty} P\Big(||Y_T^L||>M\Big)=0 \quad \textit{and}$$

$$\text{(3.5)}\qquad (ii)\quad \forall\,\epsilon>0\qquad \lim_{\delta\to 0}\limsup_{L\to\infty} P\Big(\sup_{t,t'\in[0,T],|t'-t|<\delta}||Y_{t'}^L-Y_t^L||>\epsilon\Big)=0\,.$$

In this paper, $\mathbb{X}$ is $\mathbb{R}$ with the Euclidean norm and the process (Y_t^L) is (y_t^L) from (2.1) as well as the associated processes form the differential formula (Ito's formula) (3.1).

4. Proof of Theorem 2

Theorem 2 will be proven in several steps. First, we show that the scaled process y_t^L is C-tight. By Prokhorov's theorem, tightness is pre-compactness in the space of probability measures. Denote by $\mathcal{L}^L$ the probability law of the processes $y_\cdot^L$, indexed by $L\in\mathbb{Z}_+$, with values in $\mathcal{D}([0,\infty),\mathbb{R})$. If the measures are tight, then there exists at least a limit point $\mathcal{L}$ and denote by (y_t) the process with this probability law. In subsection 4.3 we show that $\mathcal{L}$ solves the martingale problem associated with the generator $\mathcal{U}$ defined in (2.3) (Theorem 4.28 in [9], also Corollary 4.29 in [5]). The aim is then to prove that for any $g\in C_c^\infty(\mathbb{R})$, the process

$$M_t^g = g(y_t)-g(y_0)-\int_0^t \mathcal{U}g(y_s)\,ds$$

is a continuous $(\mathcal{F}_t)$ - martingale under the probability measure $\mathcal{L}$. Finally, this process is unique, coinciding with the Ornstein-Uhlenbeck process. This concludes the proof of convergence in distribution.

All functions considered belong to the family of test functions of the form $f(u) = g(y)$, $y = \sqrt{L}(u - u')$, $g \in C_c^3(\mathbb{R})$.

Tightness requires a square norm bound, uniform in L. This is done in subsection 4.1, Proposition 1, which also implies (3.4). As a consequence, both the generator $\mathcal{A}^L g(y_s^L)$ (Proposition 2) and the quadratic variation of the martingale $M_t^{g,L}$ (Proposition 3), are uniformly bounded in L^2. These bounds are needed in subsection 4.2 which establishes the modulus of continuity, necessary for (3.5). Proposition 6 concludes the proof of tightness. Finally subsection 4.3 shows that any limit point solves the martingale problem for the Ornstein-Uhlenbeck generator (Proposition 8).

Several times we shall apply the differential formula (1.3) -(3.1) to a function $f \in C_c^\infty(\mathbb{R})$ together with the expansion of H around the point $u_s^L = u' + \frac{y_s^L}{\sqrt{L}}$, based on Taylor's formula with remainder in integral form. If $\phi \in C_c^{n+1}(\mathbb{R})$ such that ϕ^{n+1} exists and is continuous on an open interval containing $[u', u' + \frac{y_s^L}{\sqrt{L}}]$, then by Taylor's formula with remainder in the integral form

$$\phi(u' + \frac{y_s^L}{\sqrt{L}}) = \sum_{k=0}^{n} \frac{1}{k!} \phi^{(k)}(u') \left(\frac{y_s^L}{\sqrt{L}} \right)^k + R_n(u' + \frac{y_s^L}{\sqrt{L}}) \tag{4.1}$$

where

$$R_n(u' + \frac{y_s^L}{\sqrt{L}}) = \frac{1}{n!} \left(\frac{y_s^L}{\sqrt{L}} \right)^{n+1} \int_0^1 w^n \phi^{(n+1)} \left(\left(u' + (1-w) \frac{y_s^L}{\sqrt{L}} \right)^k \right) dw. \tag{4.2}$$

Since ϕ^{n+1} is bounded on $[u', u' + \frac{y_s^L}{\sqrt{L}}]$ by some constant $c(\phi)$, then R_n satisfies

$$|R_n(u' + \frac{y_s^L}{\sqrt{L}})| \leq \frac{1}{(n+1)!} \left(\frac{y_s^L}{\sqrt{L}} \right)^{n+1} c(\phi). \tag{4.3}$$

4.1. **Uniform L^2 Bound.** Due to the initial condition (2.2), the initial value has uniformly bounded second moment. We prove a slightly stronger result.

Proposition 1. *Assume that the initial values are random variables and there exists a positive C, independent of L, such that $E[(y_0^L)^2] \leq C$. Let $T > 0$ be arbitrary but fixed. Then the process y_t^L is square integrable. Moreover, we have the bounds*

$$\mathbb{E}\left[\left(y_t^L \right)^2 \right] \leq K_1 \exp\left(K_2 \, t \right) \qquad 0 \leq t \leq T, \tag{4.4}$$

where K_1, K_2 are constants independent of t and L, and are given explicitly by $K_1 = 3C + 3\left(1 + \frac{2}{N}\right) T$ and $K_2 = 3T(c_1(H))^2$, where $c_1(H)$ is bound of the function H'.

Proof. Notice that $u_t^L \in [0,1]$ and so $y_t^L = \sqrt{L}(u_t^L - u') \in [-2\sqrt{L}, 2\sqrt{L}]$. Fix $T > 0$. Adopt a test function $f(x) = I(x)$ (as in identity function), with $I(x) = x$ on $[-3\sqrt{L}, 3\sqrt{L}]$ with compact support. We need uniform bounds in L. However, the estimates obtained below only use the fact that I equal the identity on the range of the process. For the generator (1.3) and $0 \leq s \leq t \leq T$

(4.5)

$$\begin{aligned}\mathcal{A}^L I(y_s^L) = \mathcal{A}^L y_s^L &= L\left[\frac{1}{N}\left((y_s^L - \frac{1}{\sqrt{L}}) - y_s^L\right) u_s^L + \gamma(u_s^L)\left((y_s^L + \frac{1}{\sqrt{L}}) - y_s^L\right)(1-u_s^L)\right] \\ &= \frac{1}{N}\left(-\frac{1}{\sqrt{L}}\right) L u_s^L + \gamma(u_s^L)\left(\frac{1}{\sqrt{L}}\right) L(1-u_s^L) \\ &= H(u_s^L)\sqrt{L} \\ &= y_s^L \int_0^1 H'(u' + (1-w)\frac{y_s^L}{\sqrt{L}})dw,\end{aligned}$$

where the last line is obtained form (4.1) applied to $\phi = H$, $n = 0$. We obtain that

$$|\mathcal{A}^L y_s^L| \leq c_1(H)\,|y_s^L| \tag{4.6}$$

with $c_1(H)$ the supremum of H' on $[0,1]$ which is independent of s and L.

A similar calculation for the quadratic variation (3.3) of the martingale $M_t^{I,L}$ shows

$$\begin{aligned}\langle M^{I,L}\rangle_t &= L\int_0^t \left\{\left[\frac{1}{N}\left((y_s^L - \frac{1}{\sqrt{L}}) - y_s^L\right)^2 (u_s^L) \;+ \gamma(u_s^L)\left((y_s^L + \frac{1}{\sqrt{L}}) - y_s^L\right)^2 (1-u_s^L)\right]\right\} ds \\ &= L\int_0^t \left\{\frac{1}{N}\left(-\frac{1}{\sqrt{L}}\right)^2 u_s^L + \gamma(u_s^L)\left(\frac{1}{\sqrt{L}}\right)^2 (1-u_s^L)\right\} ds \\ &= \int_0^t H(u_s^L) + \frac{2}{N} u_s^L \, ds.\end{aligned}$$

Since $|H(u)| \leq 1$, $u \in [0,1]$, then

$$\mathbb{E}\left[\langle M^{I,L}\rangle_t\right] \leq \left(1 + \frac{2}{N}\right) T \qquad 0 \leq t \leq T. \tag{4.7}$$

Recall that by formula (3.1) we can write

$$y_t^L = y_0^L + \int_0^t \mathcal{A}^L y_s^L \, ds + M_t^{I,L} \tag{4.8}$$

where $\mathcal{A}_t^L y_s^L$ is given by the differential formula (4.5). Thus, we obtain that

$$(y_t^L)^2 \leq 3\left((y_0^L)^2 + \left(\int_0^t |\mathcal{A}^L y_s^L| \, ds\right)^2 + \left(M_t^{I,L}\right)^2\right).$$

Remember also, that

$$N_t^I = \left(M_t^I\right)^2 - \langle M^{I,L}\rangle_t \tag{4.9}$$

is a martingale. Thus, by Cauchy-Schwarz, Fubini's Theorem, the initial condition on $\mathbb{E}[(y_0^L)^2]$, and relations (4.6) and (4.7) we obtain that

$$\begin{aligned}
\mathbb{E}\left[\left(y_t^L\right)^2\right] &\le 3\left(\mathbb{E}\left[\left(y_0^L\right)^2\right] + \mathbb{E}\left[T\int_0^t \left(|\mathcal{A}^L y_s^L|\right)^2\, ds\right] + \mathbb{E}\left[\left(M_t^{I,L}\right)^2\right]\right) \\
&\le 3\mathbb{E}\left[\left(y_0^L\right)^2\right] + 3(c_1(H))^2\, T\int_0^t \mathbb{E}\left[\left(y_t^L\right)^2\right]\, ds + 3\mathbb{E}\left[\langle M^{I,L}\rangle_t\right] \qquad (4.6) \\
&\le 3C + 3(c_1(H))^2\, T\int_0^t \mathbb{E}\left[\left(y_s^L\right)^2\right]\, ds + 3\left(1+\frac{2}{N}\right)T. \qquad (3.2),\ (4.7)
\end{aligned}$$

Hence

$$\mathbb{E}\left[\left(y_t^L\right)^2\right] \le K_1 + K_2\int_0^t \mathbb{E}\left[\left(y_s^L\right)^2\right] ds$$

where $K_1 = 3C + 3\left(1+\frac{2}{N}\right)T$, $K_2 = 3T(c_1(H))^2$.

By Gronwall's inequality

$$\mathbb{E}\left[\left(y_t^L\right)^2\right] \le K_1 \exp\left(K_2\, t\right) \qquad 0\le t\le T. \tag{4.10}$$

□

Proposition 2. *For every function $g\in C_c^3(\mathbb{R})$ and $H\in C^2(\mathbb{R})$ the generator of the process y_t^L is in L^2. Moreover, we have the estimate*

$$\mathbb{E}\left(\left|\mathcal{A}^L g(y_s^L)\right|^2\right) \le C_1\mathbb{E}\left(|y_s^L|^2\right) + C_2\,, \qquad s\ge 0\,,$$

where C_1 and C_2 are constants depending on the functions g, H and their derivative but not depending on L, t.

Proof. We choose a test function $g\in C_c^3(\mathbb{R})$, and apply the differential formula (1.3)-(3.2). We get the formula

(4.11)

$$\begin{aligned}
\mathcal{A}^L g(y_s^L) &= L\left[\frac{1}{N}\left(g(y_s^L - \frac{1}{\sqrt{L}}) - g(y_s^L)\right)u_s^L + \gamma(u_s^L)\left(g(y_s^L + \frac{1}{\sqrt{L}}) - g(y_s^L)\right)(1-u_s^L)\right] \\
&= \frac{1}{N}\Big[\left(g(y_s^L - \frac{1}{\sqrt{L}}) - g(y_s^L)\right)L\Big]u_s^L \\
&+ \gamma(u_s^L)\Big[\left(g(y_s^L + \frac{1}{\sqrt{L}}) - g(y_s^L)\right)L\Big](1-u_s^L).
\end{aligned}$$

We expand each difference in the integrand using the Taylor expansion (4.1) of order two $n=2$ around the point y_s^L with remainder in the integral form

$$
\begin{aligned}
g(y_s^L \pm \frac{1}{\sqrt{L}}) - g(y_s^L) &= g'(y_s^L)\left(\pm\frac{1}{\sqrt{L}}\right) + \frac{1}{2}g''(y_s^L)\left(\pm\frac{1}{\sqrt{L}}\right)^2 \\
&\quad + \frac{1}{2}\left(\pm\frac{1}{\sqrt{L}}\right)^3 \int_0^1 w^2 g'''\left(y_s^L + (\pm\frac{1}{\sqrt{L}})(1-w)\right)^2 dw.
\end{aligned} \tag{4.12}
$$

Replacing these expressions back in (4.11) we obtain that the generator takes the form

$$
\begin{aligned}
\mathcal{A}^L g(y_s^L) = \frac{1}{N}&\left(g'(y_s^L)\left(-\frac{1}{\sqrt{L}}\right) + \frac{1}{2}g''(y_s^L)\left(-\frac{1}{\sqrt{L}}\right)^2\right) L u_s^L \\
&+ \gamma(u_s^L)\left(g'(y_s^L)\left(\frac{1}{\sqrt{L}}\right) + \frac{1}{2}g''(y_s^L)\left(\frac{1}{\sqrt{L}}\right)^2\right) L(1-u_s^L) + R(\omega, g, L, s)\,.
\end{aligned}
$$

Where $R(\omega, g, L, s)$ is a function obtained by collecting all expressions involving the remainder expressions in the Taylor formulas (4.12). The random element $\omega \in \Omega$ is added to emphasize the random nature of the intermediate points in the remainder theorem. Collecting all terms involving g' and recalling that $H(u')=0$

$$
\begin{aligned}
\mathcal{A}^L g(y_s^L) &= g'(y_s^L)\Big[\sqrt{L}\left(H(u' + \frac{y_s^L}{\sqrt{L}}) - H(u')\right)\Big] \\
&+\frac{1}{2}g''(y_s^L)\left(H(u_s^L) + \frac{2}{N}u_s^L\right) + R(\omega, g, L, s)\,.
\end{aligned} \tag{4.13}
$$

Since the function g has compact support there exists a constant $c_3(g)$, depending on the function g (in fact, on g''') only, and not L or s, such that the part of the remainder appearing in (4.12) is bounded by

$$
\left|\frac{1}{2}\left(\pm\frac{1}{\sqrt{L}}\right)^3 \int_0^1 w^2 g'''\left(y_s^L + (\pm\frac{1}{\sqrt{L}})(1-w)\right)^2 dw\right| \le \frac{1}{6}\left(\frac{1}{\sqrt{L}}\right)^3 c_3(g). \tag{4.14}
$$

Now, bound (4.14) together with $u_t^L \in [0,1]$ imply that the function $R(\omega, g, L, s)$ can be estimated by

$$
|R(\omega, g, L, s)| \le \frac{1}{6}\left(\frac{1}{\sqrt{L}}\right)\left(H(u_s^L) + \frac{2}{N}u_s^L\right) c_3(g) \le \frac{1}{6}\frac{1}{\sqrt{L}}\left(1 + \frac{2}{N}\right) c_3(g). \tag{4.15}
$$

At this point, we take a close look at expression (4.13) and observe that the third term in the sum is bounded by (4.15) and the second term is bounded since g has compact support, while $H(u_s^L)$ and u_s^L belong to $[0,1]$. In the first term, the factor $\sqrt{L}$ would be too large unless $H(u')=0$. The key point of the approximation we study is exactly that it makes no semse at non-stationary points. Since u' is stationary by hypothesis, we develop further the function H by Taylor formula of order one, to obtain

$$H(u' + \frac{y_s^L}{\sqrt{L}}) = H(u') + \frac{y_s^L}{\sqrt{L}} \int_0^1 H'(u' + w\,\frac{y_s^L}{\sqrt{L}})dw. \tag{4.16}$$

Plugging back into (4.13) we obtain that the generator is

$$\mathcal{A}^L g(y_s^L) = g'(y_s^L) y_s^L \int_0^1 H'(u' + w\,\frac{y_s^L}{\sqrt{L}})dw + \frac{1}{2} g''(y_s^L)\left(H(u_s^L) + \frac{2}{N}u_s^L\right) + \, R\,(\omega, g, L, s)\,. \tag{4.17}$$

Since H' is continuous, the integral remainder in (4.16) can be bounded by a constant $c_1(H)$ depending on the function H'. Denote by $c_1(g)$, $c_2(g)$ uniform bounds for the function g' and g'' respectively. More precisely, we obtain the L^2 bound

$$\begin{aligned}
\mathbb{E}\left[\left(\mathcal{A}^L g(y_s^L)\right)^2\right] &\leq \mathbb{E}\left[3\left(\left(g'(y_s^L)c_1(H)y_s^L\right)^2 + \frac{1}{4}\left(g''(y_s^L)\left(H(u_s^L) + \frac{2}{N}u_s^L\right)\right)^2 + \,\left(R\left(\omega, g''', L, s\right)\right)^2\right)\right] \\
&\leq \mathbb{E}\left[3c_1^2(g)c_1^2(H)\left(y_s^L\right)^2 + 3\left(1 + \frac{2}{N}\right)^2\left(\frac{1}{4}c_2^2(g) + c_3^2(g)\frac{1}{L}\right)\right].
\end{aligned}$$

Since L and N can be taken bigger than 1 we obtain that,

$$\mathbb{E}\left[\left(\mathcal{A}^L g(y_s^L)\right)^2\right] \leq C_1 \mathbb{E}\left[\left(y_s^L\right)^2\right] + C_2$$

where $C_1 = 3c_1^2(g)c_1^2(H)$ and $C_2 = 27\left(\frac{1}{4}c_2^2(g) + c_3^2(g)\right)$. □

Proposition 3. *For every function $g \in C_c^3(\mathbb{R})$ and every $t \in [0, T]$, the quadratic variation of $M_t^{g,L}$ satisfies the bound*

$$\langle M^{g,L}\rangle_t \leq \left(M_1 + \frac{1}{\sqrt{L}}M_2\right)t \tag{4.18}$$

where M_1, M_2 are constants depending on g, H and their derivatives but not depending on $L > 0$ and $t \geq 0$.

Proof. We apply formula (3.3) and obtain that the quadratic variation can be written as

$$\begin{aligned}
\langle M^{g,L}\rangle_t = \int_0^t \Big[\frac{1}{N}\left(g(y_{s-}^L - \frac{1}{\sqrt{L}}) - g(y_{s-}^L)\right)^2 L\, u_t^L \\
+ \gamma(u_{s-}^L)\left(g(y_{s-}^L + \frac{1}{\sqrt{L}}) - g(y_{s-}^L)\right)^2 L(1 - u_t^L)\Big]\, ds.
\end{aligned} \tag{4.19}$$

We develop the differences in the integrand by Taylor's formula

$$g(y_s^L \pm \frac{1}{\sqrt{L}}) - g(y_s^L) = g'(y_s^L)\left(\pm\frac{1}{\sqrt{L}}\right) + r_{\pm}(\omega, g, L, s)$$

where $r_\pm(\omega, g, L, s)$ are the remainders of the Taylor expansion of order one in integral form

$$r_\pm(\omega, g, L, s) = \left(\pm\frac{1}{\sqrt{L}}\right)^2 \int_0^1 w\, g''\left(y_s^L + (\pm\frac{1}{\sqrt{L}})(1-w)\right) dw$$

These expressions are bounded uniformly in time t and ω by

$$|r_\pm(\omega, g, L, s)| \ \le\ \frac{1}{2}\left(\frac{1}{\sqrt{L}}\right)^2 c_2(g) \tag{4.20}$$

Replacing the Taylor expansion in (4.19) we obtain that, the quadratic variation of the martingale is

(4.21)

$$\begin{aligned}\langle M^{g,L}\rangle_t &= L\int_0^t \left(\frac{1}{N}\left(-\frac{1}{\sqrt{L}}g'(y_s^L) + r_-\right)^2 u_t^L \ + \gamma(u_{s-}^L)\left(\frac{1}{\sqrt{L}}g'(y_s^L) + r_+\right)^2 (1-u_t^L)\right) ds \\ &= \int_0^t \left(\left(g'(y_s^L)\right)^2\left(H(u_s^L) + \frac{2}{N}u_s^L\right) + \bar{R}\left(g''', L, s\right)\right) ds\end{aligned}$$

where $\bar{R}\left(g''', L, s\right)$ collects all terms involving the Taylor formula remainders. We do not need the exact form of $\bar{R}\left(g''', L, s\right)$. It is only important that

$$\bar{R}\left(g''', L, s\right) = O(\frac{1}{\sqrt{L}}),$$

more precisely, there is a constant M_2 such that $\bar{R}\left(g''', L, s\right) \le \frac{1}{\sqrt{L}}M_2$. Thus, we obtain that

$$0 \le \langle M^{g,L}\rangle_t \le \left(c_1^2(g)\left(1+\frac{2}{N}\right) + \left|\bar{R}\left(g''', L, s\right)\right|\right) t \le \left(c_1^2(g)\left(1+\frac{2}{N}\right) + \frac{1}{\sqrt{L}}M_2\right) t.$$

Hence, by denoting $M_1 = c_1^2(g)\left(1+\frac{2}{N}\right)$ we obtain

$$0 \le \langle M^{g,L}\rangle_t \le \left(M_1 + \frac{1}{\sqrt{L}}M_2\right) t.$$

□

4.2. **Modulus of Continuity.** With the same convention of notation as in the preceding sections $g(y) = f(\sqrt{L}(u-u'))$, the process satisfies the differential formula (3.1)

$$g(y_t^L) = g(y_0^L) + \int_0^t \mathcal{A}^L g(y_s^L) + M_t^{g,L}\,. \tag{4.22}$$

First, we show that $g(y_t^L)$, $L>0$ is a tight family by showing that the three terms on the right hand side are tight. Notice that only condition (ii) (3.5) is non-trivial when g has compact support.

Proposition 4. *The processes $d_t^{g,L} := \int_0^t \mathcal{A}^L g(y_s^L)\, ds$ is C-tight.*

Proof. Condition (i) in definition (1) is easily satisfied by $d_t^{g,L}$ since g has compact support. In order to prove condition (ii) for $d_t^{g,L}$ we let $\delta > 0$. On the interval $[0, T]$, we choose $0 \le t < t' \le T$ such that $t' - t < \delta$. For the process $d_t^{g,L}$ we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{\substack{0\le t<t'\le T\\ t'-t<\delta}} \left|d_{t'}^{g,L} - d_t^{g,L}\right| > \epsilon\right) &\le \frac{1}{\epsilon^2}\mathbb{E}\left(\sup_{\substack{0\le t<t'\le T\\ t'-t<\delta}} \left(\int_t^{t'} \mathcal{A}^L g(y_s^L)\, ds\right)^2\right) && \text{(Chebyshev)}\\
&\le \frac{1}{\epsilon^2}\mathbb{E}\left(\delta \sup_{\substack{0\le t<t'\le T\\ t'-t<\delta}} \int_t^{t'} \left(\mathcal{A}^L g(y_s^L)\right)^2 ds\right) && \text{(Cauchy-Schwarz)}\\
&\le \frac{\delta}{\epsilon^2}\mathbb{E}\left(\int_0^T \left(\mathcal{A}^L g(y_s^L)\right)^2 ds\right) && \text{(positivity of the integrand)}\\
&= \frac{\delta}{\epsilon^2}\int_0^T \mathbb{E}\left[\left(\mathcal{A}^L g(y_s^L)\right)^2\right] ds && \text{(Fubini)}\\
&\le \frac{3\delta}{\epsilon^2}\int_0^T \left(C_1 \mathbb{E}\left[\left(y_s^L\right)^2\right] + C_2\right) ds && \text{by (2)}\\
&\le \frac{3\delta}{\epsilon^2}\int_0^T \left[C_1 K_1 \exp\left(K_2\, s\right) + C_2\right] ds && \text{by Proposition 1}\\
&= \frac{3\delta}{\epsilon^2}\left[C_1 K_1 \frac{\exp\left(K_2\, T\right) - 1}{K_2} + C_2 T\right],
\end{aligned}$$

hence

$$\lim_{\delta\to 0}\limsup_{L\to\infty} \mathbb{P}\left(\sup_{\substack{0\le t<t'\le T\\ t'-t<\delta}} \left|d_{t'}^{g,L} - d_t^{g,L}\right| > \epsilon\right) = 0.$$

□

Proposition 5. *The process $M_t^{g,L}$ is C-tight.*

Proof. As before, condition (i) in definition (1) is easily satisfied by $M_t^{g,L}$ since g has compact support. Again, we let $\delta > 0$. On the interval $[0, T]$, we choose $0 \le t < t' \le T$ such that $t' - t < \delta$. For the process $M_t^{g,L}$ we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{\substack{0\le t<t'\le T\\ t'-t<\delta}} \left|M_{t'}^{g,L} - M_t^{g,L}\right| > \epsilon\right) &\le \frac{1}{\epsilon^2}\mathbb{E}\left(\left|M_{t'}^{g,L} - M_t^{g,L}\right|^2\right) && \text{(Doob's } L^2 \text{ maximal inequality)}\\
&= \frac{1}{\epsilon^2}\mathbb{E}\left(\left|\langle M^{g,L}\rangle_{t'} - \langle M^{g,L}\rangle_t\right|\right) && \text{by (3.2)}\\
&\le \frac{\delta}{\epsilon^2}\left(M_1 + \frac{1}{\sqrt{L}} M_2\right) && \text{by (3)}\,,
\end{aligned}$$

hence

$$\lim_{\delta\to 0}\limsup_{L\to\infty}\mathbb{P}\left(\sup_{\substack{0\le t<t'\le T\\ t'-t<\delta}}\left|M^{g,L}_{t'}-M^{g,L}_{t}\right|>\epsilon\right)=0.$$

□

Proposition 6. *The process y^L_t is C-tight.*

Proof. Condition (i) (3.4) is immediately satisfied from Proposition 1, eq. (4.4). Inspecting (4.22), we see that Propositions 4, 5 imply that $(g(y^L_t)$ is tight for $g\in C^3_c(\mathbb{R})$. The modulus of continuity is obtained by localization. □

4.3. **Solution of the Martingale Problem.** We know that any process (y_t) with probability law a limit point of the $C-tight$ family of processes $(y^L_t)_{L>0}$, has continuous paths. In addition, we would like to show it satisfies the martingale problem for $\mathcal{U}$ defined in (2.3). It is well known ([5]) that the problem is well posed, and the solution has the law of the solution to the stochastic differential equation

$$dy_t=-H'(u')y_t dt+\frac{2u'}{N}dB_t\,, \tag{4.23}$$

with initial value y_0 defined in (2.2). Here (B_t) is a standard Brownian motion.

Proposition 7. *There exist D_1, D_2 and D_3 independent of s and L (but dependent on T) such that*

$$\limsup_{L\to\infty}E\left[|\mathcal{A}^L g(y^L_s)-\mathcal{U}g(y^L_s)|\right]=0\,. \tag{4.24}$$

Proof. From (4.17) and (2.3) we have that

$$\begin{aligned}\mathcal{A}^L g(y^L_s)-\mathcal{U}g(y^L_s)=&\underbrace{g'(y^L_s)y^L_s\int_0^1\left(H'(u'+w\,\frac{y^L_s}{\sqrt{L}})-H'(u')\right)dw}_{(I)}\;+\\ &+\underbrace{\frac{1}{2}g''(y^L_s)\left(H(u^L_s)+\frac{2}{N}u^L_s-\frac{2}{N}u'\right)}_{(II)}\;+\\ &+\underbrace{R\left(\omega,g''',L,s\right).}_{(III)}\end{aligned}$$

The function γ, and thus H are of class C^1, and the argument appearing in H is $u'+(1-w)\frac{y^L_s}{\sqrt{L}}\in[0,1]$ because it is a point between u' and u^L_s. We shall prove that for $0\le s\le T$

$$\limsup_{L\to\infty}E\left[\left|y^L_s\int_0^1(H'(u'+w\,\frac{y^L_s}{\sqrt{L}})-H'(u'))\,dw\right|\right]=0\,. \tag{4.25}$$

Choose $M > 0$ arbitrary. We shall split the quantity in absolute value in two. The first will be I_1, over the event $\{|y_s^L| > M\}$ and I_2, over its complement. Pick $\epsilon > 0$. H' is continuous on $[0,1]$, thus uniformly continuous. There exists $\delta > 0$ such that $|u_1 - u_2| < \delta$ then $|H'(u_1) - H(u_2)| < \epsilon$. Take L so large that $|y_s^L/\sqrt{L}| \leq M/\sqrt{L} < \delta$. Then

$$E[I_1 + I_2|] \leq M\epsilon + 2(\sup_{u \in [0,1} |H'(u)|) \frac{E[(y_s^L)^2]}{M^2}$$

where we used Chebyshev's inequality for I_2. $E[(y_s^L)^2] \leq K_1 e^{K_2 T}$ from (4.4) in Proposition 1. Denote the supremum norm of H' by $c_1(H)$. We obtained that the bound for all L satisfying $M/\sqrt{L} < \delta$, i.e. $L > (M/\delta)^2$. This implies that $\limsup_{L\to\infty} E[I_1 + I_2|] \leq M\epsilon + Const(T)/M^2$ for arbitrary M, ϵ. Let $\epsilon \to 0$ and the $M \to \infty$, and we are done.

In expression (II), we write $u_t^L = u' + \frac{y_s^L}{\sqrt{L}}$ and expand the function H around the point u' using Taylor formula with remainder, in integral form, of order one. Thus, obtain that

$$H(u_s^L) = \frac{y_s^L}{\sqrt{L}} \int_0^1 H'\left(u' + (1-w)\frac{y_s^L}{\sqrt{L}}\right) dw$$

$$\begin{aligned}(II) &= \frac{1}{2} g''(y_s^L) \left(\frac{y_s^L}{\sqrt{L}} \int_0^1 H'\left(u' + (1-w)\frac{y_s^L}{\sqrt{L}}\right) dw + \frac{2}{N}(u' + \frac{y_s^L}{\sqrt{L}}) - \frac{2}{N} u' \right) \\ &= \frac{1}{2} g''(y_s^L) \frac{y_s^L}{\sqrt{L}} \left(\int_0^1 H'\left(u' + (1-w)\frac{y_s^L}{\sqrt{L}}\right) dw + \frac{2}{N} \right)\end{aligned}$$

Since H' is continuous, it is bounded, and let $c_1(H)$ be supremum norm on $[0,1]$ for H', and $c_2(g)$ a positive bound for $\frac{1}{2}g''$. Then

$$|(II)| \leq c_2(g) \frac{|y_s^L|}{\sqrt{L}} \left(c_1(H) + \frac{2}{N} \right) \tag{4.26}$$

Finally, (4.15) gives

$$|(III)| = \left| R\left(\omega, g^{(3)}, L, s\right) \right| \leq \frac{1}{\sqrt{L}} \left(1 + \frac{2}{N} \right) c_3(g). \tag{4.27}$$

Where $c_3(g)$ is a positive bound for g'''. Now bounds (4.25), (4.26) and (4.27) show that the difference

$$\begin{aligned}&|\mathcal{A}^L g(y_s^L) - \mathcal{U} g(y_s^L)| \leq c_1(g)(I_1 + I_2) \\ &+ \frac{1}{\sqrt{L}} \left(c_2(g) |y_s^L| \left(c_1(H) + \frac{2}{N} \right) + \left(1 + \frac{2}{N} \right) c_3(g) \right).\end{aligned} \tag{4.28}$$

Taking the expected value and letting $L \to \infty$ proves the proposition. ☐

Proposition 8. *Let $(y_t)_{t\geq 0}$ with probability law $\mathcal{L}$ be a limit point of the C-tight family $(y_t^L)_{t\geq 0}$ with probability laws $\mathcal{L}^L$, indexed by $L \geq 1$. Then $(y_t)_t$ has continuous paths almost* *surely and for any* $g \in C_c^3(\mathbb{R})$

$$g(y_t) - g(y_0) - \int_0^t \mathcal{U} g(y_s)\, ds \tag{4.29}$$

is a $(\mathcal{F}_t)$ *- martingale, where* $\mathcal{U}$ *is the Ornstein-Uhlenbeck generator in (2.3).*

Proof. Define $M_{s,t}^{g,\eta}$, $g \in C_c^3(\mathbb{R})$ the difference of expressions from (2.3) taken at two times $0 \le s \le t \le T$.

$$M_{s,t}^{g,\eta} = g(\eta_t) - g(\eta_s) - \int_s^t \mathcal{U} g(\eta_{s'})\, ds' \qquad \eta \in \mathcal{D}([0,T], \mathbb{R})\,. \tag{4.30}$$

Let $\psi(\eta_\cdot)$ a $\mathcal{F}_s$ - measurable function, equal to a finite product of continuous bounded functions applied at a finite number of times s', $0 \le s' \le s$. Define $\Psi : \mathcal{D}([0,T], \mathbb{R}) \to \mathbb{R}$

$$\Psi(\eta_\cdot) = M_{s,t}^{g,\eta} \psi(\eta_\cdot)\,. \tag{4.31}$$

Ψ is a bounded, continuous functional. Since g has compact support, the only term that appears not bounded is the drift term in $\mathcal{U}$. However, a closer look shows it is equal to $-ryg'(y)$. If $supp(g) \subset [-A, A]$ then this term is, for a fixed g, bounded by $|r| A \sup_{y \in [-A,A]} |g'(y)| < \infty$. It only remains to prove that, if (y_t), with probability law $\mathcal{L}$ is a limit point of the tight sequence of processes (y_t^L), with probability laws $\mathcal{L}^L$, then $E[\Psi(y_\cdot)] = 0$. This will show that $\mathcal{L}$, which is supported on the subspace of continuous paths, solves the martingale problem for $\mathcal{U}$. This reasoning is standard but theparticulars of the functional involved diffrs from model to model. See [11] for a similar proof.

We have

$$\begin{aligned}
E[\Psi(y_\cdot^L)] &= \mathbb{E}\left[\left(g(y_t^L) - g(y_s^L) - \int_s^t \mathcal{U} g(y_{s'}^L)\, ds'\right)\psi(y_\cdot^L)\right] \\
&= \mathbb{E}\left[\left(g(y_t^L) - g(y_s^L) - \int_s^t \mathcal{A}^L g(y_{s'}^L)\, ds' + \int_s^t (\mathcal{A}^L - \mathcal{U}) g(y_{s'}^L)\, ds'\right)\psi(y_\cdot^L)\right] \\
&= \mathbb{E}\left[(M_t^{g,L} - M_s^{g,L})\psi(y_\cdot^L)\right] + \mathbb{E}\left[\left(\int_s^t (\mathcal{A}^L - \mathcal{U}) g(y_{s'}^L)\, ds'\right)\psi(y_\cdot^L)\right] \\
&= \mathbb{E}\left[\left(\int_s^t (\mathcal{A}^L - \mathcal{U}) g(y_{s'}^L)\, ds'\right)\psi(y_\cdot^L)\right]
\end{aligned}$$

where the last equality follows from the fact that $M_t^{g,L}$ is a martingale. From (4.24) we have that the error term

$$\begin{aligned}
\left|E[\Psi(y_\cdot^L)]\right| &\le \left|\mathbb{E}\left[\left(\int_s^t (\mathcal{A}^L - \mathcal{U}) g(y_{s'}^L)\, ds'\right)\psi(y_\cdot^L)\right]\right| \\
&\le (\sup|\psi|) \int_s^t E[|(\mathcal{A}^L - \mathcal{U}) g(y_{s'}^L)|]\, ds'\,.
\end{aligned}$$

Proposition 7 says that the integrand, as a function of s' converges to zero as $L \to \infty$. We recall the bound on $\mathcal{A}^L g(y^L_s)$ from Proposition 2. A similar bound on $\mathcal{U}g(y^L_{s'})$ is immediate from Proposition 1 , eq. (4.4). Together, these allow to use dominated convergence to conclude that $\lim_{L\to\infty} E[\Psi(y^L_\cdot)] = 0$.

The proof of $C - tightness$ show that $(y^L_t) \Rightarrow y(\cdot)$ (converges in distribution) and (y_t) which has continuous paths almost surely. Portmanteau Theorem implies that

$$E[\Psi(y_\cdot)] = \lim_{L\to\infty} E[\Psi(y^L_\cdot)] = 0\,.$$

Keeping in mind the choice of ψ, as a bounded $\mathcal{F}_s$ - measurable function, we obtained that

$$M^g_t = g(y_t) - g(y_0) - \int_0^t \mathcal{U}g(y_s)\,ds$$

is a continuous $(\mathcal{F}_t)$ - martingale and the probability law $\mathcal{L}$ of the process (y_t) solves the martingale problem associated with the generator

$$\mathcal{U}g(x) = -rxg'(x) + \frac{1}{2}\sigma^2 g''(x)\,, \qquad r = -H'(u')\,, \quad \sigma^2 = \frac{2u'}{N}\,.$$

The parameters r and σ^2 are constant and the coefficients are Lipschitz. By Theorem 4.28 and Corollary 4.29 in Karatzas et al. [5] (and Stroock & Varadhan [9]), the martingale problem is well posed. The unque solution is the Ornstein-Uhlenbeck process described in Theorem 2. This concludes the proof that the sequence of processes (y^L_t) converge in distribution, as $L \to \infty$, to (y_t). □

Acknowledgement. Part of the work presented here is based on the first author's doctoral dissertation [3].

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