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RESEARCH ARTICLE

Rigidity of outermost MOTS: the initial data version

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Abstract In the paper Commun Anal Geom 16(1):217–229, 2008, a rigidity result was obtained for outermost marginally outer trapped surfaces (MOTSs) that do not admit metrics of positive scalar curvature. This allowed one to treat the "borderline case" in the author's work with R. Schoen concerning the topology of higher dimensional black holes (Commun Math Phys 266(2):571–576, 2006). The proof of this rigidity result involved bending the initial data manifold in the vicinity of the MOTS within the ambient spacetime. In this note we show how to circumvent this step, and thereby obtain a pure initial data version of this rigidity result and its consequence concerning the topology of black holes.

Keywords Marginally outer trapped surfaces · Rigidity · Black hole topology

1 Introduction

The aim of this note is to obtain pure initial data versions of the main results in [6] concerning the topology and rigidity of marginally outer trapped surfaces. This settles problem (18) in the open problem section in [4]. In order to describe our results and put them in context, we begin with some basic definitions and background.

Let (M, g) be an $n+1$, $n \geq 3$, dimensional spacetime (time oriented Lorentzian manifold). By an initial data set in (M, g) we mean a triple (V, h, K), where V is a smooth spacelike hypersurface, h is its induced (Riemannian) metric and K is its

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second fundamental form. To set sign conventions, we have $K(X, Y) = g(\nabla_X u, Y)$, where $X, Y \in T_p V$, ∇ is the Levi-Civita connection of M, and u is the future directed timelike unit vector field to V.

Recall that the *spacetime* dominant energy condition is the requirement,

$$G(X, Y) \geq 0 \quad \text{for all future directed causal vectors } X, Y \in TM, \tag{1.1}$$

where $G = Ric_M - \frac{1}{2} R_M g$ is the Einstein tensor. Given an initial data set (V, h, K), the spacetime dominant energy condition implies

$$\mu \geq |J| \quad \text{along } V, \tag{1.2}$$

where the scalar $\mu = G(u, u)$ is the *local energy density* and the one-form $J = G(u, \cdot)$ is the *local momentum density* along V. It is a basic fact that μ and J can be expressed solely in terms of initial data. When referring to an initial data set, the inequality (1.2) is what is meant by the dominant energy condition.

We now recall the key concept of a marginally outer trapped surface. Consider an initial data set (V, h, K) in a spacetime (M, g). Let Σ be a closed (compact without boundary) two-sided hypersurface in V. Then Σ admits a smooth unit normal field ν in V, unique up to sign. By convention, refer to such a choice as outward pointing. Let u be the future pointing timelike unit vector field orthogonal to M. Then $l_+ = u + \nu$ (resp. $l_- = u - \nu$) is a future directed outward (resp., future directed inward) pointing null normal vector field along Σ.

Associated to l_+ and l_-, are the two *null second fundamental forms*, χ_+ and χ_-, respectively, defined as, $\chi_\pm : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$, $\chi_\pm(X, Y) = g(\nabla_X l_\pm, Y)$. The *null expansion scalars* (or *null mean curvatures*) $\theta_\pm$ of Σ are obtained by tracing $\chi_\pm$,

$$\theta_\pm = \operatorname{tr} \chi_\pm = \operatorname{tr}_\Sigma K \pm H, \tag{1.3}$$

where in the latter expressions, which depend only on initial data, H is the mean curvature of Σ in V and $\operatorname{tr}_\Sigma K$ is the partial trace of K along Σ. Physically, θ_+ (resp., θ_-) measures the divergence of the outgoing (resp., ingoing) light rays emanating from Σ.

In regions of spacetime where the gravitational field is strong, one may have both $\theta_- < 0$ and $\theta_+ < 0$, in which case Σ is called a *trapped surface*, the important concept introduced by Penrose. Focusing attention on the outward null normal only, we say that Σ is an *outer trapped surface* if $\theta_+ < 0$. Finally, we say that Σ is a *marginally outer trapped surface* (MOTS) if θ_+ vanishes identically. MOTSs arise naturally in a number of situations. For example, cross sections of the event horizon in stationary black holes spacetimes, such as the Kerr solution, are MOTSs. MOTSs may also occur as the boundary of the 'trapped region' (cf. [1], and references therin). We note that in the time-symmetric case ($K = 0$), a MOTS is a minimal surface in V.

A basic step in the proof of the "no hair theorem" (i.e., the uniqueness of the Kerr solution) is Hawking's theorem on the topology of black holes [10], which asserts that compact cross-sections of the event horizon in $3 + 1$-dimensional asymptotically flat stationary black hole space-times obeying the dominant energy condition are topologically 2-spheres. The discovery of Emparan and Reall [5] of a $4+1$ dimensional asymptotically flat stationary vacuum black hole space-time with horizon topology $S^1 \times S^2$, the so-called "black ring", showed that black hole uniqueness fails in higher dimensions and, moreover, that horizon topology need not be spherical. This naturally led to the question as to what horizon topologies are allowed in higher dimensional black hole space-times. This question was addressed in a paper with Schoen [9], in which a generalization of Hawking's black hole topology theorem was obtained.

Let Σ be a MOTS in an initial data set (V, h, K), and suppose Σ separates V into an "inside" and an "outside" (the side into which ν points). We shall say that Σ is *outermost* in V if there are no outer trapped ($\theta_+ < 0$) or marginally outer trapped ($\theta_+ = 0$) surfaces outside of and homologous to Σ. Outermost MOTS are necessarily *stable*. (The important concept of the stability of MOTS [2,3] is reviewed in the next section.) In [9] the author and Schoen proved the following.

Theorem 1.1 *Let (V, h, K) be an n-dimensional initial data set, $n \geq 3$, satisfying the dominant energy condition (DEC), $\mu \geq |J|$. If Σ is a stable MOTS in V (in particular if Σ is outermost) then, apart from certain exceptional circumstances, Σ must admit a metric of positive scalar curvature.*

The 'exceptional circumstances' are ruled out if, for example, the DEC holds strictly at some point of Σ or Σ is not Ricci flat. Apart from such exceptional circumstances, Σ is admits a metric of positive scalar curvature, which implies many well known restrictions on the topology; see [7] for a discussion. In particular, in the case $\dim M = 4+1$, so that $\dim \Sigma = 3$ (and assuming orientablity), Σ must be diffeomorphic to either a spherical space (quotient of a 3-sphere) or to $S^1 \times S^2$, or to a connected sum of these two types.

One drawback of Theorem 1.1 is that it allows certain possibilities that one would like to rule out. For example, it does not rule out the possibility of a vacuum black hole spacetime with toroidal topology. However, in [6], we were able to eliminate these exceptional cases for outermost MOTSs provided the initial data set can be embedded into a spacetime obeying the spacetime DEC (1.1).

Theorem 1.2 [6] *Let (V^n, h, K), $n \geq 3$, be an initial data set in a spacetime obeying the DEC. If Σ^{n-1} is an outermost MOTS in (V^n, h, K) then Σ admits a metric of positive scalar curvature.*

Thus, in particular, there can be no stationary vacuum black hole spacetime with toroidal horizon topology. Theorem 1.2 is an immediate consequence of the following rigidity result.

Theorem 1.3 [6] *Let (V^n, h, K), $n \geq 3$, be an initial data set in a spacetime obeying the DEC. Suppose Σ is a separating MOTS in V such that there are no outer trapped surfaces ($\theta_+ < 0$) outside of, and homologous, to Σ. If Σ does not admit a metric of positive scalar curvature, then there exists an outer half-neighborhood $U \approx [0, \epsilon) \times \Sigma$ of Σ in V such that each slice $\Sigma_t = \{t\} \times \Sigma$, $t \in [0, \epsilon)$ is a MOTS. In fact, each Σ_t has vanishing null second fundamental form, with respect to the outward null normal, and is Ricci flat.*

An unsatisfactory feature of Theorems 1.2 and 1.3, from both a conceptual and practical point of view, is that they are not pure initial data results: The proof of Theorem 1.3 in [6] requires the DEC (1.1) to hold in a spacetime neighborhood of Σ. However, many fundamental results in general relativity, such as the positive mass theorem, are statements about initial data sets; no assumptions about the evolution of the data are required. From this point of view, it would be desirable to obtain a pure initial data version of Theorem 1.3, one that only requires the initial data version (1.2) of the DEC. Such a version is presented in Sect. 3. Some preliminary results are presented in Sect. 2.

2 Preliminaries

Let (Σ, γ) be a compact Riemannian manifold. We will be considering operators $\mathcal{L} : C^\infty(\Sigma) \to C^\infty(\Sigma)$ of the form

$$\mathcal{L}(\phi) = -\Delta\phi + 2\langle X, \nabla\phi\rangle + (\mathcal{Q} + \operatorname{div} X - |X|^2)\phi, \tag{2.1}$$

where $\mathcal{Q} \in C^\infty(\Sigma)$, X is a smooth vector field on Σ and $\langle\ ,\ \rangle = \gamma$.

Although the operator $\mathcal{L}$ is not self-adjoint in general, it nevertheless has the following properties (see [3]).

Lemma 2.1 *The following holds for the operator $\mathcal{L}$.*

1. *There is a real eigenvalue $\lambda_1 = \lambda_1(\mathcal{L})$, called the* principal eigenvalue *of $\mathcal{L}$, such that for any other eigenvalue μ, $\mathrm{Re}(\mu) \geq \lambda_1$. The associated eigenfunction ϕ, $L\phi = \lambda_1\phi$, is unique up to a multiplicative constant, and can be chosen to be strictly positive.*
2. *$\lambda_1 \geq 0$ (resp., $\lambda_1 > 0$) if and only if there exists $\psi \in C^\infty(\Sigma)$, $\psi > 0$, such that $\mathcal{L}(\psi) \geq 0$ (resp., $\mathcal{L}(\psi) > 0$).*

The following is proved in [6] (based on the main argument in [9]).

Lemma 2.2 *Consider the operator $\mathcal{L}$ such that,*

$$\mathcal{Q} = \frac{1}{2}S - P, \tag{2.2}$$

where S is the scalar curvature of (Σ, γ) and $P \geq 0$. If $\lambda_1(\mathcal{L}) \geq 0$ then Σ admits a metric of positive scalar curvature, unless $\lambda_1(\mathcal{L}) = 0$, $P \equiv 0$ and (Σ, γ) is Ricci flat.

MOTSs admit an important notion of stablilty, as introduced by Andersson et al. [2,3], which we now recall. In what follows, to simplify notation, we drop the plus sign, and denote $\theta = \theta_+$, $\chi = \chi_+$, and $l = l_+$.

Let Σ be a MOTS in the initial data set (V, h, K) with outward unit normal ν. We consider a normal variation of Σ in V, i.e., a variation $t \to \Sigma_t$ of $\Sigma = \Sigma_0$ with variation vector field $V = \frac{\partial}{\partial t}|_{t=0} = \phi\nu$, $\phi \in C^\infty(\Sigma)$. Let $\theta(t)$ denote the null expansion of Σ_t with respect to $l_t = u + \nu_t$, where u is the future directed timelike unit normal to M and ν_t is the outer unit normal to Σ_t in M. A computation as in [3] gives,

$$\left.\frac{\partial\theta}{\partial t}\right|_{t=0} = L(\phi), \tag{2.3}$$

where $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ is the operator,

$$L(\phi) = -\triangle\phi + 2\langle X, \nabla\phi\rangle + \left(Q + \operatorname{div} X - |X|^2\right)\phi, \tag{2.4}$$

and where

$$Q = \frac{1}{2}S_\Sigma - (\mu + J(\nu)) - \frac{1}{2}|\chi|^2. \tag{2.5}$$

Here $\triangle$, ∇ and div are the Laplacian, gradient and divergence operators, respectively, on Σ, S_Σ is the scalar curvature of Σ with respect to the induced metric $\langle\, ,\, \rangle$ on Σ, X is the vector field on Σ dual to the one form $K(\nu, \cdot)|_{T\Sigma}$, and μ and J are as in the introduction.

We note that L is of the form (2.1). In the time-symmetric ($K = 0$) case, L reduces to the classical stability (or Jacobi) operator of minimal surface theory. As such, L is referred to as the MOTS stability operator. We say that a MOTS is stable provided $\lambda_1(L) \geq 0$. In the minimal surface case this is equivalent to the second variation of area being nonnegative. Lemma 2.1 and (2.4) imply that a MOTS is stable if and only there is an outward variation $t \to \Sigma_t$ such that $\frac{\partial\theta}{\partial t}|_{t=0} \geq 0$.

A basic criterion for stability is the following. We say that a separating MOTS Σ is *weakly outermost* provided there are no outer trapped ($\theta < 0$) surfaces outside of, and homologous to, Σ. Weakly outermost MOTS are necessarily stable. Indeed, if $\lambda_1(L) < 0$, (2.3), with ϕ a positive eigenfunction ($L(\phi) = \lambda_1(L)\phi$) implies that Σ can be deformed outward to an outer trapped surface.

The following was a key element in the proofs of Theorems 1.2 and 3.1 in [6] (see also [8,11]).

Lemma 2.3 *Let Σ be a MOTS in an initial data set (V, h, K). If $\lambda_1(L) = 0$, where L is the MOTS stability operator, then, up to isometry, there exists a neighborhood W of Σ such that*:

(i) *$W = (-t_0, t_0) \times \Sigma$ and $h|_W$ has the orthogonal decomposition,*

$$h|_W = \phi^2 dt^2 + \gamma_t$$

where $\phi = \phi(t, x)$ and γ_t is the induced metric on $\Sigma_t = \{t\} \times \Sigma$.

(ii) *The outward null expansion of each Σ_t is constant, i.e., $\theta = \theta(t)$, with respect to $\ell_t = u + \nu_t$, where $\nu_t = \frac{1}{\phi}\frac{\partial}{\partial t}$ is the outward unit normal to Σ_t.*

3 Main results

The main aim of this section is to prove the following.

Theorem 3.1 *Let (V^n, h, K), $n \geq 3$, be an initial data set satisfying the DEC, $\mu \geq |J|$. Suppose Σ^{n-1} is a weakly outermost MOTS in V^n that does not admit a metric of positive scalar curvature. Then there exists an outer neighborhood $U \approx [0,\epsilon) \times \Sigma$ of Σ in V such that each slice $\Sigma_t = \{t\} \times \Sigma$, $t \in [0,\epsilon)$ is a MOTS. In fact each such slice has vanishing outward null second fundamental form, and is Ricci flat.*

Theorem 3.1 was proved in [6] under the additional assumption that the mean curvature of $\tau = \operatorname{tr} K$ of V is nonpositive, $\tau \leq 0$ [cf. [6, Theorem 3.1], whose proof only requires (1.2)]. This assumption was removed in [6, Theorem 1.2], assuming that the ambient spacetime satisfies the DEC (1.1). The proof of Theorem 3.1 above turns out to be a rather mild variation of the proof of [6, Theorem 3.1].

Proof of Theorem 3.1 As observed in Sect. 2, since Σ is weakly outermost, it is stable, $\lambda_1(L) \geq 0$. Then, since Σ does not admit a metric of positive scalar curvature, Lemma 2.2 applied to L, with $P = (\mu + J(\nu)) + \frac{1}{2}|\chi|^2$, implies that $\lambda_1 = 0$. Hence, there exists a neighborhood $W = (-t_0, t_0) \times \Sigma$ of Σ with the properties specified in Lemma 2.3. In particular, for each $t \in (-t_0, t_0)$, the outward null expansion $\theta = \theta(t)$ of Σ_t is constant.

A computation similar to that leading to (2.3) (but where we can no longer assume θ vanishes) shows that the null expansion function $\theta = \theta(t)$ of the foliation obeys the evolution equation (see [1]),

$$\frac{d\theta}{dt} = -\Delta\phi + 2\langle X, \nabla\phi\rangle + \left(Q - \frac{1}{2}\theta^2 + \theta\tau + \operatorname{div} X - |X|^2\right)\phi\,, \tag{3.1}$$

where it is to be understood that, for each t, the above terms live on Σ_t, e.g., $\triangle = \triangle_t$ is the Laplacian on Σ_t, $Q = Q_t$ is the quantity (2.5) now defined on Σ_t, etc. Also, in the above, τ is the mean curvature of V.

The assumption that Σ is weakly outermost, together with the constancy of $\theta(t)$, implies that $\theta(t) \geq 0$ for all $t \in [0, t_0)$. Fixing $\epsilon \in (0, t_0)$, we will now show that $\theta(t) = 0$ for all $t \in [0, \epsilon)$. To this end, we re-express (3.1) as follows,

$$\frac{d\theta}{dt} - \tau\phi\,\theta = L_t(\phi), \tag{3.2}$$

where

$$L_t(\phi) = -\Delta\phi + 2\langle X, \nabla\phi\rangle + \left(Q - \frac{1}{2}\theta^2 + \operatorname{div} X - |X|^2\right)\phi. \tag{3.3}$$

On $[0,\epsilon] \times \Sigma$, fix a constant c such that $\tau\phi \leq c$. Then (3.2) and the nonnegativity of θ imply,

$$L_t(\phi) \geq \frac{d\theta}{dt} - c\theta = e^{ct}\frac{d}{dt}F(t), \quad \text{for all } \; t \in [0,\epsilon), \tag{3.4}$$

where $F(t) = e^{-ct}\theta(t)$. We have that $F(0) = 0$ and $F(t) \geq 0$ on $[0, \epsilon)$. To show that $F(t) = 0$ on $[0, \epsilon)$, it is sufficient to show that $F'(t) \leq 0$ for all $t \in [0, \epsilon)$.

Suppose there exists $t \in [0, \epsilon)$, such that $F'(t) > 0$. Then (3.4) implies that $L_t(\phi) > 0$, and so, by Lemma 2.1, $\lambda_1(L_t) > 0$. Applying Lemma 2.2 to the operator L_t, where, in this case, $P = P_t = (\mu + J(\nu)) + \frac{1}{2}|\chi|^2 + \frac{1}{2}\theta^2 \geq 0$, $\Sigma_t \approx \Sigma$, carries a metric of positive scalar curvature, contrary to assumption.

Thus, $F(t) = 0$, and hence, $\theta(t) = 0$ for all $t \in [0, \epsilon)$. Since, by (3.2), $L_t(\phi) = \theta' - \tau\phi\theta = 0$, Lemma 2.1 implies $\lambda_1(L_t) \geq 0$ for each $t \in [0, \epsilon)$. Hence, by Lemma (2.2), we have that for each $t \in [0, \epsilon)$, $\chi_t = 0$ and Σ_t is Ricci flat. □

Theorem 3.1 has the following immediate consequence.

Theorem 3.2 *Let (V^n, h, K), $n \geq 3$, be an initial data set satisfying the DEC, $\mu \geq |J|$. If Σ^{n-1} is an outermost MOTS in (V^n, h, K) then Σ admits a metric of positive scalar curvature.*

We remark in closing that Theorems 3.1 and 3.2 may be viewed as local results in the following sense. Let Σ be a MOTS in an initial data set satisfying the DEC. By definition, Σ is 2-sided. As such, Σ admits a neighborhood U, within which Σ is separating. To apply Theorem 3.1 (resp., Theorem 3.2) it is sufficient that Σ be weakly outermost (resp., outermost) in U.

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