Horizons non–differentiable on a dense set

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September 17, 1997

Abstract

It is folklore knowledge amongst general relativists that horizons are well behaved, continuously differentiable hypersurfaces except perhaps on a negligible subset one needs not to bother with. We show that this is not the case, by constructing a Cauchy horizon, as well as a black hole event horizon, which contain no open subset on which they are differentiable.

1 Introduction

In various works where Cauchy horizons or event horizons occur it is assumed that those objects have a fair amount of differentiability, except perhaps for a well behaved lower dimensional subset thereof. A standard example is the proof of the area theorem for black holes [7, Proposition 9.2.7], where something close to C^2 differentiability "almost everywhere" of the event horizon seems to have been assumed. An extreme example is the rigidity theorem for black-holes [5, 7], where it is assumed that the event horizon is an analytic submanifold of space–time.

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Other examples include discussions of the structure of Cauchy horizons (e.g., [7, p. 295–298], [1, 4, 6, 10, 13]). It seems reasonable to raise the question, whether there are any reasons to expect that the horizons considered in the above quoted papers should have the degree of differentiability assumed there. While we do not have an answer to that question, in this paper we construct examples of horizons the degree of differentiability of which is certainly not compatible with the considerations in the papers previously quoted.

Our examples are admittedly very artificial and display various pathological features (*e.g.*, an incomplete null infinity). One would hope that the behavior described here can not occur, in the case of event horizons in "reasonable" asymptotically flat space–times (*e.g.*, in vacuum and maximal globally hyperbolic space–times), or for compact Cauchy horizons in spatially compact space–times. While this expectation might well turn out to be true, the examples here show that we have no *a priori* reasons to expect that to be correct, and some new insights are necessary in those cases to substantiate such hopes. It is, moreover, clear from our results below that the kind of behaviour we obtain is a *generic* property of horizons as considered here, in an appropriate $C^{0,1}$ topology on the collection of sets. We shall, however, not attempt to formalize this statement. We note that a criterion for stability of differentiability, for a rather special class of compact Cauchy horizons, has been given in [2].

Let us recall some facts about, say, Cauchy horizons in globally hyperbolic space–times (M, g). Let Ω be an open subset of a Cauchy surface $\Sigma \subset M$. Then [7, Prop. 6.3.1] [12, Lemma 3.17], $\mathcal{H}^+(\Omega)$ is a Lipschitz topological submanifold of M. It then follows from a theorem of Rademacher [3, Theorem 3.1.6] that $\mathcal{H}^+(\Omega)$ is differentiable almost everywhere. (Recall that in local coordinates x^μ in which x^0 is a time function $\mathcal{H}^+(\Omega)$ can be written as a graph $x^0 = f(x^i)$, and then the "almost everywhere" above refers to the Lebesgue measure $\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$.) This shows that there is some smoothing associated with Cauchy horizons: consider a set K such that ∂K is a nowhere differentiable curve. (Take, e.g., a Mandelbrot set in $\mathbb{R}^2$, considered as the Cauchy surface for the three-dimensional Minkowski space–time $\mathbb{R}^{2,1}$.) Even in such a "bad" situation, for almost all (in the sense of Lebesgue measure on $\mathbb{R}$) later-times t the boundary of $D^+(K)$ intersected with the plane $t = \mathrm{const}$ will be differentiable almost everywhere. It could be the case that some more smoothing occurs, so that perhaps $\mathcal{H}^+(K) \cap \{t = \mathrm{const}\}$ will have nicer differentiability properties than what follows from the above considerations. The example we construct here shows that this does not happen. More precisely, we show the following:

Theorem 1.1 *There exists a connected set $K \subset \mathbb{R}^2 = \{t = 0\} \subset \mathbb{R}^{2,1}$, where $\mathbb{R}^{2,1}$ is the $2+1$ dimensional Minkowski space–time, with the following properties:*

1. *The boundary $\partial K = \bar{K} \setminus \mathrm{int}\, K$ of K is a connected, compact, Lipschitz topological submanifold of $\mathbb{R}^2$. K is the complement of a compact set of $\mathbb{R}^2$.*

2. *There exists no open set* $\Omega \subset \mathbb{R}^{2,1}$ *such that* $\Omega \cap \mathcal{H}^+(K) \cap \{0 < t < 1\}$ *is a differentiable submanifold of* $\mathbb{R}^{2,1}$.

Let us show that Theorem 1.1 implies the existence of "nowhere" differentiable (in the sense of point 2 of Theorem 1.1) Cauchy horizons $\mathcal{H}^+(\Sigma)$ in stably causal asymptotically flat space–times, with well behaved space–like surfaces Σ. (Our example satisfies even the vacuum Einstein equations, as the space–time is flat.) To do that, consider the complement $\mathcal{C}K$ of K in $\mathbb{R}^2$, and consider the set $J^+(\mathcal{C}K)$. As $\mathcal{C}K$ is compact, it follows from global hyperbolicity of $\mathbb{R}^{2,1}$ that $K' \equiv J^+(\mathcal{C}K) \cap \{t = 1\}$ is compact, hence there exists R such that K' is a subset of the open ball $B(0, R)$ of radius R centered at the origin of $\mathbb{R}^2$. Let $\mathcal{N} = J^+(\{1\} \times \overline{B(0, R)})$ and set $\hat{M} = \mathbb{R}^{2,1} \setminus (\mathcal{N} \cup \mathcal{C}K)$, equipped with the obvious flat metric coming from $\mathbb{R}^{2,1}$ (*cf.* Figure 1). Consider the space–like surface $\Sigma = \{t = -1\} \subset \hat{M}$. It is easily

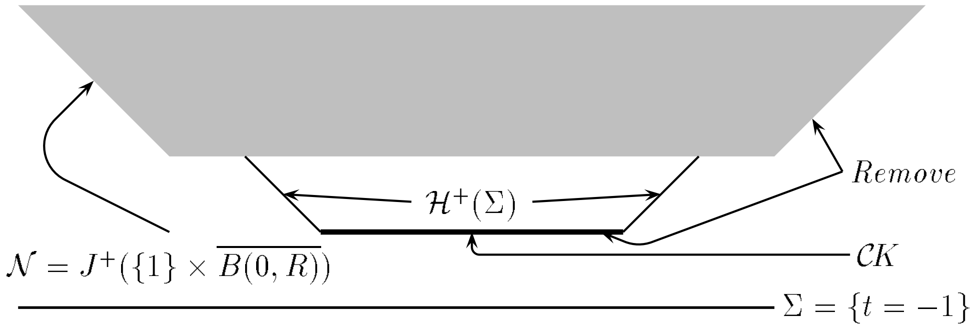

Figure 1: The space–time $\hat{M}$.

seen that the Cauchy horizon $\mathcal{H}^+(\Sigma; \hat{M})$ of Σ in $\hat{M}$ coincides with the intersection of the Cauchy horizon $\mathcal{H}^+(K)$ in $\mathbb{R}^{2,1}$ with the "slab" $\{0 < t < 1\} \subset \mathbb{R}^{2,1}$. Hence there exists no open subset of $\mathcal{H}^+(\Sigma; \hat{M})$ such that this subset is a differentiable submanifold of $\hat{M}$. We note that the construction guarantees global hyperbolicity of M.

Let us show that Theorem 1.1 also implies the existence of black hole space–times with "nowhere" differentiable event horizons ("nowhere" in the sense of point 2 of Theorem 1.1). The construction should be clear from Figure 2, and can be formally described as follows: Construct first a compact set $K' \subset \mathbb{R}^3 = \{t = 1\} \subset \mathbb{R}^{2,1}$ with the property that the past Cauchy horizon $\partial\mathcal{D}^-(K')$ of K' is "nowhere differentiable" for $0 \leq t \leq 1$. This is easily achieved using the same construction as below, by adding *exterior* ripples to the disc $B(0, 2)$ of radius 2 centered at 0. Consider the space–time $\tilde{M}$ obtained by removing from the three–dimensional Minkowski space–time the set K' together with $J^-(\overline{\mathcal{D}^-(K') \cap \{t = 0\}})$. Clearly those points in $\tilde{M}$ which are in $\overline{\mathcal{D}^-(K')}$ cannot be seen from the usual future null infinity $\mathscr{I}^+(\tilde{M})$ of $\tilde{M}$ (which coincides with that of $\mathbb{R}^{2,1}$), so that $\overline{\mathcal{D}^-(K'; \tilde{M})}$ lies in the black hole region of $\tilde{M}$. It is also easily seen that the black hole horizon $\mathcal{B}$ in $\tilde{M}$ coincides with $\mathcal{H}^-(K') \cap \{0 < t < 1\}$.

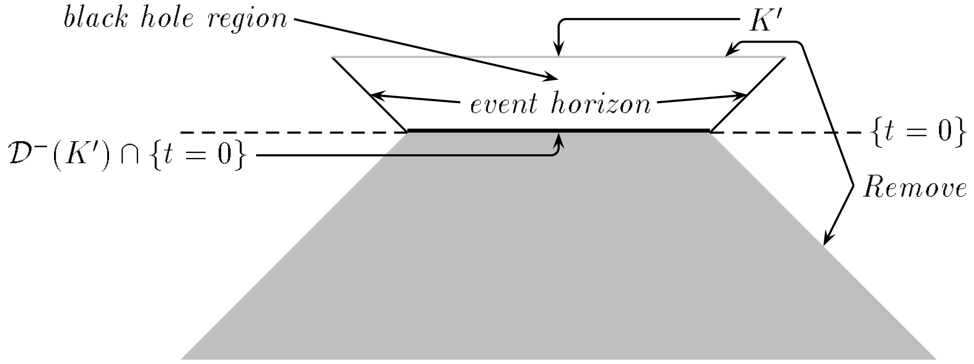

Figure 2: A flat space–time with a black hole region.

It follows again that there exists no open subset of $\mathcal{B}$ which is a differentiable submanifold of $\tilde{M}$.

We note that the black–hole space–time $\tilde{M}$ constructed in the last paragarph has a complete $\mathscr{I}^+$ and is stably causal, but is not globally hyperbolic, moreover the domain of outer communications is not globally hyperbolic either. We could obtain a globally hyperbolic space–time *and* a globally hyperbolic domain of outer communications *and* a flat metric by further removing the set $J^+(K') \cap \{t > 1\}$ from $\tilde{M}$, but this would lead to an incomplete $\mathscr{I}^+$.A way of improving global causal properties of the space–time so constructed is to consider the manifold $\tilde{M}$ defined as the interior of $J^-(K'; \mathbb{R}^{1+2})$ with the metric being the flat metric multiplied by a conformal factor which is strictly positive on the interior of $J^-(K'; \mathbb{R}^{1+2})$ and which vanishes on its boundary, in a way such that $\partial J^-(K'; \mathbb{R}^{1+2}) \setminus K'$ becomes the null conformal infinity. Such a construction does not, however, guarantee that the Ricci tensor of the new metric will satisfy any positivity conditions.

This paper is organized as follows. In Section 5 we present the construction of K. While all the assertions made there can be seen, with some effort, by inspection of the pictures, it seems difficult to give a rigorous proof based on that. For that reason we develop, in the following sections, a framework which allows us to handle various problems one needs to address in the construction presented in Section 5. We believe that the results obtained in those sections are of independent interest, and shed light on various properties of Cauchy horizons. In particular the notion of the distribution of semi–tangents introduced below seems to be rather useful in understanding the structure of horizons. In Section 2 we analyze continuity properties of the map which to a set $K \subset \Sigma$, where Σ is a Cauchy surface in a globally hyperbolic space–time, assigns its Cauchy horizon $\mathcal{H}^+(K)$, in various topologies. Some related results can be found in [1]. In Section 3 we introduce the notion of the distribution of semi-tangents to a Cauchy horizon, which plays a crucial role in our analysis. We also study some properties of that distribution. In Section 4 we analyze continuity properties of the map which to a set $K \subset \Sigma$ assigns its distribution of semi-tangents $\mathcal{N}^+(K)$. We show that this map is continuous both in a "clustering" topology and in a Hausdorff-distance topology. In the concluding section we briefly discuss how our construction generalizes to higher dimensional Minkowski space–times, or, in fact, to any globally hyperbolic space–times.

2 Convergence Properties of Cauchy Horizons

In this section, and subsequent sections, we establish some convergence properties of Cauchy horizons. For this purpose we make use of the notion of Hausdorff convergence.

Let X be a metric space with distance function d. For any subset $A \subset X$, let $U_\varepsilon(A)$ be the ε-neighbourhood of A,

$$U_\varepsilon(A) = \{x \in X :\ d(x, A) < \varepsilon\}$$

where

$$d(x, A) = \inf_{y \in A} d(x, y).$$

For subsets $A,\ B \in X$, the Hausdorff distance between A and B, denoted $D(A, B)$, is defined as follows,

$$D(A, B) = \inf\{\varepsilon > 0 :\ A \subset U_\varepsilon(B) \text{ and } B \subset U_\varepsilon(A)\}.$$

Then, a sequence of subsets $A_n \subset X$ is said to (Hausdorff) converge to a subset $A \subset X$ provided $D(A_n, A) \to 0$ as $n \to \infty$. Note that $D(A_n, A) \to 0$ if and only if $\sup_{x \in A_n} d(x, A) \to 0$ and $\sup_{y \in A} d(y, A_n) \to 0$.

There is an auxilliary convergence notion that turns out to be useful in our study, as well. Let $\{A_n\}$ be a sequence of subsets of X. Then the *cluster limit* of A_n, denoted $\text{c}-\lim A_n$, is defined as follows,

$$\begin{aligned} \text{c}-\lim A_n = \{x \in X :\quad & \exists \text{ sequence } x_j \in A_{n_j} \text{ such that} \\ & n_j \to \infty \text{ and} \lim\nolimits_{j \to \infty} x_j = x\}. \end{aligned}$$

Proposition 2.1 $\text{c}-\lim A_n$ *is closed.*

PROOF: Assume $x_j \in \text{c}-\lim A_n$ and $x_j \to x$. Now, $x_j \in \text{c}-\lim A_n$ implies $\exists x_{j_k} \to x_j$, $x_{j_k} \in A_{n(j,k)}$, where, for each j, $n(j, k) \to \infty$ as $k \to \infty$.

For each j there exists $K(j)$ such that

$$d(x_{j_k}, x_j) < \frac{1}{j} \text{ for all } k \geq K(j).$$

Hence, we can construct a sequence $\{K(j)\}$ so that $n(j, K(j)) \to \infty$ as $K(j) \to \infty$ and

$$d(y_j, x_j) < \frac{1}{j},$$

where $y_j = x_{n(j,K(j))} \in A_{n(j,K(j))}$. Hence, $\lim_{j\to\infty} y_j = \lim_{j\to\infty} x_j = x$. Thus, $x \in$ c$-\lim A_n$. □

Under fairly general circumstances, Hausdorff convergence implies cluster limit convergence.

Proposition 2.2 *If $A \subset X$ is closed and if $D(A_n, A) \to 0$, then* c$-\lim A_n = A$.

PROOF: c$-\lim A_n \subset A$: Let $x \in$ c$-\lim A_n$. Then $\exists x_j \in A_{n_j}$ such that $x = \lim x_j$. We have that $\sup_{y\in A_n} d(y, A) \to 0$. Hence, $d(x_j, A) \to 0$. Since $z \to d(z, A)$ is continuous, $d(x, A) = \lim_{j\to\infty} d(x_j, A) = 0$. Thus, since A is closed, $x \in A$.

$A \subset$ c$-\lim A_n$: $D(A_n, A) \to 0$ implies that $\sup_{y\in A} d(y, A_n) \to 0$. Hence for any $x \in A$, $d(x, A_n) \to 0$. It follows that there exists a sequence $y_n \in A_n$ such that $d(x, y_n) \to 0$, i.e., $x = \lim_{n\to\infty} y_n \in$ c$-\lim A_n$. □

We will have occasion to make use of the following simple fact concerning Hausdorff distance.

Proposition 2.3 *Let (X, d) be a locally compact metric space such that all distance balls $\{y \in X : d(x, y) < r\}$ are connected (e.g. a connected Riemannian manifold). Suppose we are given subsets A_n, $A \subset X$ such that $D(A_n, A) \to 0$ and $D(\partial A_n, \partial A) \to 0$. Let C be compact subset of X.*

(a) *If $C \subset \operatorname{int} A$ then $C \subset \operatorname{int} A_n$ for all n sufficiently large.*

(b) *If $C \cap \overline{A} = \emptyset$ then $C \cap \overline{A_n} = \emptyset$ for all n sufficiently large.*

PROOF: We prove only (a), as the proof of (b) is similar. To prove (a), it is sufficient to consider the case in which C is a closed metric ball, $C = \{x : d(x_0, x) \leq r\}$. The proof for general C follows by taking an appropriate finite cover of C. Hence, in particular, we may assume C is connected.

Since C is compact and contained in the interior of A, we have that $\inf_{x\in C} d(x, \partial A) = \delta > 0$. The convergence $D(\partial A_n, \partial A) \to 0$ then implies that $\inf_{x\in C} d(x, \partial A_n) \geq \delta/2$ for all n sufficiently large. Since we are assuming C is connected, it follows that for all n sufficiently large, $C \subset \operatorname{int} A_n$ or $C \subset X \setminus A_n$. If

$C \subset X \setminus A_n$ for some n then $\inf_{x \in C} d(x, A_n) \geq \delta/2$. Since $D(A_n, A) \to 0$, this cannot happen for infinitely many n. Hence, $C \subset \operatorname{int} A_n$ for all n sufficiently large. $\square$

We now pass to the space–time setting. Let (M, g) be a globally hyperbolic space–time, and Σ be a smooth space–like Cauchy surface for M. We wish to emphasize that we shall assume global hyperbolicity throughout. Fix a future directed time–like vector field T on M. Introduce a complete Riemannian metric h on M. Let d be the associated distance function, and let D be the associated Hausdorff distance on subsets of M. Normalize T so that T has unit length with respect to h.

Each integral curve of T meets Σ once and only once. Let $\Pi : M \to \Sigma$ be the projection onto Σ along the integral curves of T, i.e., $\Pi(p)$ is the point in Σ where the integral curve of T through p meets Σ. Let $\Psi : M \to \mathbb{R} \times \Sigma$ be the diffeomorphism determined by the integral curves of T, i.e., $\Psi(p) = (t, x)$, where $x = \Pi(p)$ and $t = t(p)$ is the signed h-length of the integral curve of T from x to p. When convenient, we will surpress the map Ψ and simply identify M with $\mathbb{R} \times \Sigma$. Thus, for example, by the *graph* of a function $u : \Sigma \to \mathbb{R}$ we mean the following subset of M,

$$\operatorname{graph} u = \{(u(x), x) \in M : \ x \in \Sigma\}.$$

We now consider the Cauchy horizons of certain subsets of Σ. For any subset $A \subset \Sigma$, we let $\partial A = \overline{A} \setminus \operatorname{int} A$ denote the boundary of A in Σ. Let $\hat{\mathcal{X}}$ be the following collection of subsets of Σ,

$$\begin{aligned}\hat{\mathcal{X}} = \{A \subset \Sigma : \ & A \text{ is the closure of an open set in } \Sigma,\ A \text{ is connected} \\ & \text{and } \partial A \text{ is a topological submanifold of co-dimension one in } \Sigma\}.\end{aligned}$$

For any $A \in \hat{\mathcal{X}}$, the future Cauchy horizon $\mathcal{H}^+(A)$ is an achronal Lipschitz hypersurface in M such that $\operatorname{edge} \mathcal{H}^+(A) = \operatorname{edge} A = \partial A$ [12, Propositions 5.8 and 5.11]. Each integral curve of T meets $\mathcal{H}^+(A)$ at most once. Henceforth it will be convenient to define a slightly more restricted class of sets $\mathcal{X}$ by

$$\begin{aligned}\mathcal{X} = \{A \in \hat{\mathcal{X}} : \quad & \text{each integral curve of } T \\ & \qquad \text{that meets } A \text{ also meets } \mathcal{H}^+(A)\}.\end{aligned} \tag{2.1}$$

In this case $\mathcal{H}^+(A)$ may be viewed as a graph over A. More precisely, there exists a locally Lipschitz function $u : A \to \mathbb{R}$ such that,

$$\mathcal{H}^+(A) = \{(u(x), x) \in M : \ x \in A\}.$$

For the proof of the following lemma, let Σ_A denote the achronal surface obtained by "gluing together" $\mathcal{H}^+(A)$ and $\Sigma \setminus A$ along ∂A. Then by setting $u(x) = 0$ for all $x \in \Sigma \setminus A$, we obtain a locally Lipschitz function $u : \Sigma \to \mathbb{R}$ such that,

$$\Sigma_A = \{(u(x), x) \in M : \ x \in \Sigma\}.$$

Lemma 2.4 *Let A_n, $A \in \mathcal{X}$ be such that $D(A_n, A) \to 0$ and $D(\partial A_n, \partial A) \to 0$. Then for each compact set $C \subset A$,*

$$D(\mathcal{H}^+(A_n) \cap \Pi^{-1}(C), \mathcal{H}^+(A) \cap \Pi^{-1}(C)) \to 0.$$

PROOF: There exist locally Lipschitz functions $u_n : \Sigma \to \mathbb{R}$, $n = 1, 2, \ldots$, and $u : \Sigma \to \mathbb{R}$ such that $\Sigma_{A_n} = \operatorname{graph} u_n$ and $\Sigma_A = \operatorname{graph} u$. To prove the lemma, it is sufficient to show that $u_n \to u$ pointwise, for then $u_n \to u$ uniformly on C, and the lemma easily follows.

First suppose $x \in \Sigma \setminus A$, so that $u(x) = 0$. Then, by Proposition 2.3, $x \in \Sigma \setminus A_n$ for all n sufficiently large, and hence $u_n(x) = 0$ for all such n. Thus, $\lim_{n\to\infty} u_n(x) = 0 = u(x)$.

Now suppose $x \in \operatorname{int} A$, so that $(u(x), x) \in \mathcal{H}^+(A) \setminus \operatorname{edge} A$ and $u(x) > 0$. For any $\varepsilon > 0$ sufficiently small, $q = (u(x) - \varepsilon, x) \in \operatorname{int} D^+(A)$. Hence $J^-(q) \cap \Sigma$ is compact and contained in $\operatorname{int} A$. Thus, by Proposition 2.3, $J^-(q) \cap \Sigma$ is contained in A_n for all n sufficiently large. It follows that $q \in D^+(A_n)$ for all such n. Hence, $(u(x) - \varepsilon, x)$ lies to the past of $(u_n(x), x) \in \mathcal{H}^+(A_n)$ along the integral curve of T through x. This implies that $u(x) - \varepsilon \le u_n(x)$. A similar argument shows that $u_n(x) \le u(x) + \varepsilon$. Combining these inequalities we obtain $|u(x) - u_n(x)| \le \varepsilon$ for all n sufficiently large. Hence, $\lim_{n\to\infty} u_n(x) = u(x)$.

Finally, suppose $x \in \partial A$. Since each hypersurface Σ_{A_n} is achronal, there exists a neighbourhood W of x in Σ and a uniform Lipschitz constant K such that for all n and all x_1, $x_2 \in W$,

$$|u_n(x_2) - u_n(x_1)| \le K\rho(x_1, x_2), \tag{2.2}$$

where ρ is the distance function in Σ determined by the induced metric on Σ. The condition, $D(\partial A_n, \partial A) \to 0$ implies that there exists a sequence of points $y_n \in \partial A_n$ such that $y_n \to x$. Then (2.2) implies,

$$|u_n(x)| = |u_n(x) - u_n(y_n)| \le K\rho(y_n, x).$$

Since $\rho(y_n, x) \to 0$, it follows that $\lim_{n\to\infty} u_n(x) = 0 = u(x)$. □

Remarks. We note that every subset B of $\mathcal{H}^+(A)$ is of the form $\mathcal{H}^+(A) \cap \Pi^{-1}(C)$, where $C = \Pi(B)$. In particular, if B is compact, C is compact. We note also that Lemma 2.4 remains valid if C is merely assumed to be a subset of A with compact closure.

The following corollary is an immediate consequence of Proposition 2.2 and Lemma 2.4.

Corollary 2.5 *Consider A_n, $A \in \mathcal{X}$. Let $C \subset A$ be compact. If $D(A_n, A) \to 0$ and $D(\partial A_n, \partial A) \to 0$ then*

$$\text{c}-\lim[\mathcal{H}^+(A_n) \cap \Pi^{-1}(C)] = \mathcal{H}^+(A) \cap \Pi^{-1}(C).$$

The next lemma to be presented, which we refer to as the localization lemma, shows that, under suitable circumstances, "localized" changes in the boundary ∂A produce only "localized" changes in $\mathcal{H}^+(A)$. The proof is a consequence of the following simple fact which will arise in other situations, as well. We use the convention that$\mathcal{D}^+(A)$, for an achronal set A, is the collection of all points p such that every past inextendible *time–like* curve through p intersects A.

Proposition 2.6 *Let A be a closed achronal subset of a space–time M. Let η be a future directed causal curve from $p \in \operatorname{edge} A$ to $q \in \mathcal{D}^+(A) \setminus \operatorname{edge} A$. Then η is contained in $\mathcal{H}^+(A)$ and hence is a null geodesic generator (or segment thereof) of $\mathcal{H}^+(A)$.*

PROOF: $q \in \mathcal{D}^+(A)$ forces η to be contained in $\mathcal{D}^+(A)$. If η were to enter the interior of $\mathcal{D}^+(A)$ then p would have to be a point in $A \setminus \operatorname{edge} A$ (see, e.g. [12, Prop. 5.16, p. 45]). It follows that $\eta \subset \mathcal{H}^+(A)$. Moreover, to avoid an achronality violation of $\mathcal{H}^+(A)$, η must be a null geodesic segment. □

Lemma 2.7 (The Localization Lemma) *Consider A, $\hat{A} \in \mathcal{X}$, with $A \subset \hat{A}$. For any subset $\partial_0 A$ of ∂A, let*

$$\mathcal{H}^+_0(A) = \{x \in \mathcal{H}^+(A) : \textit{ x lies on a null geodesic generator with past end point on } \partial_0 A\}.$$

If $\partial_0 A \subset \partial \hat{A}$ then $\mathcal{H}^+_0(A) \subset \mathcal{H}^+(\hat{A})$.

PROOF: Let $y \in \mathcal{H}^+_0(A)$. If $y \in \partial_0 A$, there is nothing to show. Hence, we may assume $y \in \mathcal{H}^+_0(A) \setminus \partial A$. Let η be a null geodesic generator of $\mathcal{H}^+(A)$ from $x \in \partial_0 A$ to y. Since $A \subset \hat{A}$, $\mathcal{D}^+(A) \subset \mathcal{D}^+(\hat{A})$. Hence, η is a null geodesic segment from $x \in \partial \hat{A} = \operatorname{edge} \hat{A}$ to $y \in \mathcal{D}^+(\hat{A})$. By Proposition 2.6, η is a null geodesic generator of $\mathcal{H}^+(\hat{A})$, and hence $y \in \mathcal{H}^+(\hat{A})$. □

3 Semi-tangents

We introduce the notion of semi-tangents. Let $A \in \mathcal{X}$. A semi-tangent of $\mathcal{H}^+(A)$ is a vector $X \in TM$ based at a point $p \in \mathcal{H}^+(A) \setminus \partial A$, which is tangent to a null generator η of $\mathcal{H}^+(A)$, is past-directed, and is normalized so that $h(X, X) = 1$[1].

Let $\mathcal{N}^+{}_p(A)$ denote the collection of semi-tangents of $\mathcal{H}^+(A)$ based at p, and let $\mathcal{N}^+(A)$ denote the collection of all semi-tangents of $\mathcal{H}^+(A)$. Note that $\mathcal{N}^+(A) \subset UM$, where UM is the unit tangent bundle of M with respect to the

[1]A more elegant and essentially equivalent approach would be *not* to impose the condition $h(X, X) = 1$. That would avoid the usage of an auxiliary Riemannian metric h. The current definition is more convenient for technical reasons, as it fixes the parametrisation of the null geodesics under consideration.

Riemannian metric h. If we let $M_A = M \setminus \partial A$, then M_A is an open submanifold of M and UM_A is an open sub–bundle of UM. Since, in the definition of $\mathcal{N}^+(A)$, vectors based at ∂A are excluded, we have that $\mathcal{N}^+(A) \subset UM_A$.

Lemma 3.1 *$\mathcal{N}^+(A)$ is closed in UM_A.*

PROOF: Let $X_n \in \mathcal{N}^+(A)$ such that $X_n \to X$ in UM_A. Let X_n be based at the point $p_n \in \mathcal{H}^+(A)$ and X be based at p. Since $p_n \to p$, $p \in \mathcal{H}^+(A) \setminus \partial A$. Let η_n be the past directed null generator of $\mathcal{H}^+(A)$ starting at p_n with initial tangent X_n. Let $q_n \in \partial A$ be the past end point of η_n. The η_n's converge, as geodesics, to a past directed null geodesic η which starts at p with initial tangent X. η will meet Σ at a point $q \in \mathcal{D}^+(A)$. Convergence properties of geodesics guarantee that $q_n \to q$. Hence, $q \in \partial A$. Thus η is a null geodesic segment from $p \in \mathcal{H}^+(A)$ to $q \in \partial A$. By Proposition 2.6, this forces η to be a null geodesic generator of $\mathcal{H}^+(A)$. Thus, $X \in \mathcal{N}^+(A)$. □

We now establish some connections between the distribution of semi-tangents and the regularity of Cauchy horizons. Lemma 3.1 may be used to show that the distribution of semi-tangents is continuous at *interior* points (non-end points) of null generators. For $A \in \mathcal{X}$, consider the set,

$$S = \{x \in \mathcal{H}^+(A) : \ x \text{ is an interior point of a null generator of } \mathcal{H}^+(A)\},$$

endowed with the subspace topology. We note the following well known result:

Lemma 3.2 *At each point $p \in S$ there exists a* unique *semi-tangent $X(p) \in \mathcal{N}^+(A)$.*

PROOF: Indeed, if this were not the case, there would be one null generator meeting another at an interior point. Then, traveling around the corner at which they meet would produce an achronality violation. □

Lemma 3.2 shows that there is a well-defined map $\Psi : S \to \mathcal{N}^+(A)$ defined by $\Psi(p) = X(p)$. If we endow $\mathcal{N}^+(A) \subset TM$ with the subspace topology, we obtain:

Proposition 3.3 *$\Psi : S \to \mathcal{N}^+(A)$ is continuous.*

PROOF: Given $p \in S$, consider any sequence $\{p_n\} \subset S$ such that $p_n \to p$. Let$X_n = \Psi(p_n)$ and $X = \Psi(p)$. We want to show that $X_n \to X$. Indeed, since the X_n's are h-unit vectors, there exists a subsequence X_n, that converges to an h-unit vector Y. By Lemma 3.1, Y is a semi-tangent based at the interior point p. But since semi-tangents at interior points are unique, we must have $Y = X$, which had to be established. □

The interest in the distribution of semi-tangents lies in the fact, that it encodes most — if not all — information about differentiability of $\mathcal{H}^+(A)$. Indeed, at any point of $\mathcal{H}^+(A)$ where there is more than one semi-tangent, the Cauchy horizon must fail to be differentiable.

Proposition 3.4 *Consider the Cauchy horizon $\mathcal{H}^+(A)$, $A \in \mathcal{X}$. If there exist two distinct semi-tangents at $p \in \mathcal{H}^+(A) \setminus \partial A$ then $\mathcal{H}^+(A)$ is not differentiable at p.*

The heuristic idea is clear. If $\mathcal{H}^+(A)$ were differentiable at p it would have a tangent hyper-plane Π_p at p. Since $\mathcal{H}^+(A)$ is achronal and generated by null geodesics, Π_p should be a null hyper-plane. On the other hand, Π_p should contain the two distinct semi-tangents at p, forcing Π_p to be a time–like hyper plane. The proof presented below carries this argument out rigorously.

PROOF: Suppose to the contrary that $\mathcal{H}^+(A)$ is differentiable at p. Introduce geodesic normal coordinates $(x^0, x^1, \ldots, x^n) = (x^0, \vec{x})$ centered at p, defined in a neighborhood $\mathcal{U}$ of p such that $\left.\frac{\partial}{\partial x^0}\right|_p$ is future pointing time–like. Choose $\mathcal{U}$ sufficiently small so that $\frac{\partial}{\partial x^0}$ is time–like on $\mathcal{U}$. Any geodesic γ through p, when expressed in these coordinates takes the form,

$$\gamma : \quad x^\mu(s) = \lambda^\mu s, \quad \lambda^\mu \in \mathbb{R}, \quad \mu = 0, \ldots, n.$$

γ is time–like, space–like, null if and only if $-|\lambda^0|^2 + |\vec{\lambda}|^2 < 0, > 0, = 0$. It follows that the future (past) null cone at p is described by the graph $x^0 = |\vec{x}|$ ($x^0 = -|\vec{x}|$).

Since $\mathcal{H}^+(A)$ is achronal, it can be expressed in $\mathcal{U}$ as a graph over the slice $\mathcal{V} = \{x^0 = 0\}$,

$$\mathcal{H}^+(A) \cap \mathcal{U} = \{ \quad x^0 = f(\vec{x}), \quad \vec{x} \in \mathcal{V}\} \ .$$

$\mathcal{H}^+(A)$ is differentiable at p if and only if the function $f(\vec{x})$ is differentiable at $\vec{0}$, i.e. iff there exists a vector $\vec{C} = (C^1, \ldots, C^n)$ such that,

$$\lim_{\vec{x} \to 0} \frac{f(\vec{x}) - \vec{C} \cdot \vec{x}}{|\vec{x}|} = 0 \ . \tag{3.1}$$

Claim. $|\vec{C}| \leq 1$.

Otherwise, as we show, the achronality of $\mathcal{H}^+(A)$ is violated. Suppose, then, that $|\vec{C}| > 1$. By (3.1), with $\vec{x} = t\vec{C}$, we have that for any $\varepsilon > 0$ $\exists$ $t_0 > 0$ such that for all $t \in (0, t_0)$,

$$\frac{|f(t\vec{C}) - |\vec{C}|^2 t|}{|\vec{C}|t} < \varepsilon \ ,$$

which implies

$$f(t\vec{C}) > t|\vec{C}|(|\vec{C}| - \varepsilon) > t|\vec{C}|,$$

provided ε is sufficiently small. But this implies that $\mathcal{H}^+(A)$ enters into the future null cone at p, which violates the achronality of $\mathcal{H}^+(A)$.

Let $\overline{A} = (A^0, \vec{A})$ and $\overline{B} = (B^0, \vec{B})$ be any two semi-tangents at p, scaled so that $|\vec{A}| = |\vec{B}| = 1$. Since $\overline{A}$ and $\overline{B}$ are past directed null, $A^0 = B^0 = -1$. Then $s \to (sA^0, s\vec{A})$ and $s \to (sB^0, s\vec{B})$ are past directed null geodesic generators of $\mathcal{H}^+(A)$. It follows that $f(s\vec{A}) = sA^0 = -s$, and $f(s\vec{B}) = -s$. Setting $\vec{x} = s\vec{A}$ in (3.1) gives,

$$\lim_{s\to 0} \frac{s + s\vec{C}\cdot\vec{A}}{s} = 0$$

which implies that

$$-1 = \vec{C}\cdot\vec{A} = |\vec{C}|\cos\theta.$$

Hence $\cos\theta = -\frac{1}{|\vec{C}|} \le -1$, forcing $\theta = \pi$ and $|\vec{C}| = 1$. Thus, $\vec{A} = -\vec{C}$. But the same argument also shows that $\vec{B} = -\vec{C}$. Hence, $\vec{A} = \vec{B}$, which contradicts the assumption that there are distinct semi-tangents at p. $\Box$

Proposition 3.4 shows what happens at those points at which $\mathcal{N}^+(A)$ consists of more than one element. We do not know what happens at all points at which $\mathcal{N}^+(A)$ contains only one vector (and it would be of interest to settle that question).[2] However, when those points are interior points of generators, we have the following result.

Proposition 3.5 *At interior points of null generators the Cauchy horizon $\mathcal{H}^+(A)$ is differentiable.*

Related results concerning the regularity of Cauchy horizons (or, more generally, achronal sets) at interior points of generators have been obtained by I. Major [8] and K. Newman [9].

PROOF: Let γ be a null generator of $\mathcal{H}^+(A)$ and let p be any of its interior points. Let $\mathcal{U}$, x^μ, f, etc., be as in the proof of Proposition 3.4. Note that any point $p = (t, \vec{x}) \in \mathcal{U} \cap \mathcal{D}^+(A)$ satisfies $t \le f(\vec{x})$. Similarly if $p = (t, \vec{x}) \in \mathcal{U}$ is not in $\mathcal{D}^+(A)$ then $t \ge f(\vec{x})$. We can find $q^+ \in \gamma \cap J^+(p)$ such that, passing to a subset of $\mathcal{U}$ if necessary, $J^-(q^+)$ is a smooth submanifold of $\mathcal{U}$, given as a graph $x^0 = f_+(\vec{x})$. Similarly we can find $q^- \in \gamma \cap J^-(p)$ such that $J^+(q^-)$ is a smooth submanifold of $\mathcal{U}$, given as a graph $x^0 = f_-(\vec{x})$. As every point to the past of q^+ lies inside of $\mathcal{D}^+(A)$ we have

$$f_+(\vec{x}) \le f(\vec{x}).$$

Similarly every point to the future of q^- lies outside of $\mathcal{D}^+(A)$, so that we have

$$f(\vec{x}) \le f_-(\vec{x}).$$

At the origin we have $f(0) = f_+(0) = f_-(0)$. Moreover $J^+(q^-)$ and $J^-(q^+)$ are tangent there, so that there exists a vector $\vec{C}$ such that $\vec{C} = df_+(0) = df_-(0)$. It follows that

$$\vec{C}\vec{x} + o(|\vec{x}|) = f_+(\vec{x}) \le f(\vec{x}) \le f_-(\vec{x}) = \vec{C}\vec{x} + o(|\vec{x}|),$$

[2] Added in proof: This has been recently settled by Beem and Królak (preprint), who show that the horizon has a tangent plane at such points.

which establishes (3.1) and shows differentiability of $\mathcal{H}^+(A)$ at p. □

4 Convergence in the Distribution of Semi–tangents

In this section we establish some further properties of the distribution of semi–tangents. We retain all the notational conventions from previous sections. In particular, here as elsewhere, global hyperbolicity of the space–time is assumed. Let $\hat{\Pi} = \Pi \circ P : UM \to \Sigma$, where $P : UM \to M$ is the natural projection map. Thus, if $C \subset \Sigma$, $\hat{\Pi}^{-1}(C)$ is the set of all h-unit vectors over $\Pi^{-1}(C) \subset M$.

Lemma 4.1 *Consider A_n, $A \in \mathcal{X}$ such that $D(A_n, A)$ and $D(\partial A_n, \partial A)$ tend to zero as n tends to infinity.*

(a) If C is compact and $C \subset \operatorname{int} A$ then

$$\text{c--}\lim[\mathcal{N}^+(A_n) \cap \hat{\Pi}^{-1}(C)] \subset \mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C). \tag{4.1}$$

(b) If C is open (as a subset of Σ) and $C \subset A$ then,

$$\mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C) \subset \text{c--}\lim[\mathcal{N}^+(A_n) \cap \hat{\Pi}^{-1}(C)]. \tag{4.2}$$

Remark. Following the proof of Lemma 4.1 we present an example which shows that the inclusion in (4.1) (resp. (4.2)) can be strict (even if, in (4.2), one takes the *closure* of $\mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C)$).

PROOF: (a): Let $X_j \in \mathcal{N}^+(A_{n_j}) \cap \hat{\Pi}^{-1}(C)$ such that $X_j \to X$ in UM. Since $\hat{\Pi}^{-1}(C)$ is closed, $X \in \hat{\Pi}^{-1}(C)$. Let X_j be based at $p_j \in \mathcal{H}^+(A_{n_j})$, and let X be based at p. Let η_j be the past directed null geodesic generator of $\mathcal{H}^+(A_{n_j})$ with initial point p_j and initial tangent X_j, and let $q_j \in \partial A_{n_j}$ be its past end point. Let η be the past directed null geodesic with initial point p and initial tangent X. Since $p_j \to p$, Corollary 2.5 implies that $p \in \mathcal{H}^+(A) \setminus \partial A$. Then η meets Σ at some point $q \in A$. Since $\eta_j \to \eta$ in the sense of geodesics it follows that $q_j \to q$. The assumption $D(\partial A_n, \partial A) \to 0$ implies that $q \in \partial A$. Hence, by Proposition 2.6, η is a null geodesic generator of $\mathcal{H}^+(A)$, and thus $X \in \mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C)$.

(b): Consider $X \in \mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C)$ based at $p \in \mathcal{H}^+(A) \cap \Pi^{-1}(C)$ (in particular, $p \notin \partial A$). Let η be the past directed null generator of $\mathcal{H}^+(A)$ starting at p with initial tangent X, and let $q \in \partial A$ be the past end point of η. Let $y \in \mathcal{H}^+(A) \cap \Pi^{-1}(C)$ be an interior point of η. Since C is open, y can be chosen arbitrarily close to p. By Corollary 2.5 (with $C = \Pi(y)$) there exists $y_j \in \mathcal{H}^+(A_{n_j}) \cap \Pi^{-1}(C)$ such that $y_j \to y$. Let μ_j be a past directed null generator of $\mathcal{H}^+(A_{n_j})$ starting at y_j with initial tangent $Y_j \in \mathcal{N}^+(A_{n_j})$. Assume μ_j meets ∂A_{n_j} at z_j. For any $p' \in I^+(p)$, the sequence $\{z_j\}$ eventually enters the set $J^-(p') \cap \Sigma$. Since, by the global hyperbolicity of M, this set is compact, $\{z_j\}$ has an accumulation point z. Proposition 2.2 implies that $z \in \partial A$. Thus, by passing to a subsequence if necessary, we may assume the μ_j converge to a null geodesic μ with future end point y and past end point z. Proposition 2.6 implies that μ is a null geodesic generator of $\mathcal{H}^+(A)$. In order to avoid there being two distinct semi–tangents at the interior point y, μ must coincide with η, *cf.* Lemma 3.2. Hence, the tangent of η at y coincides with the tangent of μ at $y \in \text{c}-\lim \mathcal{N}^+(A_n)$. Since y can be chosen arbitrarily close to p, Proposition 2.1 implies that $X \in \text{c}-\lim \mathcal{N}^+(A_n)$. $\square$

Example 4.2 *Consider Minkowski 2-space, $M = \mathbb{R}^{1,1}$, with metric $\mathrm{d}s^2 = \mathrm{d}x^2 - \mathrm{d}y^2$. Let Σ be the slice $y = 0$. Set, $A = \{(x,0) : \ x \in (-2,2)\}$, $A_n = \{(x,0) : x \in (-2 - \frac{1}{n}, 2 - \frac{1}{n})\}$. If $C = \{(x,0) : \ x \in [0,1]\}$, the inclusion (4.1) is strict. If $C = \{(x,0) : \ x \in (-1,0)\}$, the inclusion (4.2) is strict (even if one takes the closure of $\mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C)$).*

We now wish to consider the Hausdorff convergence of certain subsets of the distribution of semi–tangents. For this purpose, introduce a complete Riemannian metric on the h-unit tangent bundle UM, and let ρ denote the associated distance function. With respect to subsets V, $W \subset UM$, let $\mathcal{D}(V,W)$ denote the Hausdorff distance between V and W with respect to the distance function ρ.

Given A_n, $A \in \mathcal{X}$ such that $D(A_n, A) \to 0$ and $D(\partial A_n, \partial A) \to 0$, one might have hoped that, for reasonable subsets $C \subset \text{int}A$ (e.g. C compact or open with compact closure), $\mathcal{N}^+(A_n) \cap \hat{\Pi}^{-1}(C)$ Hausdorff converges to $\mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C)$, i.e., $\mathcal{D}(\mathcal{N}^+(A_n) \cap \hat{\Pi}^{-1}(C), \mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C)) \to 0$ as $n \to \infty$. But Example 4.2 shows that this will not be the case in general. However, it is possible to establish a slightly weaker result that is sufficient for our purposes.

Theorem 4.3 *Consider A_n, $A \in \mathcal{X}$ such that $D(A_n, A)$ and $D(\partial A_n, \partial A)$ tend to zero as n tends to infinity.*

(a) If C is compact and $C \subset \text{int}A$ then,

$$\sup_{X \in \mathcal{N}^+(A_n) \cap \hat{\Pi}^{-1}(C)} \rho(X, \mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C)) \to 0 \quad \text{as} \quad n \to \infty.$$

(b) If C is open with compact closure and $\overline{C} \subset \text{int}A$ then,

$$\sup_{Y \in \mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C)} \rho(Y, \mathcal{N}^+(A_n) \cap \hat{\Pi}^{-1}(C)) \to 0 \quad \text{as} \quad n \to \infty.$$

Part (a) is a consequence of Lemma 3.1, part (a) of Lemma 4.1 and the following simple fact.

Lemma 4.4 *Let (X, d) be a metric space. Consider subsets A_n, $A \subset X$. Assume the A_n's are closed. Suppose there exists a compact set $B \subset X$ such that $A_n \subset B$ for all n. If* $\text{c}-\lim A_n \subset A$ *then,*

$$\sup_{x \in A_n} d(x, A) \to 0 \quad \text{as} \quad n \to \infty.$$

Proof: of Lemma 4.4. Note that the A_n's are compact. If the conclusion does not hold then there exists $\varepsilon > 0$ and a subsequence A_{n_k} such that,

$$\sup_{x \in A_{n_k}} d(x, A) \geq \varepsilon \quad \text{for all} \quad k.$$

By the compactness of A_{n_k}, $\exists x_k \in A_{n_k}$ such that $d(x_k, A) \geq \varepsilon$. Since $\{x_k\} \subset B$, by passing to subsequence if necessary, we may assume $x_k \to y$. Since $\text{c}-\lim A_n \subset A$, $y \in A$. On the other hand,

$$d(y, A) = \lim_{k \to \infty} d(x_k, A) \geq \varepsilon,$$

and hence $y \notin A$, a contradiction. $\square$

Proof: of part (a) of Theorem 4.3. The aim is to apply Lemma 4.4. The set $B := \mathcal{H}^+(A) \cap \Pi^{-1}(C)$ is compact and does not meet ∂A. Note that $\mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C) = \mathcal{N}^+(A) \cap P^{-1}(B)$, where, recall, $P : UM \to M$ is the natural projection map. Since $P^{-1}(B)$ is compact and $\mathcal{N}^+(A)$ is closed in $\mathcal{U}M_A$ (recall that $M_A = M \setminus \partial A$) by Lemma 3.1, $\mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C)$ is compact. Similarly, using Proposition 2.3, for sufficiently large n, $\mathcal{N}^+(A_n) \cap \hat{\Pi}^{-1}(C)$ is compact. Since, by Lemma 2.4, $D(\mathcal{H}^+(A_n) \cap \Pi^{-1}(C), \mathcal{H}^+(A) \cap \Pi^{-1}(C)) \to 0$, it follows that $\mathcal{H}^+(A_n) \cap \hat{\Pi}^{-1}(C)$ (n sufficiently large) is contained in a compact set $K \subset M$. (One may take K to be the closure of an ε-neighborhood of $\mathcal{H}^+(A) \cap \hat{\Pi}^{-1}(C)$). It follows that $\mathcal{N}^+(A_n) \cap \hat{\Pi}^{-1}(C)$ (for n sufficiently large) is contained in the compact set $P^{-1}(K) \subset UM$. Also, part (a) of Lemma 4.1 implies that $\text{c}-\lim[\mathcal{N}^+(A_n) \cap \hat{\Pi}^{-1}(C)] \subset \mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C)$. Part (a) of Theorem 4.3 now follows directly from Lemma 4.4.

Proof: of part (b) of Theorem 4.3. Suppose our claim does not hold. Then there exists $\varepsilon > 0$ and a sequence n_j such that,

$$\sup_{Y \in \mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C)} \rho(Y, \mathcal{N}^+(A_{n_j}) \cap \hat{\Pi}^{-1}(C)) \geq 4\varepsilon. \tag{4.3}$$

Let $Y_j \in \mathcal{U}_{p_j}M$ be such that the sup in (4.3) is obtained at $Y = Y_j$. By the compactness of $\overline{\mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C)}$, there exists $Y \in \overline{\mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C)}$ such that, passing to a subsequence if necessary, we have $Y_j \to Y$. It follows that for j large enough we shall have,

$$\rho(Y, \mathcal{N}^+(A_{n_j}) \cap \hat{\Pi}^{-1}(C)) \geq 2\varepsilon. \tag{4.4}$$

Since every neighborhood of Y meets $\mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C)$, we can choose $\tilde{Y} \in \mathcal{N}^+(A) \cap \Pi^{-1}(C)$ such that $\rho(Y, \tilde{Y}) \leq \varepsilon/2$. Since C is open, there will exist a semi-tangent $\hat{Y} \in \mathcal{N}^+(A) \cap \hat{\Pi}^{-1}(C)$ at an interior point $\hat{p}$ of the null geodesic generator of $\mathcal{H}^+(A)$ determined by $\tilde{Y}$, such that $\rho(\hat{Y}, \tilde{Y}) \leq \varepsilon/2$. It follows that for all j large enough,

$$\rho(\hat{Y}, \mathcal{N}^+(A_{n_j}) \cap \hat{\Pi}^{-1}(C) \geq \varepsilon. \tag{4.5}$$

By Corollary 2.5 there exists a sequence $q_j \in \mathcal{H}^+(A_{n_j}) \cap \hat{\Pi}^{-1}(C)$ such that $q_j \to \hat{p}$. (4.5) implies there exists a sequence $Z_j \in \mathcal{N}^+(A_{n_j}) \cap \mathcal{U}_{p_j}M$ such that $\rho(\hat{Y}, Z_j) \geq \epsilon$ for all j sufficiently large. By passing to a subsequence if necessary, we have that $Z_j \to Z \in \overline{U}_{\hat{p}}M$. Proposition 2.1 and part (b) of Lemma 4.1 imply that $Z \in \mathcal{N}^+(A)$. Clearly $Z \neq \hat{Y}$ by (4.5), which contradicts the fact that there can be only one semi-tangent at an interior point of a null geodesic generator. □

5 The proof of Theorem 1.1

Let us start by giving an outline of the construction of the set K, the existence of which is asserted in Theorem 1.1. The idea is to construct a sequence of sets K_i, converging in a precise sense to the desired set K, such that

1. $K_0 = \mathbb{R}^3 \setminus B(0,1)$, where $B(x_0, r)$ denotes an open ball of radius r centered at x_0.
2. ∂K_{i+1} is obtained by adding a certain number of "ripples" to ∂K_i. The process of adding ripples consists of adding a certain number of circular arcs to ∂K_i. It follows that ∂K_i is the union of a finite number of circular arcs.
3. The horizon $\mathcal{H}^+(K_i)$ has several "creases" – curves along which $\mathcal{H}^+(K_i)$ is not differentiable. The addition of ripples to ∂K_i does not affect the existence of the "old" creases, and leads to the occurrence of new ones. The construction guarantees that the set of points which lie on creases of $\mathcal{H}^+(K)$ is dense.
4. The boundaries of the K_i's are radial graphs, that is, ∂K_i is described by an equation $r = f_i(\theta)$, $\theta \in [0, 2\pi]$, for some Lipschitz continuous function $f_i(\theta)$. The construction guarantees that ∂K will also have that property, hence will be a Lipschitz continuous topological manifold.

Before we proceed further, let us recall some elementary facts from Lorentzian geometry. Consider a globally hyperbolic space–time (M, g) with Cauchy surface Σ. Let Ω be an open subset of Σ, then $\mathcal{H}^+(\overline{\Omega})$ is a union of null geodesic segments γ ([12, Theorem 5.12], [11, Prop. 53, p. 430]), called generators of $\mathcal{H}^+(\overline{\Omega})$. Each such generator has a past end point $p \in \partial\Omega = \overline{\Omega} \setminus \Omega$, and either no future end point, or a future end point $q \in \mathcal{H}^+(\overline{\Omega})$. If $\partial\Omega$ is C^2 near p, then γ is the unique null geodesic orthogonal to $\partial\Omega$ pointing towards Ω [11, Lemma 50, p. 298].

Let us now pass to the description of the process that we call "adding a ripple". More precisely, we will be adding a ripple at a point y_0 lying in the "middle" of a circular arc $\mathcal{A}$. Consider thus an arc $\mathcal{A}$ of a circle $S(x_0, r_0)$ of radius r_0 centered at x_0, thus $\mathcal{A}$ is given by the equation $\mathcal{A} = \{x : \ x - x_0 = r_0 e^{i\varphi}, \varphi \in [\varphi_-, \varphi_+]\}$. We will be interested in the structure of a Cauchy horizon of a set which has first $\mathcal{A}$, and then "$\mathcal{A}$ plus a ripple" as part of its boundary. Now for any set Ω and any conformal isometry Ψ we have $\mathcal{H}^+(\Psi(\Omega)) = \Psi(\mathcal{H}^+(\Omega))$. We can always find a conformal isometry Ψ of the three-dimensional Minkowski space–time $\mathbb{R}^{2,1}$ so that $x_0 = 0$, $r_0 = 1$, and $\varphi \in [\frac{\pi}{2} - \varphi_0, \frac{\pi}{2} + \varphi_0]$ on $\Psi(\mathcal{A})$, and it follows that for our purposes it is sufficient to consider this case. Then the addition of a ripple to $\Psi(\mathcal{A})$ at $y_0 = e^{i\frac{\pi}{2}}$ proceeds as follows. Let $0 < r_1 < 1$, $0 < \varphi_1 < \varphi_0$, and let $x_1 = (1 - r_1)e^{i(\frac{\pi}{2} - \varphi_1)}$, $x_2 = (1 - r_1)e^{i(\frac{\pi}{2} + \varphi_1)}$. Consider the circles $S(x_1, r_1)$ and $S(x_2, r_1)$, they are tangent to $\mathcal{A}$ at points $y_1 = e^{i(\frac{\pi}{2} - \varphi_1)}$ and $y_2 = e^{i(\frac{\pi}{2} - \varphi_2)}$. If r_1 is large enough compared with $|x_1 - x_2|$, $r_1 \geq a = \frac{|x_1 - x_2|}{2}$, then $S(x_1, r_1)$ will intersect $S(x_2, r_1)$, and henceforth we will assume that this is the case. The process of adding a ripple to $\mathcal{A}$ is made clear by Figure 3, and can formally be described as follows: "Ψ($\mathcal{A}$ with a ripple added)" will be the curve

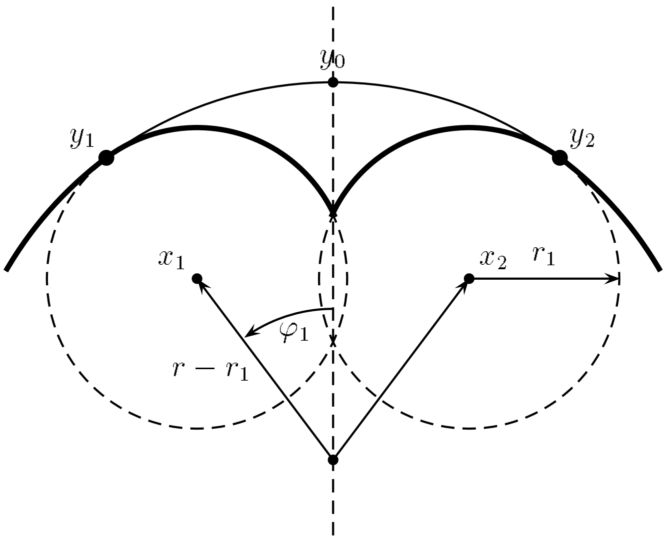

Figure 3: The process of adding a ripple: the thin solid line represents the original arc, the thick solid line corresponds to "an arc with a ripple added".

obtained by following $\Psi(\mathcal{A})$ from $e^{i(\frac{\pi}{2} - \varphi_0)}$ to $e^{i(\frac{\pi}{2} - \varphi_1)}$, then following $S(x_1, r_1)$ from $e^{i(\frac{\pi}{2} - \varphi_1)}$ to the y-axis where $S(x_1, r_1)$ intersects $S(x_2, r_1)$, then following $S(x_2, r_1)$ to $e^{i(\frac{\pi}{2} + \varphi_1)}$, and finally following $\Psi(\mathcal{A})$ to its end point. It should be seen that this construction is invariant under reflections across the y-axis.

The parameter r_1 in the construction will be called the radius of the ripple. The distance $a = \frac{1}{2}|x_1 - x_2|$ will be called the diameter of the ripple. It should be noted that "$\mathcal{A}$ with a ripple" can be described as a radial graph $|x - x_0| = g(\varphi)$, $\varphi \in [\varphi_-, \varphi_+]$. $g(\varphi)$ is Lipschitz continuous and piecewise smooth. Let us for further use note that the modulus of Lipschitz continuity $L = L(r_1, a)$ of $g(\varphi)$ satisfies $L \le \lim_{\varphi \to \frac{\pi}{2}} |g'(\varphi)|$. L is a strictly decreasing function of a, with $\lim_{a \to 0} L(r_1, a) = 0$.

Let us now address the question, how does the addition of a ripple affect the Cauchy horizon of $K_0 = \mathbb{R}^2 \setminus B(0,1)$. As discussed at the beginning of this section, generators of the horizon of any set are null geodesics, which in our case are null straight lines in $\mathbb{R}^{2,1}$. Every such line is determined uniquely by its projection on $\mathbb{R}^2$ under the projection map $\Pi : \mathbb{R}^{2,1} \to \mathbb{R}^2$, $\Pi(t,x) = x$, and this projection is itself a straight line segment in $\mathbb{R}^2$. The generators emanating from points p lying on a smooth piece of the boundary of the set under consideration are thus straight lines orthogonal to the boundary. It follows that the generators of $\mathcal{H}^+(K_0)$ project down to the radial lines $\{x = re^{i\varphi}, r \in [1, \infty)\}$.

Consider now a set $K(a, r_1)$, defined by the addition of a ripple to $S(0,1)$ at $y_0 = e^{i\frac{\pi}{2}}$, of radius r_1 and diameter a, with $a < r_1 < 1/2$. (By that we mean that $K(a, r_1)$ is that set which has as boundary $S(0,1)$ with a ripple added, and $\mathbb{R}^3 \setminus B(0,1) \subset K(a, r_1)$). From what has been said so far it follows that those generators of $\mathcal{H}^+(K(a, r_1))$ which emanate from points lying on $S(0,1)$ project on the radial half lines $\{re^{i\varphi}, r \in [1, \infty)\}$. Those generators of $\mathcal{H}^+(K(a, r_1))$ which emanate from points x lying on $S(x_1, r_1) \setminus \{re^{i\frac{\pi}{2}}, r \in \mathbb{R}\}$ project to line segments contained in the lines $\{r(x - x_1), r \in \mathbb{R}\}$. Note that each such segment, except for the point y_1 where $S(x_1, r_1)$ meets $S(0,1)$, intersects the axis $\{re^{i\frac{\pi}{2}}, r \in \mathbb{R}\}$ precisely at one point. By invariance of $K(a, r_1)$ under reflections across that axis it follows that each such generator must have an end point precisely on the plane $\{t \in \mathbb{R}, x = re^{i\frac{\pi}{2}}, r \in \mathbb{R}\}$. The projection of some of the generators of $\mathcal{H}^+(K(a, r_1))$ can be found in Figure 4.

We wish to point out the following:

1. The generators can always be parameterized so that they are given by an equation of the form $\gamma = \{x^\mu(s), \dot{x}^0 = 1, |\dot{x}| = 1\}$, with $x^\mu(0) \in \partial K(a, r_1)$. In everything that follows all generators will always be parameterized in that manner.

2. Let $C = \mathcal{H}^+(K(a, r_1)) \cap \{(t,x), t \in \mathbb{R}, x = re^{i\frac{\pi}{2}}, r \in \mathbb{R}\}$. Every point $p \in C$ is the future end point of two generators of $\mathcal{H}^+(K(a, r_1))$, while every point $p \in \mathcal{H}^+(K(a, r_1)) \setminus C$ is an interior point of precisely one generator of $\mathcal{H}^+(K(a, r_1))$. C will be called the crease set of $\mathcal{H}^+(K(a, r_1))$.

We wish now to set up an iterative scheme for constructing a sequence $(K_i, \varepsilon_i, \delta_i)$, where ε_i and δ_i are positive numbers and K_i is an increasing sequence

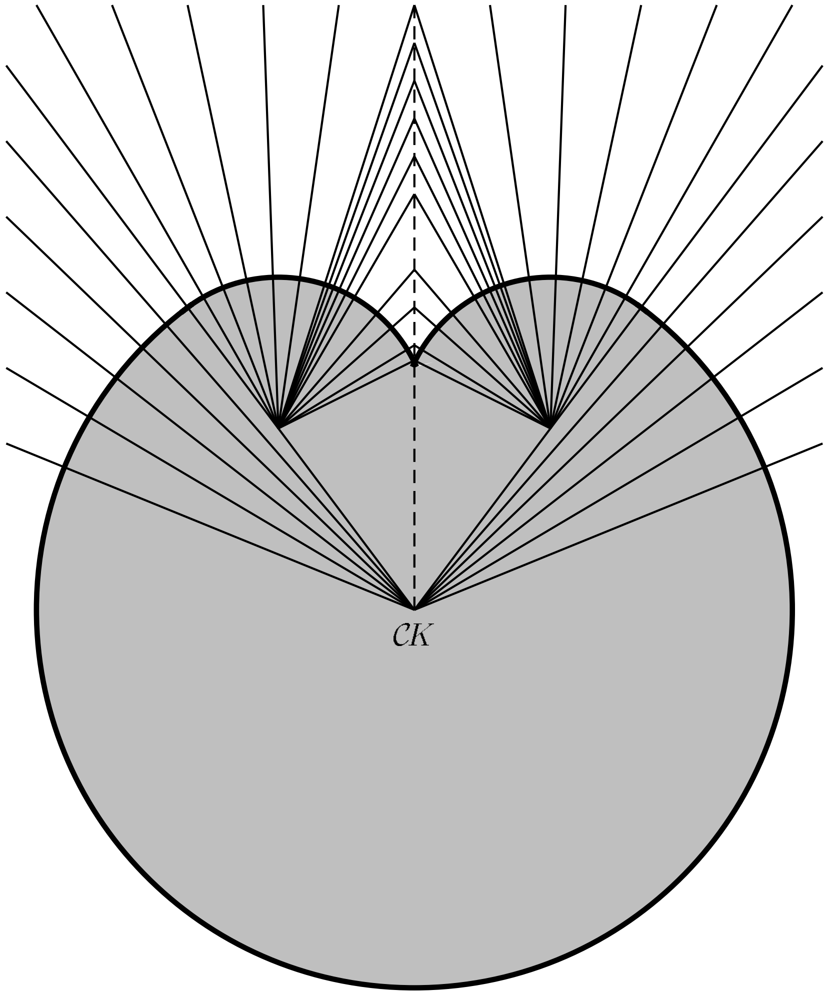

Figure 4: The portion of the solid segments that lies *outside* of the shaded area is the projection of the generators of $\mathcal{H}^+(K(a,r_1))$ on the plane $t=0$.

of sets. The reader is referred to Section 3 for the definition of semi–tangents and of the distribution of semi–tangents. For any set $A \in \mathcal{X}$, with $\mathcal{X}$ as in (2.1), we define *the crease set* of $\mathcal{H}^+(A)$ as the set of points of $\mathcal{H}^+(A)$ at which there is more than one semi-tangent. We set $K_0 = \mathbb{R}^3 \setminus B(0,1)$, $\varepsilon_0 = \infty$, $\delta_0 = \infty$. The future Cauchy horizon $\mathcal{H}^+(K_0)$ of K_0 is shown in Figure 5. The number ε_i will be a measure of "how close to each other are distinct semi–tangents on the crease set of $\mathcal{H}^+(K_i)$". The number δ_i will be a measure of "how far apart are the distributions of semi–tangents $\mathcal{N}^+K_{i-1}$ and $\mathcal{N}^+K_i$". Clearly the crease set C_0 of K_0 is empty.

Let $\{y_n = 2\mathrm{e}^{\frac{i\pi n}{2}}, n = 1, \ldots, 4\}$, and let x_n be the starting points of those generators of $\mathcal{H}^+(K_0)$ which pass through y_n, thus $x_n = \mathrm{e}^{\frac{i\pi n}{2}}$. We define K_1 by adding a ripple of radius r_1 and diameter a_1 at each x_n. r_1 and a_1 are arbitrary except for the requirement that the ripples do not intersect, and that ∂K_1 be the radial graph of a function f_1, $\partial K_1 = \{r = f_1(\theta), \theta \in [0, 2\pi]\}$, such that the modulus of Lipschitz continuity of f_1 is smaller than $1/2$. A possible set K_1 has been shown in Figure 6. By the localization principle, Lemma 2.7, the horizon $\mathcal{H}^+(K_1)$ will coincide with that of $\mathcal{H}^+(K_0)$ away from those circular arcs at which ripples have been added. Again by the localization principle in a small neighborhood of each ripple the horizon will look like the horizon of the set $K(a, r_1)$ as previously considered – $\mathbb{R}^3 \setminus B(0,1)$ with a single ripple added. Thus $\mathcal{H}^+(K_1)$ will look as shown in Figure 7. It follows that the crease set C_1 of $\mathcal{H}^+(K_1)$ projects down on $\mathbb{R}^2$ under Π to half–infinite intervals included in the four half-lines $\{r\mathrm{e}^{i\frac{\pi n}{2}}, r \in [1, \infty)\}$, $n = 1, \ldots, 4$. It should be noted that every point in the annulus $\overline{B(0,2) \setminus B(0,1)}$ is a distance not further than $\frac{\pi}{2}$ from ΠC_1. Letting ρ be the distance on $T\mathbb{R}^{2,1}$ as defined after the example 4.2 in Section 2, we set

$$\varepsilon_1 = \inf\{\rho(X, Y) | \, X, Y \in \mathcal{N}_p K_1, X \neq Y, p \in C_1\},$$
$$\delta_1 = \varepsilon_1/4.$$

This finishes our first inductive step.

The remainder of the construction consists in adding ripples in a controlled way. The second step is shown in Figure 8, and that figure makes it also clear how the induction proceeds; the formal description goes as follows: Suppose that $(K_j, \varepsilon_j, \delta_j)$ have been constructed for all $j \leq i$, with the property that $\Pi C_j \subset \Pi C_{j+1}$, and

$$\delta_{j+1} \leq \frac{1}{4}\min(\varepsilon_j, \delta_j),$$

and with the property that for $j \geq 1$ every point $p \in \overline{B(0,2) \setminus B(0,1)}$ is a distance not further than $2^{-j}\pi$ from ΠC_j. Let us further assume that for $j = 1, \ldots, i$ the K_j's have the property that ∂K_j is a radial graph $\partial K_j = \{r = f_j(\theta), \theta \in [0, 2\pi]\}$ of a Lipschitz continuous function, the modulus of continuity of which is smaller than $\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^j}$. We construct K_{i+1} as follows: we can find points $z_k \in C_i \cup (\mathcal{H}^+(K_i) \cap \Pi^{-1}(\partial B(0,2))$ $k = 1, \ldots, I$, such that every point

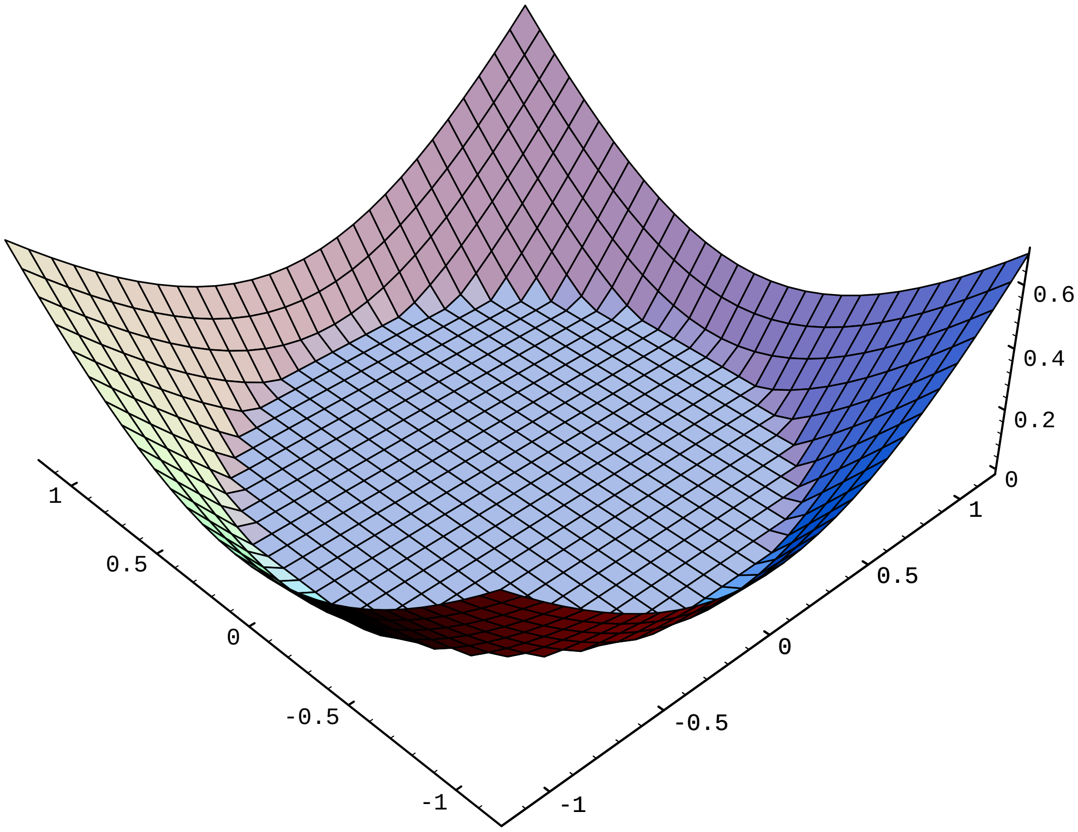

Figure 5: The future Cauchy horizon $\mathcal{H}^+(K_0)$ of K_0.

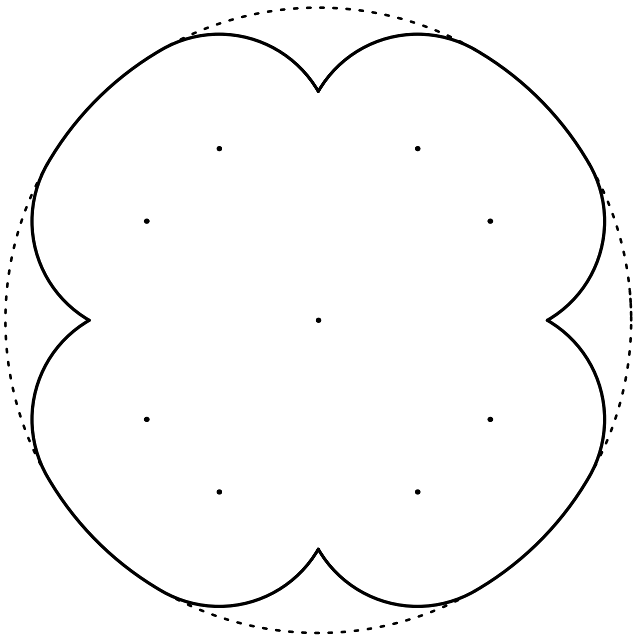

Figure 6: The dotted line is the boundary of K_0, the solid line that of K_1. For graphical clarity the parameters describing the ripples have *not* been chosen so that the modulus of Lipschitz continuity of the function representing K_1 be smaller than one half, as required in the construction described below.

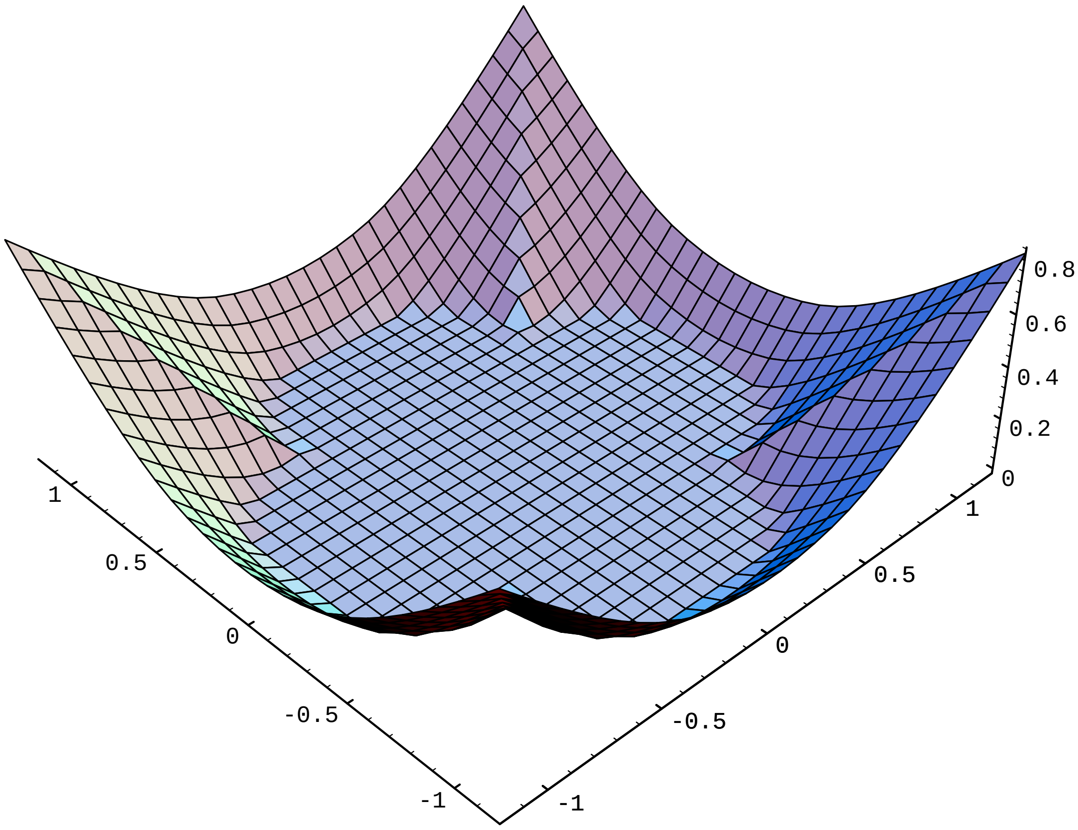

Figure 7: The future Cauchy horizon $\mathcal{H}^+(K_1)$ of K_1. (K_1 here has been rotated by 45 degrees, as compared to Figure 6.)

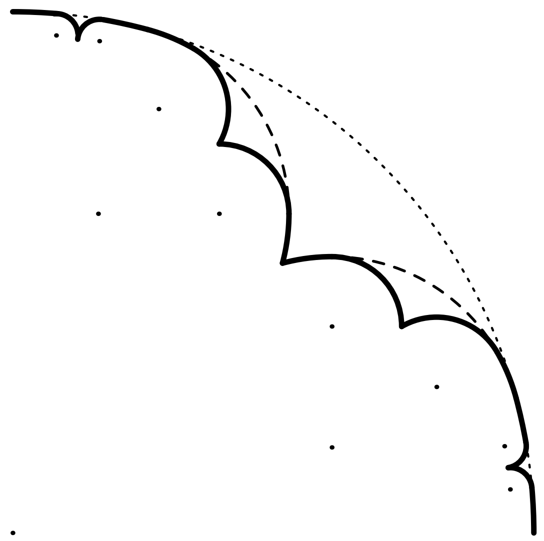

Figure 8: The dotted line represents the intersection of the boundary of K_0 with the first quadrant, the dashed one that of K_1, and the solid one that of K_2. The dots are the centers of the corresponding circles. K_0 and K_1 have been rotated by 45^o, as compared to Fig. 6. For graphical clarity the radii of the "second-generation-ripples" have not been chosen to be equal, as done in the construction below.

$p \in \mathcal{H}^+(K_i) \cap \Pi^{-1}(\overline{B(0,2)})$ is a distance not larger than $2^{-i-1}\pi$ from a generator of $\mathcal{H}^+(K_i)$ passing through one of the z_k's, or having its end point there. Let $\{y_n\}_{n=1}^I$ denote the set of the starting points of those generators. We may choose the z_k's so that none of the y_n's lies on the end point of a circular arc on ∂K_i. We can also require that $\#\mathcal{N}_{z_k} K_i \leq 2$.

Define $K_{i+1}(a)$ by adding ripples of radius r_{i+1} and diameter a, $a \leq a_{i+1} \leq r_{i+1}$, to ∂K_i at all the y_n's. The radius r_{i+1} and threshold diameter a_{i+1} are chosen small enough so that ∂K_{i+1} is a radial graph of a function f_{i+1}, the modulus of Lipschitz continuity of which is smaller than or equal to $\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{i+1}}$. Moreover a_{i+1} is chosen small enough so that the new ripples do not intersect each other. Consider now a point $z_k \in C_i$, such that $z_k \in C_j$ and $z_k \notin C_{j-1}$ for some j. There exist precisely two points, say x_1 and x_2, on ∂K_i, which are initial points of generators of $\mathcal{H}^+(K_i)$, say γ_1 and γ_2, with end point at z_k. Moreover x_1 and x_2 are symmetric images of each other under the reflection across the line which contains the projection of that generator r_k of $\mathcal{H}^+(K_{j-1})$ which passes through z_k. By the localization principle, Lemma 2.7, the addition of a sufficiently small ripple at x_1 and x_2 will only affect $\mathcal{H}^+(K_i)$ in a small neighborhood of γ_1 and γ_2. Because the ripples are added symmetrically with respect to r_k, the generators emerging from points on the ripples can only meet on a set, the projection of which contains the projection of r_k. It follows that for a_{i+1} small enough the projection on $\mathbb{R}^2$ of the crease set $C_{i+1}(a)$ of $\mathcal{H}^+(K_{i+1}(a))$, $a \leq a_{i+1}$, will contain the projection of the crease set C_i.

Note that the above discussion also shows that there exists a compact neighborhood D_i of $\{z_k\}$, with $D_i \cap \partial K_i = \emptyset$, such that the crease set of $\mathcal{H}^+(K_i)$ is contained in that of $\mathcal{H}^+(K_{i+1}(a))$, except for points in D_i, for all $0 \leq a < a_{i+1}$. Moreover, by choosing a_{i+1} sufficiently small, D_i may be expressed as a disjoint union, $D_i = \cup_k D_{i,k}$, where $D_{i,k}$ is an arbitrarily small compact neighborhood of z_k. Then, continuity properties of geodesics, as applied to the null geodesics orthogonal to ∂K_{i+1}, and the structure of the crease sets imply the following: For any $\epsilon > 0$, a_{i+1} (and, in turn, the $D_{i,k}$'s) may be chosen sufficiently small so that for any $Z \in \mathcal{N}^+ K_{i+1}(a) \cap \hat{\Pi}^{-1}(\Pi D_{i,k})$ and for any $q \in \mathcal{H}^+(K_{i+1}(a)) \cap \Pi^{-1}(\Pi(C_i \cap D_{i,k}))$ there exists $Y \in \mathcal{N}^+_q K_{i+1}(a)$ such that $\rho(Z, Y) < \epsilon$.

Define

$$\delta_{i+1}(a) = \sup\{\inf\{\rho(X,Y) | \; Y \in \mathcal{N}^+_{\Pi^{-1}p} K_{i+1}(a)\} | \; p \in \Pi C_i, X \in \mathcal{N}^+_{\Pi^{-1}p} K_i\}. \tag{5.1}$$

Here $\Pi^{-1}p$ denotes that point which is the lift of p to the appropriate Cauchy horizon. By our previous remark, in (5.1) we can replace the condition $p \in \Pi C_i$ by: $p \in \Pi(C_i \cap D_i)$, where D_i is as above. Hence, from our previous remark and part (b) of Theorem 4.3 it follows that $\delta_{i+1}(a) \to 0$ as $a \to 0$. (This should anyway be clear by inspecting what happens in Figures 4 or 5 when the diameter of the ripples tends to zero). We can thus choose a_{i+1} small enough so that

$$\delta_{i+1}(a_{i+1}) \leq \frac{1}{4}\min(\delta_i, \varepsilon_i).$$

Set $K_{i+1} = K_{i+1}(a_{i+1})$, and define

$$\varepsilon_{i+1} = \inf\{\inf\{\rho(X,Y) \mid X, Y \in \mathcal{N}^+_{\Pi^{-1}p}K_{i+1}, X \neq Y\} \mid p \in C_{i+1}\} > 0,$$

where, C_{i+1} is the crease set of $\mathcal{H}^+(K_{i+1})$. This completes our inductive step.

Consider the family of functions $f_i(\theta)$ which represent ∂K_i as radial graphs. For any θ we have $f_{i+1}(\theta) \leq f_i(\theta)$, so that $f(\theta) = \lim_{i\to\infty} f_i(\theta)$ exists. Set

$$K = \{re^{i\theta} \mid r \geq f(\theta)\}.$$

As the modulus of Lipschitz continuity of all the f_i's is less than $\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^i} < 1$, f is a Lipschitz continuous function and ∂K is thus a Lipschitz continuous topological manifold.

Let C be the crease set of $\mathcal{H}^+(K)$, we wish to show that ΠC is dense in the annulus $\overline{B(0,2)\setminus B(0,1)}$. This will follow if we show that ΠC contains ΠC_i for each i.

Consider then a point $p \in C_i$, there exist at least two vectors X and Y in TM which are semi-tangent to $\mathcal{H}^+(K_i)$, with $\rho(X,Y) \geq \varepsilon_i$, let $q = \Pi p$. By construction there exist $X_1, Y_1 \in \mathcal{N}^+_{p_1}K_{i+1}$, $\Pi p_1 = \Pi p$, such that $\rho(X, X_1) \leq \delta_{i+1} \leq \frac{\varepsilon_i}{4}$, $\rho(Y, Y_1) \leq \frac{\varepsilon_i}{4}$. Similarly there exist $X_2, Y_2 \in \mathcal{N}^+_{p_2}K_{i+2}$ such that $\rho(X_1, X_2) \leq \frac{\delta_{i+1}}{4} \leq \frac{\varepsilon_i}{16}$, $\rho(Y_1, Y_2) \leq \frac{\varepsilon_i}{16}$. This gives $\rho(X, X_2) \leq \frac{\varepsilon_i}{4} + \frac{\varepsilon_i}{16} < \frac{\varepsilon_i}{3}$, $\rho(Y, Y_2) < \frac{\varepsilon_i}{3}$. Continuing this way one obtains a sequence of vectors X_i, Y_i such that $\rho(X, X_i) < \frac{\varepsilon_i}{3}$, $\rho(Y, Y_i) < \frac{\varepsilon_i}{3}$. By compactness a converging sequence can be chosen, converging to vectors X_∞, Y_∞. By the triangle inequality we have $\rho(X_\infty, Y_\infty) \geq \frac{\varepsilon}{3}$, so that $X_\infty \neq Y_\infty$. By the cluster-limit Lemma 4.1, $X_\infty, Y_\infty \in \mathcal{N}^+_{\Pi^{-1}q}K$. By Proposition 3.4 $\mathcal{H}^+(K)$ cannot be smooth at $\Pi^{-1}q$, and the result follows. □

6 Higher dimensional examples

Let us now address the question of existence of "nowhere" differentiable horizons in more than 2+1 space–time dimensions. First, consider the space–times $M^{p,q} = \mathbb{R}^{2,1} \times \mathbb{R}^p \times S_1^q$, where S_1 denotes a circle, p, q are non-negative integers. Here $\mathbb{R}^p$ and S_1^q are equipped with the obvious metric, and $M^{p,q}$ is equipped with the product metric. We claim that

$$\mathcal{H}^+(K \times \mathbb{R}^p \times S_1^q; M^{p,q}) = \mathcal{H}^+(K; \mathbb{R}^{2,1}) \times \mathbb{R}^p \times S_1^q\ . \tag{6.1}$$

Here K is the set constructed in Theorem 1.1, $\mathcal{H}^+(\Omega; M)$ denotes the Cauchy horizon of an acausal set Ω in a space–time M. It follows that $\mathcal{H}^+(K \times \mathbb{R}^p \times$

$S^q; M^{p,q})$ provides us with a "nowhere" differentiable Cauchy horizon, via the constructions discussed in Section 1. To prove (6.1) it is sufficient to note that $\mathbb{R}^p \times S_1^q$ acts transitively on itself by isometries, which consist of translations in the $\mathbb{R}^p$ factor and rotations of each S_1 factor. Since $\mathcal{H}^+(\Psi(\Omega); M) = \Psi(\mathcal{H}^+(\Omega; M))$ for any isometry Ψ, it follows that $\mathcal{H}^+(K \times \mathbb{R}^p \times S^q; M^{p,q})$ is of the form $\mathcal{U} \times \mathbb{R}^p \times S^q$ for some set $\mathcal{U}$. Clearly $\mathcal{U} = \mathcal{H}^+(K; \mathbb{R}^{2,1})$ and (6.1) follows.

Note that if $p > 0$ then $K \times \mathbb{R}^p \times S^q$ does not have a compact boundary, while if $q > 0$ then $M^{p,q}$ is not asymptotically flat in the usual sense. It is therefore of interest to try to mimic our construction in higher dimension, to obtain a set $K \subset \mathbb{R}^n$, with compact boundary ∂K, having the properties listed in Theorem 1.1, when $\mathbb{R}^n$ is considered as the $t = 0$ hyper-surface in $n + 1$-dimensional Minkowski space–time $\mathbb{R}^{n,1}$. Now the results proved in Sections 2 and 3 are dimension-independent, let us then isolate the essential elements of our construction.

The key element was the procedure of adding a ripple to the boundary of a set, which then produced a crease on the event horizon. Thus we need a building block which will have an effect similar to the one that our two-dimensional ripple had. Such a higher dimensional ripple can be easily constructed as follows: let us start with 3+1 dimensions. Let $K(a, r)$ be the complement of a disc with a ripple added, as described at the beginning of Section 2 and shown in Figure 4, lying in the plane $z = 0$ in $\mathbb{R}^3$. Let $\hat{K}(a, r) \subset \mathbb{R}^3$ be the set obtained by rotating $K(a, r)$ around the axis $z = x = 0$. ($K(a, r)$ is clearly diffeomorphic to an apple without its stalk.) $\hat{K}(a, r)$ is invariant under rotations around that axis, so must therefore be its Cauchy horizon in $\mathbb{R}^{3,1}$. It is then easily seen that $\mathcal{H}^+ \equiv \mathcal{H}^+(\hat{K}(a, r); \mathbb{R}^{3,1})$ is obtained by rotating $\mathcal{H}^+(K(a, r); \mathbb{R}^{2,1})$ around the $z = x = 0$ axis. This shows that the crease set of $\mathcal{H}^+$ is non-trivial and projects on $\mathbb{R}^3$ under the canonical projection $\Pi : \mathbb{R}^{3,1} \to \mathbb{R}^3$ to a subset of the $z = x = 0$ axis. Clearly one can now start an iterative scheme of adding new ripples, and obtain a desired set K by passing to a limit.

Let us mention that a useful bookkeeping procedure in the two-dimensional case was to ensure that the projections ΠC_i of the crease sets C_i formed an increasing sequence of sets. Presumably a way of ensuring that could also be conceived in the three-dimensional case. Rather than doing that let us note that this aspect of the construction is not necessary, and can be bypassed as follows: At the i-th step of the construction add the new "end points" $\{z_j\}_{j=1}^{N_i}$ of the generators γ_j, $j = 1, \ldots, N_i$, so that every point of $\overline{B(0, 2) \setminus B(0, 1)} \subset \mathbb{R}^3$ be a distance not larger than 2^{-i-2} from the projection of some generator with future end point in the set $\{z_j\}_{j=1}^{N_i}$, or passing through a point in $\{z_j\}_{j=1}^{N_i}$. Choose then the size of the ripples added at the z_j's, $j = 1, \ldots, N_i$ so small that ΠC_{i+1} lies a distance not larger than 2^{-i-2} from one of the distinguished generators γ_i or from ΠC_i. The other arguments go through with some rather obvious modifications.

Let us finally describe one of the many possible processes of "adding a ripple" in n-dimension. Let $f_{a,r}$ be the function which describes the "rippled circle" $\partial K(a,r)$ as a radial graph, $\partial K(a,r) = \{f_{a,r}(\varphi)\mathrm{e}^{i\varphi}, \varphi \in [0,2\pi]\}$; $K(a,r)$ is as described at the beginning of Section 2. Then a "rippled $n-1$-dimensional sphere" $\partial\tilde{K}(a,r)$ is given by the equation

$$\begin{aligned} \partial\tilde{K}(a,r) = \qquad & \{x_n = f_{a,r}(\theta)\cos\theta, \\ & \quad x_1^2 + x_2^2 + \ldots + x_{n-1}^2 = f_{a,r}(\theta)\sin\theta, \qquad \theta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]\}. \end{aligned}$$

The reader should be able to check, by isometry considerations, that for all $t \geq 0$ the cross-sections of $\mathcal{H}^+ \equiv \mathcal{H}^+(\tilde{K}(a,r); \mathbb{R}^{n,1})$ with the hyper-planes $t =$ const are sets the boundaries of which are again rippled $n-1$ spheres. The crease set of $\mathcal{H}^+$ projects down to a subset of the axis $x^1 = x^2 = \ldots = x^{n-1} = 0$.

Acknowledgments P.T.C. acknowledges useful discussions with R. Geroch, K. Newman, P. Tod and R. Wald. He also wishes to thank the Department of Mathematics of University of Miami for hospitality during part of work on this paper. We are grateful to V.E. Dubau and C. Georgelin for producing the MATHEMATICA figures.

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