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The codimension 2 null cut locus with applications to spacetime topology

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ABSTRACT

In this paper, we review and extend some results in the literature pertaining to spacetime topology while naturally utilizing properties of the codimension 2 null cut locus. Our results fall into two classes, depending on whether or not one assumes the presence of horizons. Included among the spacetimes we consider are those that apply to the asymptotically (locally) anti-de sitter (AdS) setting.

I. INTRODUCTION

In this paper, we review and extend some results in the literature pertaining to spacetime topology from the point of view of the codimension 2 null cut locus, i.e., the null cut locus associated with a codimension 2 spacelike hypersurface in spacetime. Our results fall into two classes, depending on whether or not one assumes the presence of horizons. The results in which we do not assume the presence of horizons (as in Refs. 6, 7, 10, 14, and 22) may be interpreted as singularity results: Nontrivial topology (in a suitable sense) leads to future incompleteness. The results in which we do assume the presence of horizons (as in Refs. 4 and 18) pertain to the notion of topological censorship, which has to do with the idea that, outside of all black holes, one expects the topology to be simple (in a suitable sense) at the fundamental group level.

In Ref. 6, Costa e Silva presented a higher dimensional generalization of a well-known result of Gannon[10] and Lee,[14] in which the "enclosing surface" S (within a given spacelike hypersurface) is not required to be simply connected. We present a version of this result, in both the horizon and (as in Ref. 6) no horizon case. An essential step in the proof is to construct a certain covering spacetime. An issue arises in the covering construction in Ref. 6 (see also Ref. 22), which we address here (see Sec. III A).

An interesting effort in Ref. 6 was made to lower the causality assumption from global hyperbolicity to causal simplicity. However, subsequently, Costa e Silva and Minguzzi showed that causal simplicity does not necessarily lift to covers.[7] To address this, they assume that the spacetime and its covers are past reflecting (a property weaker than causal simplicity). As shown in Ref. 7, this assumption holds for spacetimes, which admit a past complete conformal timelike Killing field.

Our results apply to globally hyperbolic spacetimes and also to a class of spacetimes that are not, in general, globally hyperbolic, but which admit an "exterior foliation" by globally hyperbolic spacetimes-with-timelike-boundary; see Sec. III for details. The latter spacetimes model spacetimes that are asymptotically locally anti-de Sitter, without having to introduce conformal completions. Our proofs involving these spacetimes make use of a reflectivity assumption in the "base" but do not require that this assumption lifts to covers. As such, we circumvent the difficulties mentioned in the previous paragraph.

Let S be a closed separating hypersurface in a spacelike hypersurface V. In Sec. II, we study the null cut locus of S with respect to the "inward" future directed null normal geodesics to S. In later applications, we make a distinction between S-cut points and S-focal points. While S-focal points lift to covers, S-cut points, in general, do not. In later proofs, we use the fact that S-focal points lift to covers and that the existence of an S-focal point implies the existence of an S-cut point. One can then apply properties of the null cut locus of S, as discussed in Sec. II. Apart from its analytic properties, the null cut locus has a natural relationship to achronal boundaries. In Secs. III and IV, we discuss applications to spacetime topology in the no horizon and horizon case, respectively.

II. THE CODIMENSION 2 NULL CUT LOCUS AND COMPACTNESS RESULTS

A. Null cut locus and the s function

Unless otherwise stated, all objects are considered smooth.

Let (M, g) be a spacetime and let S be a codimension 2 spacelike submanifold. We define the *future null tangent bundle* of S as

$$\mathcal{N}S^+ = \{v \in T_pM \mid p \in S \text{ and } v \text{ is future directed null}\}.$$

By our convention, the zero vector is not null, and hence, the zero section does not lie in $\mathcal{N}S^+$. We give $\mathcal{N}S^+$ the subspace topology inherited from the tangent bundle TS.

Define $s : \mathcal{N}S^+ \to [0, \infty]$ by

$$s(v) = \sup\{t \geq 0 \mid d(S, \exp(tv)) = 0\},$$

where tv lies in the maximal domain of exp and where $d(S, \cdot) : J^+(S) \to [0, \infty]$ is the Lorentzian distance function restricted to S defined via

$$d(S, p) = \sup\{L(\gamma) \mid \gamma \text{ is a future directed causal curve from } S \text{ to } p\}.$$

Here, L is the Lorentzian arclength given by $L(\gamma) = \int \sqrt{-g(\gamma', \gamma')}$. If $v \in \mathcal{N}S^+$ and $\gamma(t) := \exp(tv)$ extends to $[0, s(v)]$, then we say $\gamma(s(v))$ is the *future null cut point of S along γ*. We will abbreviate this to $\gamma(s(v))$, *which is the S-cut point along γ.*

Proposition 1. Suppose $s(v) > 0$, and if $s(v) < \infty$, suppose $\gamma(s(v))$ is the S-cut point along γ, where $\gamma(t) = \exp(tv)$. Then, s is upper semicontinuous at v.

Proof. s is upper semicontinuous at v, provided $s(v) \geq \limsup_{v' \to v} s(v')$. This holds if and only if $s(v) \geq \limsup s(v_n)$ for all sequences $v_n \to v$. If $s(v) = \infty$, then we are done. Assume $s(v) < \infty$. Seeking a contradiction, suppose $v_n \to v$ and $s(v) < A := \limsup s(v_n)$. By moving to a subsequence, we can assume $s(v_n) \to A$. Define $\gamma_n(t) = \exp(tv_n)$. Choose $\delta > 0$ such that $b := s(v) + \delta < A$ and γ still extends to $[0, b]$. By moving to a subsequence, we can assume that γ_n is defined on $[0, b]$ as well. Since $b > s(v)$, there is a timelike curve from S to $\gamma(b)$. Therefore, $d(S, \gamma(b)) > 0$. However, by lower semicontinuity of the Lorentzian distance function (Ref. 2, Lemma 4.4), which holds for $d(S, \cdot)$ as well, we have

$$0 < d(S, \gamma(b)) \leq \liminf d(S, \gamma_n(b)) = \liminf 0 = 0,$$

which is a contradiction. □

Note that this proposition does not require any causality assumptions. It is used in the proofs of later results, in particular, in an essential way, in the Proof of Lemma 11.

The following proposition is given in Ref. 13 (Theorem 6.1). While not actually needed, in what follows: we include the proof for completeness, adding some relevant references.

Proposition 2. If (M, g) is globally hyperbolic and S is compact, then s is lower semicontinuous.

Proof. s is lower semicontinuous at v, provided $s(v) \leq \liminf_{v' \to v} s(v')$. This holds if and only if $s(v) \leq \liminf s(v_n)$ for all sequences $v_n \to v$. This holds trivially for those sequences that $\liminf s(v_n) = \infty$. Therefore, seeking a contradiction, suppose $v_n \to v$ and $s(v) > A := \liminf s(v_n)$ with $A < \infty$. By moving to a subsequence, we can assume $s(v_n) \to A$. Fix $\delta > 0$ so that $b := A + \delta < s(v)$. Define $b_n = s(v_n) + \delta$. Define $\gamma_n(t) = \exp(tv_n)$ and $\gamma(t) = \exp(tv)$. Since γ is defined at b, by restricting to a subsequence, we can assume that each γ_n is defined at b_n. Set $p = \gamma(0)$, $q = \gamma(A + \delta)$, $p_n = \gamma_n(0)$, and $q_n = \gamma_n(b_n)$. Since (M, g) is globally hyperbolic, a variation of Ref. 20 (Theorem 14.44) implies the existence of timelike geodesics σ_n from p_n to q_n issuing orthogonally from S, which maximize the Lorentzian distance from S to q_n. Since S is compact, by restricting to a further subsequence, we can assume $p_n \to p' \in S$. By a limit curve argument (Ref. 2, Corollary 3.32), there is a causal curve σ from p' to q.

Claim: $\sigma \neq \gamma|_{[0,b]}$. If this was not true, then the presence of σ_n implies that the normal exponential map on S would fail to be injective in a neighborhood of $\gamma|_{[0,b]}$. However, Proposition 10.30 of Ref. 20 implies that there must have existed a focal point of S along γ at or before $\gamma(b)$. Then, Theorem 10.51 of Ref. 20 implies that there is a timelike curve from S to $\gamma(b)$; hence, $s(v) < b$, which is a contradiction. This proves the claim. Since $\sigma \neq \gamma|_{[0,b]}$, a "cutting the corner" argument within a normal neighborhood of $\gamma(b)$ implies $s(v) \leq A + \delta$, which is a contradiction. □

Corollary 3. Suppose (M, g) is globally hyperbolic, and for all $v \in \mathcal{N}S^+$ with $s(v) < \infty$, suppose that $\gamma(s(v))$ is the S-cut point along γ, where $\gamma(t) = \exp(tv)$. If S is compact and acausal, then s is continuous.

Proof. By Propositions 1 and 2, it suffices to show $s(v) > 0$ for all $v \in \mathcal{N}S^+$. Suppose, to the contrary, $s(v) = 0$ for some $v \in \mathcal{N}S^+$. Since the normal exponential map is a local diffeomorphism, there is a neighborhood U of $\gamma(0)$ such that γ is the unique geodesic from S to any point on γ within U. By strong causality, there is neighborhood $V \subset U$ about $\gamma(0)$ such that if λ is a causal curve with endpoints in V, then $\lambda \subset U$. If $s(v) = 0$, then there would be a sequence of causal geodesics $\sigma_n : [0, b_n] \to M$ from S to $\gamma(1/n)$. By taking a subsequence, we can assume $\sigma_n(0) \to p$ for some $p \in S$. We have $p \neq \gamma(0)$ since the strong causality implies $\sigma_n(0) \notin V$. Therefore, a limit curve argument (Ref. 2, Corollary 3.32) implies the existence of a causal curve from p to $\gamma(0)$, but this contradicts acausality of S. □

B. Compactness results—the no horizon case

A key ingredient in the proofs of our spacetime topology results is to establish the compactness of certain regions within a given spacelike hypersurface (specifically, the set E_1 described below). While our strategy here is similar to some other works (e.g., Refs. 4, 6, 10, 14, and 18), our approach emphasizes the codimension 2 null cut locus.

Let (M, g) be a spacetime. Let V be an acausal connected spacelike hypersurface. Let $S \subset V$ be a compact connected separating codimension 2 surface. Then, $V \backslash S$ is disconnected. Let E_1' and E_2' form a separation for $V \backslash S$. Let $E_1 = E_1' \cup S$ and $E_2 = E_2' \cup S$. Then, E_1 and E_2 are closed sets. Moreover, $V = E_1 \cup E_2$ with $S = E_1 \cap E_2 = \partial_V E_1 = \partial_V E_2$. Connectedness of E_1 and E_2 follows from connectedness of V and S. Let n denote the unit future directed timelike normal vector field on V. Let ν denote the outward unit spacelike normal vector field on S where "outward" means ν points into the direction of E_2. Physically, one should think of S as a surface near "infinity."

For each $x \in S$, set $v_x = n_x - \nu_x$ and define $\gamma_x(t) = \exp_x(tv_x)$. Define

$$t_x = s(v_x) = \sup\{t \geq 0 \mid d(S, \gamma_x(t)) = 0\}.$$

Note that if γ_x is defined at t_x, then $\gamma_x(t_x)$ is the S-cut point along γ_x.

Definition 4. We define

$$W = \bigcup_{x \in S} \gamma_x(I_x),$$

where $I_x = [0, t_x]$ if γ_x is defined at t_x and $I_x = [0, t_x)$ if γ_x is not defined at t_x. Note that W is an achronal set.

Lemma 5. Assume that for each $x \in S$, there exists an S-cut point along γ_x. Then, W is compact.

Proof. We will show sequential compactness of W. Let q_n be any sequence in W. There are $t_n \in [0, t_{p_n}]$ such that $q_n = \gamma_{p_n}(t_n)$. By restricting to a subsequence, we can assume $p_n \to p \in S$. By upper semicontinuity of the s function (Proposition 1), the map $x \mapsto t_x$ is upper semicontinuous on S, and hence, there is a $t_{\max}$ such that $t_n \leq t_{\max}$ for all n. Therefore, by restricting to a further subsequence, we can assume $t_n \to t$. Since $t_n \leq t_{p_n}$, we have $t \leq t_p$ by upper semicontinuity. Let $q = \gamma_p(t)$. Then, $t \leq t_p$ implies $q \in W$. That $q_n \to q$ follows by continuity of the exponential map. □

Following Costa e Silva,[6] we say that a spacetime (M, g) admits a *piercing* for V if there is a future directed timelike vector field X on M such that each maximal integral curve of X intersects V at exactly one parameter value. We call X a *piercing vector field* for V. A spacetime is *past reflecting* if "$I^+(q) \subset I^+(p)$ implies $I^-(p) \subset I^-(q)$ for all p and q," which is equivalent to "$q \in \overline{J^+(p)}$ implies $p \in \overline{J^-(q)}$ for all p and q." See Ref. 19. Note that past reflectivity is implied by closure of $J^+(p)$ for all p, which holds, for example, in globally hyperbolic spacetimes.

Proposition 6. Suppose (M, g) is past reflecting and admits a piercing for V. Then,

$$W = \partial I^+(E_2) \backslash \mathrm{int}_V E_2.$$

Remark. Any future directed timelike vector field in a globally hyperbolic spacetime is a piercing for any of its Cauchy surfaces. Therefore, Proposition 6 holds for globally hyperbolic spacetimes.

Proof. Let X be a piercing vector field for V normalized to $h(X, X) = 1$, where h is a background complete Riemannian metric on M. Since the maximal integral curves are inextendible as continuous curves, the integral curves of X have domain $\mathbb{R}$. Let $\phi : \mathbb{R} \times M \to M$ denote the flow of X. Let $\phi_V : \mathbb{R} \times V \to M$ denote the restriction of ϕ to $\mathbb{R} \times V$. The piercing assumption and the fact that integral curves do not intersect imply that ϕ_V is bijective and hence is a diffeomorphism (Ref. 16, Theorem 9.20). Let $\pi : \mathbb{R} \times V \to V$ denote the natural projection onto V. Put $r = \pi \circ \phi_V^{-1}$. Then, $r : M \to V$ is a smooth retraction of M onto V. Finally, let $\mathcal{C}$ denote the timelike cylinder formed by the image of the integral curves of X through S. That is,

$$\mathcal{C} = r^{-1}(S) = \{\phi(t,x) \mid t \in \mathbb{R} \text{ and } x \in S\}.$$

Set $T = \partial I^+(E_2) \setminus \mathrm{int}_V E_2$. We want to show $T = W$.

We first show $W \subset T$. Fix $q \in W$. Then, $q = \gamma_x(t)$ for some $t \in [0, t_x]$. If $t = 0$, then $q \in S$ and so acausality of V implies $q \in T$. Now, suppose $t > 0$. Then, $q \notin \mathrm{int}_V\, E_2$ since V is acausal; hence, it suffices to show $q \notin I^+(E_2)$. Suppose $q \in I^+(E_2)$. We will obtain a contradiction by showing $q \in I^+(S)$. Let $\lambda : [0,b] \to M$ denote a timelike curve from E_2 to q. If $\lambda(0) \in S$, then we are done, so we can assume $\lambda(0) \notin S$. Either $r(q) \in E_1$ or $r(q) \in E_2$. First assume $r(q) \in E_1$. Then, $r \circ \lambda$ is a path in V that begins in E_2 and ends in E_1. Since S separates, there is a $t_* \in (0, b]$ such that $r \circ \lambda(t_*) \in S$. Thus, $\lambda(t_*) \in \mathcal{C}$. Therefore, $q \in I^+(S)$ follows by concatenating the integral curve on $\mathcal{C}$ from S to $\lambda(t_*)$ with the rest of λ. Now, suppose $r(q) \in E_2$. Since $r \circ \gamma_x(t') \in E_1$ for sufficiently small t', there is a $t_* \in (0, t_x)$ such that $r \circ \gamma_x(t_*) \in S$. Hence, $\gamma_x(t_*) \in \mathcal{C}$, which implies $q \in I^+(S)$.

Next, we show $T \subset W$. Define $\mathcal{H}^+$ to be those points in $J^+(S) \setminus I^+(S)$, which can be reached by γ_x for some $x \in S$. Using past reflectivity, it follows from a key argument in the Proof of Theorem 3.5 in Ref. 7 that $T \subset \mathcal{H}^+$. Therefore, $T \subset J^+(S) \setminus I^+(S)$. Fix $q \in T$. If $q \in S$, then $q = \gamma_q(0)$ and so $q \in W$. If $q \notin S$, then there is a null geodesic γ_x such that $q = \gamma_x(t)$ for some $t > 0$. It suffices to show $t \le t_x$. If $t_x = \infty$, then we are done. If $t_x < \infty$ and $t > t_x$, then there is a timelike curve from S to q, which is a contradiction. □

Proposition 7. Suppose (M, g) *is past reflecting and admits a piercing for* V*. In addition, suppose that for each* $x \in S$*, there is an* S*-cut point along* γ_x*. Then,* E_1 *is compact.*

Proof. Let $r : M \to V$ denote the retraction from M to V from the Proof of Proposition 6. Let $W' = W \setminus S$ and $E_1' = E_1 \setminus S$. Let $\tau = r|_W$ and $\tau' = r|_{W'}$ denote the restrictions of r to W and W'. From Proposition 6, we have $\tau(W) \subset E_1$. Moreover, τ is injective since the integral curves do not intersect. Since W' is a topological hypersurface, Brower's invariance of domain theorem implies that τ' is an open map. Since τ is just the identity on S, it follows that τ is an open map as well. Therefore, $\tau(W)$ is open in E_1. Since W is compact by Lemma 5, it follows that $\tau(W)$ is closed in E_1. Thus, $\tau(W) = E_1$ by connectedness of E_1, and hence, E_1 is compact. □

In applications to spacetime topology, it is important to identify which notions on a spacetime manifold lift to covers (cf. Ref. 7). An S-focal point, as defined in Ref. 20 (Definition 10.29), will lift to covers by Proposition 10.30 of Ref. 20. S-cut points, on the other hand, do not necessarily lift to covers, as illustrated in the example below. However, by Proposition 10.48 of Ref. 20, an S-focal point implies the existence of an S-cut point, which is frequently used in our proofs.

Let S^2 be the standard round sphere, and let $V = \mathbb{R} \times S^2$, with standard product metric h. By identifying each of the points $(s, p) \in \mathbb{R} \times S^2$ with $(-s, -p) \in \mathbb{R} \times S^2$, we obtain the manifold V', which is diffeomorphic to RP^3 minus a point and which inherits a metric h' from h. The sphere at $s = 0$ now corresponds to an RP^2. Now, consider the spacetime $(M', g') = (\mathbb{R} \times V', -dt^2 \oplus h')$, and let S' be a sphere in $V' \equiv \{0\} \times V'$ "parallel" to the RP^2. Each inward future directed null normal geodesic to S' comes back to S' in the future, which implies that there is an S'-cut point along each inward future directed null normal geodesic to S'. Now, consider the universal cover $(M, g) = (\mathbb{R} \times V, -dt^2 \oplus h)$. Let S be one of the two 2-spheres in $V \equiv \{0\} \times V$ covering S'. Now, the inward future directed null normal geodesics to S never return to S, and there is no S-cut point along any inward future directed null normal geodesic to S. All cut points "disappear" in the cover.

Proposition 7 along with Proposition 10.48 (Ref. 20) imply the following corollary:

Corollary 8. Suppose (M, g) *is past reflecting and admits a piercing for* V*. In addition, suppose that for each* $x \in S$*, there is an* S*-focal point along* γ_x*. Then,* E_1 *is compact.*

Remark. Typical ways one ensures the existence of an S-focal point are through conditions on the (inward) null expansion on S. The most basic situation is the following. If we assume that (i) the null energy condition holds, $\mathrm{Ric}(X, X) \ge 0$ for all null vector X, and (ii) S has negative inward null expansion, $\theta_- := \mathrm{div}_S\, v < 0$ (with v as in the beginning of Sec. II B), then there is an S-focal point along each future complete γ_x. More generally, there exists an S-focal point along each future complete γ_x, provided (see Ref. 9, Proposition 2),

$$\int_0^\infty \mathrm{Ric}(\gamma_x', \gamma_x')\, ds > \theta_-(x). \tag{2.1}$$

As a simple application of these ideas, we consider the following generalization of Theorem 1 of Ref. 14, (see also Ref. 10).

Proposition 9. Suppose (M, g) *is past reflecting and admits a piercing for* V*. Let* $\mathcal{C}$ *be the "timelike cylinder" formed by the union of the integral curves from* S *generated by the piercing vector field. If for all* $x \in S$*, either* γ_x *meets* $\mathcal{C} \setminus S$ *or is future complete and satisfies (2.1), then* E_1 *is compact.*

Proof. If γ_x meets $\mathcal{C}\backslash S$, then there is a $t_x < \infty$ such that $\gamma_x(t_x)$ is a future S-cut point of S along γ_x. Suppose γ_x does not meet $\mathcal{C}\backslash S$. Then, γ_x is future complete, and hence, by (2.1), there is a t such that $\gamma_x(t)$ is an S-focal point along γ_x. Therefore, γ cannot be maximizing past $\gamma_x(t)$ by Proposition 10.48 of Ref. 20, and hence, $t_x < t$. Then compactness of E_1 follows by Proposition 7. □

C. Compactness results—the horizon case

In this section, we extend the main compactness results of Sec. II B to allow for the presence of horizons (as described below) so as to model the presence of black holes. This will eventually lead to results about the topology of certain regions outside of these black holes (Fig. 1).

Throughout this section, we consider a spacetime (M, g) with the following four properties:

1. There is an acausal connected spacelike hypersurface V.
2. There exists a codimension 2 surface $\Sigma \subset V$, which separates V. (Σ may have multiple components.) Let B' and E' form a separation for $V\backslash\Sigma$. Set $B = B' \sqcup \Sigma$ and $E = E' \sqcup \Sigma$. Note that Σ is closed, and hence, B and E are closed. Then,
$$V = B \cup E \quad \text{and} \quad \Sigma = B \cap E.$$
We further assume that $B \approx \Sigma \times [0, \epsilon)$. ($V$ only slightly enters the black hole region.)
3. There exists a compact connected codimension 2 surface $S \subset V$, which separates $E\backslash\Sigma$. Let E_1' and E_2' form a separation for $E\backslash(\Sigma \sqcup S)$. Set $E_1 = E_1' \sqcup S \sqcup \Sigma$ and $E_2 = E_2' \sqcup S$. Hence,
$$E = E_1 \cup E_2 \quad \text{and} \quad S = E_1 \cap E_2.$$
Connectedness of E_1 and E_2 follows from connectedness of V and S.
4. Define $H := \partial I^+(B)\backslash\text{int}_V B$. We assume that the null generators of H never leave H when made future inextendible.

Remarks.

- When interpreting this model physically, one should view Σ as the surface of a black hole and S as a surface surrounding Σ.
- The assumption that the null generators of H never leave H holds whenever H makes up part of an event horizon in the traditional sense [i.e., whenever $H = \big(\partial I^-(\mathcal{J}) \cap M\big) \cap J^+(V)$, where $\mathcal{J}$ is a conformal boundary].

As in Definition 4, set $W = \bigcup_{x\in S}\gamma_x(I_x)$. In addition, as in Sec. II B, we define the upper semicontinuous function,

$$x \mapsto t_x = s(v_x) = \sup\{t \geq 0 \mid d(S, \gamma_x(t)) = 0\},$$

so that $\gamma_x(t_x)$ is an S-cut point along γ_x, provided γ_x is defined at t_x. Under the assumptions of past reflecting and a piercing, we have that W is an achronal topological hypersurface with boundary via the Proof of Proposition 6. This will be used in the Proof of Proposition 12.

Henceforth, we assume that for each $x \in S$, either γ_x crosses H or there exists an S-cut point along γ_x. When we say γ_x *crosses* H, we mean that γ_x intersects one of the null generators of H (and hence does so transversely).

Define $S_H = \{x \in S \mid \gamma_x \cap H \neq \emptyset\}$. *A priori* S_H could be the empty set, but under our considerations, it will be nonempty.

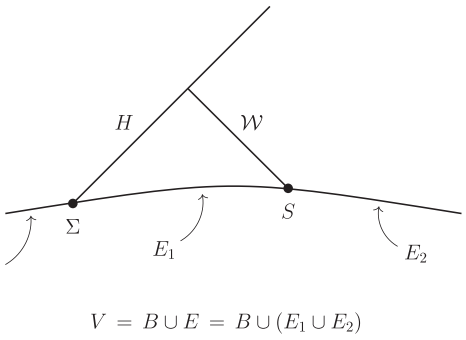

FIG. 1. The set up for this section. The black hole horizon is represented by $H = \partial I^+(B)\backslash\text{int}_V B$.

Definition 10. Let $\mathcal{W} \subset W$ be given by

$$\mathcal{W} = \bigcup_{x\in S}\gamma_x([0,\tau_x]),$$

where

$$\tau_x = \begin{cases} t_x & \text{if } x \in S_H^c, \\ \min\{t_x, h_x\} & \text{if } x \in S_H \end{cases}$$

and where h_x is the unique parameter value such that $\gamma_x(h_x) \in H$. Here, $S_H^c = S\backslash S_H$. Note that $\tau_x < \infty$ for all $x \in S$ by assumption.

Lemma 11. Suppose for each $x \in S$, either γ_x crosses H or there exists an S-cut point along γ_x. Then, $\mathcal{W}$ is compact and meets H.

Proof. We first show that $\mathcal{W}$ is compact. Then, we show that $S_H \neq \emptyset$ (which means $\mathcal{W}$ meets H).

A similar argument as used in Refs. 4 and 18 shows that S_H is open, and hence, S_H^c is closed and, therefore, compact. Since $\partial S_H \subset S_H^c$, it follows from the assumption that $\gamma_x(t_x)$ is an S-cut point along γ_x for each $x \in \partial S_H$. By continuity of the exponential map, for each $x \in \partial S_H$, there is an $\epsilon > 0$ and a neighborhood $U_x \subset S$ of x such that γ_y is defined on $[0, t_x + 2\epsilon]$ for all $y \in U_x$. By upper semicontinuity of $x \mapsto t_x$, we can assume that the neighborhood U_x also satisfies the following property: for each $y \in \overline{U}_x$, we have $t_y < t_x + \epsilon$. Let $\{U_1, \dots, U_N\} \subset \{U_x\}_{x\in\partial S_H}$ be a finite subcover of ∂S_H. Set $A = S_H^c \cup \overline{U}_1 \cup \cdots \cup \overline{U}_N$ and $A^c = S\backslash A$. Note that A is compact and $\overline{A^c} \subset S_H$ is a proper subset. Moreover, note that $\gamma_x(t_x)$ is defined for all $x \in A$ by construction.

We now prove the sequential compactness of $\mathcal{W}$. Let q_n be any sequence in $\mathcal{W}$. There are $t_n \in [0, \tau_{p_n}]$ such that $q_n = \gamma_{p_n}(t_n)$. By restricting to a subsequence, we can assume $p_n \to p \in S$. Either infinitely many points of p_n lie in A or infinitely many points of p_n lie in A^c. Hence, by restricting to a subsequence, there are two cases: either the sequence p_n lies in A or the sequence p_n lies in A^c.

Assume that the sequence p_n lies in A. In this case, we have $t_n \le \tau_{p_n} \le t_{p_n} < t_* + \epsilon$ for all n. Thus, by restricting to a subsequence, we can assume $t_n \to t \le t_* + \epsilon$. By upper semicontinuity, we have $t \le t_p$. Since A is compact, we have $p \in A$, and so $\gamma_p(t_p)$ is defined and, hence, $\gamma_p(t)$ is defined. Set $q = \gamma_p(t)$. If $p \in S_H^c$, then $\tau_p = t_p$; hence, $t \le \tau_p$, and so $q \in \mathcal{W}$. If $p \notin S_H^c$, it suffices to show that $t \le h_p$. If $t > h_p$, then $q \in I^+(B)$ (since the γ_p intersects H transversely), which implies infinitely many $q_n \in I^+(B)$, which is a contradiction.

Assume that the sequence p_n lies in A^c. As shown in Refs. 4 and 18, the function $x \mapsto h_x$ is continuous. Since $\overline{A^c} \subset S_H$, $\sup\{h_{p_n}\} < \infty$. Since $t_n \le \tau_{p_n} \le h_{p_n}$, consequently, $t_n \to t$ when restricted to a subsequence. Moreover, taking $n \to \infty$, we have $t \le h_p$. Therefore, $q := \gamma_p(t)$ is defined. Upper semicontinuity implies $t \le t_p$. Thus, $t \le \tau_p$ and so $q \in \mathcal{W}$.

We now show $S_H \neq \emptyset$. Since H is an achronal boundary, it separates $I^+(V)$ into a future and past set $F = I^+(B)$ and $P = I^+(V)\backslash(F \sqcup H)$.[21] Seeking a contradiction, suppose $\mathcal{W}$ does not meet H. Then, $\mathcal{W}' \subset P$, where $\mathcal{W}' = \mathcal{W}\backslash S$. Let $r : M \to V$ denote the flow map of the piercing of V as in Proposition 6. By properties of achronal boundaries, we have $r(P) \subset E\backslash\Sigma$, and hence, $r(\mathcal{W}) \subset E\backslash\Sigma$. By Proposition 6, we have that $\mathcal{W}$ is a compact achronal C^0 hypersurface with $\operatorname{edge}(\mathcal{W}) = S$; the contradiction now follows as in Refs. 4 and 18 since $\operatorname{edge}(E_1) = \Sigma \cup S$. □

Analogous to Proposition 7 and Corollary 8, we have the following proposition and corollary:

Proposition 12. Assume (M, g) is past reflecting and admits a piercing for V. Suppose that for each $x \in S$, either γ_x crosses H or there exists an S-cut point along γ_x. If $H \subset J^+(\Sigma)\backslash I^+(\Sigma)$; then, E_1 is compact.

Proof. The proof follows from similar arguments as in Refs. 4 and 18, which we sketch. Let H_0 denote the union of the components of H, which meet $\mathcal{W}$. Note that H_0 is nonempty by Lemma 11. Using the fact that the null generators of H never leave H when made future inextendible, it follows that H_0 is a smooth null hypersurface with boundary Σ_0 [here, we have used $H \subset J^+(\Sigma)\backslash I^+(\Sigma)$], which consists of connected components of Σ. Moreover, since the null generators of H_0 meet W transversely, it can be shown by an open and closed argument as in Refs. 4 and 18 that each null generator of H_0 issuing from Σ_0 meets $\mathcal{W}$ exactly once.

Let $H_0' \subset H_0$ denote the portion of H_0 comprising the null geodesic generators of H_0, which terminate when they intersect $\mathcal{W}$. As suggested in Fig. 1, $H_0' \cup \mathcal{W}$ forms a compact achronal topological hypersurface with boundary $\Sigma_0 \cup S$; we now sketch a proof of this. Compactness follows from compactness of $\mathcal{W}$ (Lemma 11) and compactness of H_0'; the latter follows from an analogous argument as in Refs. 4 and 18, which uses compactness of the intersection $H_0' \cap \mathcal{W}$, which implies that Σ_0 is compact. Using causal theoretic techniques, it is easily shown that $H_0' \cup \mathcal{W}$ is achronal. Finally, that $H_0' \cup \mathcal{W}$ is a topological hypersurface with boundary follows from recognizing that H_0 and W are topological hypersurfaces with boundary, and hence, away from the boundary, each can be locally modeled by graphing functions as in Ref. 20 (Lemma 14.25); then, for each point in $H_0' \cap \mathcal{W}$, there is a neighborhood in $H_0' \cup \mathcal{W}$ that agrees with the set obtained by taking the minimum of the graphing functions associated with H_0 and W.

The integral curves of the piercing vector field must map points of $H_0' \cup \mathcal{W}$ into E_1. It then follows from a connectedness argument using invariance of domain that the piercing vector field maps $H_0' \cup \mathcal{W}$ onto E_1; hence, E_1 is compact. □

Corollary 13. Assume (M,g) is past reflecting and admits a piercing for V. Suppose that for each $x \in S$, either γ_x crosses H or there exists an S-focal point along γ_x. If $H \subset J^+(\Sigma)\backslash I^+(\Sigma)$, then E_1 is compact.

The following gives a sufficient condition for when $H \subset J^+(\Sigma)\backslash I^+(\Sigma)$. This will be used in the proofs of our topological censorship results in Sec. IV.

Proposition 14. Assume (M,g) is past reflecting and admits a piercing for V. If Σ is compact, then $H \subset J^+(\Sigma)\backslash I^+(\Sigma)$.

Proof. This follows from an analogous argument as in the Proof of Proposition 6 (specifically, the part where it is shown that $T \subset W$). The essential part of that proof is based on the Proof of Theorem 3.5 in Ref. 7. □

III. APPLICATIONS TO SPACETIME TOPOLOGY—THE NO HORIZON CASE

The results we present on spacetime topology, in both the horizon and no horizon cases, use a proof strategy similar to a number results in the literature; cf. Refs. 4, 6, 7, 10, 14, 18, and 22. One of our main results, first considered by Costa e Silva[6] in the no horizon setting (see also Ref. 7), is to allow S (as in the beginning of Sec. II B) to have a nontrivial fundamental group. An essential step in the proof is to construct a certain covering spacetime. This is discussed in Sec. III A in the no horizon case. As mentioned in the Introduction, our construction addresses a certain issue with the covering construction in Ref. 6.

A. Gluing constructions

In this section, we describe the gluing constructions in the no horizon case.

Let (M,g) be a spacetime. Let V be an acausal connected spacelike hypersurface. Suppose (M,g) admits a piercing for V. Let $S \subset V$ be a compact connected separating codimension 2 surface. Then, $V\backslash S$ is disconnected. Let E_1' and E_2' form a separation for $V\backslash S$. Let $E_1 = E_1' \cup S$ and $E_2 = E_2' \cup S$. Then, E_1 and E_2 are closed sets. Moreover, $V = E_1 \cup E_2$ with $S = E_1 \cap E_2 = \partial_V E_1 = \partial_V E_2$. Connectedness of E_1 and E_2 follows from connectedness of V and S.

Let

$$i : S \to V \quad i_1 : S \to E_1 \quad i_2 : S \to E_2$$

denote the inclusion maps of S into V, E_1, and E_2.

Let $p : \widetilde{E}_1 \to E_1$ be a covering of E_1 such that $p_*\pi_1(\widetilde{E}_1, \hat{x}) = i_{1*}\pi_1(S,x)$ with $x \in S$ and $\hat{x} \in \widetilde{S} := p^{-1}(S)$ (such a covering exists by Corollary 12.19 of Ref. 15). In the general relativity literature, p is known as the *Hawking cover*. Slightly abusing notation, we wish to construct a cover $p : \widetilde{V} \to V$, which extends the covering $p : \widetilde{E}_1 \to E_1$ by gluing appropriate coverings of E_2 to each connected component of $\widetilde{S}$ along the boundary of the covering of E_2. This covering will be used to establish spacetime topology results. The next two propositions will aid us in the construction of $\widetilde{V}$.

Proposition 15.

1. *If $\hat{S}$ is the connected component of $\widetilde{S}$ containing $\hat{x}$, then $\hat{S}$ is isometric to S.*
2. *If $\widetilde{S}$ is connected, then $i_{1*} : \pi_1(S,x) \to \pi_1(E_1,x)$ is onto.*

Proof. We first prove (1). The restriction $p|_{\hat{S}} : \hat{S} \to S$ is a surjective local isometry; hence, it suffices to show that it is injective. Fix $\hat{y}, \hat{z} \in \hat{S}$ such that $p(\hat{y}) = p(\hat{z}) = y \in S$. Let $\hat{\alpha} \subset \hat{S}$ be a curve connecting $\hat{y}$ to $\hat{x}$ to $\hat{z}$. Let $\alpha = p \circ \hat{\alpha}$. Then, $\alpha \subset S$ is a loop based at y, which passes through x, and hence, we can assume that α is based at x. Since $p_*\pi_1(\widetilde{E}_1, \hat{x}) = i_{1*}\pi_1(S,x)$, there is a loop $\widetilde{\beta} \subset \widetilde{E}_1$ based at $\hat{x}$ such that $\beta = p \circ \widetilde{\beta}$ is homotopic to α. Since homotopies lift, $\widetilde{\beta}$ is homotopic to the curve formed by joining $\hat{x}$ to $\hat{z}$ and $\hat{y}$ to $\hat{x}$ via $\hat{\alpha}$. However, the latter is a curve only if $\hat{z} = \hat{y}$.

Now, we prove (2). Assume that $\widetilde{S}$ is connected. Let β be a loop in E_1 based at $x \in S$. Since $\widetilde{S}$ is isometric to S by (1), the lift of $\widetilde{\beta}$ is a loop based at $\widetilde{x} = p^{-1}(x) \in \widetilde{S}$. Then, by hypothesis, there is a loop $\alpha \subset S$ based at x such that β is homotopic to α. □

Remarks.

- Haggman *et al.*[11] observed (1) based on a common path construction technique for defining the covering $p : \widetilde{E}_1 \to E_1$. As examples show (see, e.g., Example 2.1 of Ref. 11), other components of $\widetilde{S} = p^{-1}(S)$ need not be isometric to S. This is only assured for the component of $p^{-1}(S)$ containing the base point; cf. also the proof of Proposition 14.48 of Ref. 20, where the connected component containing the base point is used. This rather subtle point appears to have been overlooked in the covering construction in Ref. 6.
- There is a situation in which each connected component of $\widetilde{S}$ is isometric to S. Specifically, if $p : \widetilde{E}_1 \to E_1$ is a normal covering, i.e., if $p_*\pi_1(\widetilde{E}_1, \hat{x})$ is a normal subgroup of $\pi_1(E_1, x)$, it follows from Theorem 11.34 of Ref. 15 that each connected component of $\widetilde{S}$ is isometric to S.

Let $\overline{S}$ be a connected component of $\widetilde{S}$. Then, $p|_{\overline{S}} : \overline{S} \longrightarrow S$ is a covering of S. Let $\phi = i_2 \circ p|_{\overline{S}}$ with $\phi(\overline{s}) = x$. Let $q : \overline{E}_2 \to E_2$ be a covering of E_2 such that $q_*\pi_1(\overline{E}_2, \overline{x}) = \phi_*\pi_1(\overline{S}, \overline{s})$ with $\overline{x} \in \partial\overline{E}_2$ satisfying $q(\overline{x}) = x$. By the general lifting criterion, there is a unique lift $\overline{\phi} : \overline{S} \to \overline{E}_2$ of $\phi : S \to E_2$ with $\overline{\phi}(\overline{s}) = \overline{x}$, as illustrated in the following diagram:

$$\begin{tikzcd}
\overline{S} \arrow[r, "\overline{\phi}"] \arrow[d, "p|_{\overline{S}}"'] \arrow[dr, "\phi"] & \overline{E}_2 \arrow[d, "q"] \\
S \arrow[r, "i_2"'] & E_2.
\end{tikzcd}$$

Proposition 16. If $i_{2*} : \pi_1(S, x) \to \pi_1(E_2, x)$ is an isomorphism, then $\overline{\phi}$ is an isometry onto $\partial\overline{E}_2$.

Proof. We first show that $\partial\overline{E}_2$ is connected. Since $\partial\overline{E}_2 = q^{-1}(\partial E_2) = q^{-1}(S)$, it suffices to show that $q^{-1}(S)$ is connected. Seeking a contradiction, suppose $\overline{S}_1$ and $\overline{S}_2$ are two disjoint connected components of $q^{-1}(S)$. Let $\overline{x}_1 \in \overline{S}_1$ and $\overline{x}_2 \in \overline{S}_2$ with $q(\overline{x}_1) = q(\overline{x}_2) = x$. Let $\overline{\alpha}$ be a curve in $\overline{E}_2$ from $\overline{x}_1$ to $\overline{x}_2$. Then, $\alpha = q \circ \overline{\alpha}$ is a loop in E_2 based at x. Since i_{2*} is onto, there is a loop β in S based at x, which is homotopic to α. Let $\overline{\beta}$ denote the lift of β starting at $\overline{x}_1$. Since homotopies lift, $\overline{\beta}$ is homotopic to $\overline{\alpha}$, and hence, $\overline{\beta}$ has endpoint $\overline{x}_2$. However, $\overline{\beta}$ cannot leave $\overline{S}_1$, which is a contradiction.

Set $p' = p|_{\overline{S}}$ and $q' = q|_{\partial\overline{E}_2}$. Note that p' and q' are covering maps (we required connectedness of $\partial\overline{E}_2$ here.). Let $\overline{\phi}' : \overline{S} \to \partial\overline{E}_2$ be the map satisfying $\overline{\phi} = \overline{i_2} \circ \overline{\phi}'$, where $\overline{i_2} : \partial\overline{E}_2 \to \overline{E}_2$ denotes the inclusion map. Then, we have the following commutative diagram:

$$\begin{tikzcd}
\overline{S} \arrow[rr, "\overline{\phi}'"] \arrow[dr, "p'"'] & & \partial\overline{E}_2 \arrow[dl, "q'"] \\
& S. &
\end{tikzcd}$$

By the covering isomorphism criterion (Ref. 15, Theorem 11.40), it suffices to show that $p'_*\pi_1(\overline{S}, \overline{s}) = q'_*\pi_1(\partial\overline{E}_2, \overline{x})$. Since

$$\begin{aligned} q_*\pi_1(\overline{E}_2, \overline{x}) &= \phi_*\pi_1(\overline{S}, \overline{s}) \\ &= i_{2*} \circ p'_*\pi_1(\overline{S}, \overline{s}), \end{aligned}$$

injectivity of i_{2*} gives

$$p'_*\pi_1(\overline{S}, \overline{s}) = i_{2*}^{-1} \circ q_*\pi_1(\overline{E}_2, \overline{x}).$$

Therefore, it suffices to show

$$i_{2*}^{-1} \circ q_*\pi_1(\overline{E}_2, \overline{x}) = q'_*\pi_1(\partial\overline{E}_2, \overline{x}).$$

Both the left and right inclusion of the above equality follow from the following commutative diagram:

$$\begin{tikzcd}
\overline{S} \arrow[r, "\overline{\phi}'"] \arrow[dr, "p'"'] & \partial\overline{E}_2 \arrow[r, "\overline{i_2}"] \arrow[d, "q'"'] & \overline{E}_2 \arrow[d, "q"] \\
& S \arrow[r, "i_2"'] & E_2.
\end{tikzcd}$$

To see the right inclusion, fix $[\alpha] \in q'_*\pi_2(\partial\overline{E}_2, \overline{x})$. Then, there is a loop $\overline{\alpha} \subset \partial\overline{E}_2$ such that $[\alpha] = [q' \circ \overline{\alpha}]$. Since i_{2*} is injective, we have $[\alpha] = i_{2*}^{-1}([q \circ \overline{i_2} \circ \overline{\alpha}])$, which shows the right inclusion.

To see the left inclusion, fix $[\alpha] \in i_{2*}^{-1} \circ q_*\pi_1(\overline{E}_2, \overline{x})$. Then, there is a loop $\overline{\alpha} \subset \overline{E}_2$ such that $[i_2 \circ \alpha] = [q \circ \overline{\alpha}]$. Since $q_*\pi_1(\overline{E}_2, \overline{x}) = \phi_*\pi_1(\overline{S}, \overline{s})$, there is a loop $\beta \subset \overline{S}$ such that $[q \circ \overline{\alpha}] = [\phi \circ \beta]$. Thus,

$$[\alpha] = [i_2^{-1} \circ \phi \circ \beta] = [i_2^{-1} \circ (q \circ \overline{i_2} \circ \overline{\phi}') \circ \beta] = [q' \circ \overline{\phi}' \circ \beta],$$

which shows the left inclusion. □

We can now construct the desired covering $p : \widetilde{V} \to V$. Let $p : \widetilde{E}_1 \to E_1$ be the covering described above Proposition 15. Let $\hat{S}$ be the connected component of $\widetilde{S}$ containing $\hat{x}$. By Proposition 15(1), $\hat{S}$ is isometric S. Therefore, we glue an isometric copy of E_2 along $\hat{S}$ in the same way they are glued in the base space V. Let $\overline{S}$ be another connected component of $\widetilde{S}$. Let $q : \overline{E}_2 \to E_2$ be the covering described above Proposition 16. Assuming i_{2^*} is an isomorphism, Proposition 16 implies that we can glue $\overline{E}_2$ along $\overline{S}$. Performing this gluing for each connected component $\overline{S}$ of $\widetilde{S}$ yields the space $\widetilde{V}$ along with the covering map $p : \widetilde{V} \to V$ (we are slightly abusing notation here). Note that $p|_{\overline{E}_2} = q$ and $p|_{\widetilde{E}_1}$ is just the covering we started with.

Finally, we note that if S is simply connected, then we do not need to assume that i_{2*} is an isomorphism (although it is automatically injective). This follows because surjectivity of i_{2*} was used in Proposition 16 to ensure that $\partial\overline{E}_2$ is connected. In the case that S is simply connected, each connected component $\overline{S}$ of $\widetilde{S}$ is isometric to S, and hence, in this case we glue a copy of E_2 along $\overline{S}$ in the same way they are glued in the base space V; we do not need to assume surjectivity of i_{2*} since ∂E_2 is automatically connected.

B. Spacetime topology à la Gannon–Lee

Let $p : \widetilde{V} \to V$ be the covering constructed in Sec. III A. We now construct a corresponding spacetime covering $P : \widetilde{M} \to M$ such that $\widetilde{V}$ embeds into $\widetilde{M}$ and $P|_{\widetilde{V}} = p$. We let $P : \widetilde{M} \to M$ be the covering such that $P_*\pi_1(\widetilde{M}, \widetilde{a}) = \psi_*\pi_1(\widetilde{V}, \widetilde{x})$, where $\psi = \iota \circ p$ and $\iota : V \to M$ is the inclusion map. From the general map lifting criterion, we obtain the following commutative diagram:

$$\begin{tikzcd}
\widetilde{V} \arrow[r, "\overline{\psi}"] \arrow[d, "p"'] \arrow[dr, "\psi"'] & \widetilde{M} \arrow[d, "P"] \\
V \arrow[r, "\iota"'] & M.
\end{tikzcd}$$

Since $\overline{\psi}$ is an embedding,[24] we can identify $\widetilde{V}$ with its image in $\widetilde{M}$. This completes the construction of the desired spacetime covering.

Since the local geometries are preserved, $\widetilde{V}$ is a connected spacelike hypersurface within $\widetilde{M}$. Moreover, $\widetilde{V}$ is acausal within $\widetilde{M}$ since any causal curve connecting two points on $\widetilde{V}$ would project down to a causal curve in M connecting two points on V, which would contradict acausality of V within M. Finally, any piercing vector field X on M for V can be lifted to a piercing vector field $\widetilde{X}$ on $\widetilde{M}$ for $\widetilde{V}$ via the local isometry P. That $\widetilde{X}$ is a piercing vector field for $\widetilde{V}$ follows from X being one for V.

As in Sec. III A, let

$$i : S \to V \quad i_1 : S \to E_1 \quad i_2 : S \to E_2$$

denote the inclusion maps of S into V, E_1, and E_2.

The following theorem generalizes a well-known result of Gannon Ref. 10, Corollary 1.20; see also Lee Ref. 14, Theorem 5. Indeed, it is an adaptation of Ref. 6, Theorem 2.1 (see also Refs. 7 and 22). Our modification of the covering space argument in Ref. 6 (see the remark after Proposition 15) has required a strengthening of the assumption on i_{2*}; see Proposition 16. The assumption of global hyperbolicity used will be weakened some in Sec. III C.

Theorem 17. *Suppose V is a Cauchy surface for (M, g), and suppose there exists an S-focal point along γ_x for each $x \in S$. In the case that S is not simply connected, assume $i_{2*}: \pi_1(S, x) \to \pi_1(E_2, x)$ is an isomorphism. If E_2 is noncompact, then $i_{1*}: \pi_1(S, x) \to \pi_1(E_1, x)$ is onto. In particular, E_1 is simply connected whenever S is simply connected and E_2 is noncompact.*

It follows from Seifert–Van Kampen that $i_{1*}: \pi_1(S, x) \to \pi_1(V, x)$ is onto.

Proof. Let $p : \widetilde{E}_1 \to E_1$ be the covering of E_1 such that $p_*\pi_1(\widetilde{E}_1, \hat{x}) = i_{1*}\pi_1(S, x)$ with $\hat{x} \in \widetilde{S} := p^{-1}(S)$. Slightly abusing notation, let $p : \widetilde{V} \to V$ denote the covering constructed at the end of Sec. III A. Let $P : \widetilde{M} \to M$ denote the spacetime covering constructed in the beginning of Sec. III B. Since V is a Cauchy surface for M, it follows that $\widetilde{V}$ is a Cauchy surface for $\widetilde{M}$. Hence $(\widetilde{M}, \widetilde{g})$ is globally hyperbolic and thus is past reflecting and admits a piercing for $\widetilde{V}$.

Seeking a contradiction, suppose i_{1*} is not onto. By Proposition 15(2), we have that $\widetilde{S}$ is disconnected. Let $\hat{S}$ denote the connected component of $\widetilde{S}$ containing $\hat{x}$, and let $\overline{S}$ denote some other connected component. Let $D_2 \subset \widetilde{V}$ denote the isometric copy of E_2 attached to $\hat{S}$. Let $D_1 = (\widetilde{V} \backslash D_2) \sqcup \hat{S}$. Since focal points lift, we can apply Corollary 8 in the covering spacetime $\widetilde{M}$ to conclude that D_1 is compact. However, D_1 contains the covering $\overline{E}_2$ of E_2, which is attached along $\overline{S}$. Since E_2 is noncompact, $\overline{E}_2$ is noncompact, and hence, D_1 is noncompact, which is a contradiction. □

Theorem 17 can be interpreted as an incompleteness theorem in the following way. Assume that E_2 is noncompact, i_{2*} is an isomorphism, the null energy condition holds, and S has negative inward null expansion. Then, if i_{1*} is not onto, some inward null geodesic from S must be future incomplete since otherwise there would be an S-focal point along each γ_x for each $x \in S$ (see the remark after Corollary 8). More generally, if the inward null geodesics are future complete, then (2.1) must be violated along at least one of the null geodesics.

The following examples illustrate how Theorem 17 can be interpreted as an incompleteness theorem.

Examples.

1. Consider the Schwarzschild $\mathbb{R}P^3$ geon described in Ref. 8. In this case, $V \approx \mathbb{R}P^3 \backslash \{\text{pt}\}$. Take $S \approx S^2$ to be a surface of constant latitude above the $\mathbb{R}P^2$ equator. S separates V. Then, E_1 corresponds to "inside," which includes $\mathbb{R}P^2$, and let E_2 be the "outside." In this case, S is simply connected, but E_1 is not simply connected. As in Schwarzschild, S has negative inward null expansion, and since the spacetime is vacuum, the null energy condition holds. Hence, by Theorem 17, there must be some future directed null geodesic from S, which is incomplete, which, of course, holds for Schwarzschild $\mathbb{R}P^3$ geon.
2. We construct a time-symmetric three-dimensional spatial slice V as follows: Let $\mathcal{V} = (S^1 \times S^2) \# S^3$. Fix a point $P \in \mathcal{V}$, and let $V = \mathcal{V} \backslash \{P\}$. Since $\mathcal{V}$ admits a metric of positive scalar curvature, there is a metric h on V with zero scalar curvature such that V is asymptotically flat (see Refs. 17 and 23), where P represents the point of infinity. We can pick a surface $S \approx S^2$ surrounding P such that S has positive h-mean curvature with respect to the outward unit normal (i.e., towards P). Then, $E_2 \approx S \times [0, \infty)$. A couple of applications of the Seifert–Van Kampen theorem show that $\pi_1(E_1) \approx \pi_1(V) \approx \pi_1(\mathcal{V}) \approx \mathbb{Z}$. Since (V, h) has zero scalar curvature, it satisfies the constraint equations for a time-symmetric initial data slice. Let (M, g) denote the maximal globally hyperbolic development of $(V, h, K = 0)$ for the vacuum Einstein equations. Since S has positive mean curvature and $K = 0$, S is inner trapped within (M, g). (M, g) is vacuum, so the null energy condition holds. Finally, i_{1^*} is not onto since S is simply connected but E_1 is not. Thus, by Theorem 17, (M, g) must contain an incomplete future directed null geodesic emanating inward from S. In fact, this incompleteness conclusion can be derived from any closed, non-simply connected manifold $\mathcal{V}$ with positive scalar curvature in dimensions ≥ 3.
3. In examples (1) and (2), S is topologically a sphere. It is easy to modify example (2) to show how Theorem 17 can be used to produce an incompleteness theorem even when S is not simply connected. Let V, S, E_1, and E_2 be as in example (2). Let h denote the metric on V with zero scalar curvature. Let $\bar{V} = S^1 \times V$ and $\bar{h} = ds^2 \oplus h$. Then, $(\bar{V}, \bar{h})$ has zero scalar curvature as well. Let $\bar{S} = S^1 \times S$. Then, $\bar{S}$ separates $\bar{V}$ via $\bar{E}_1 = E_1 \times S^1$ and $\bar{E}_2 = E_2 \times S^1$. The $\bar{h}$-mean curvature of $\bar{S}$ within $\bar{V}$ equals the h-mean curvature of S within V. Therefore, $\bar{S}$ is inner trapped within the maximal globally hyperbolic development of the new initial dataset, $(\bar{V}, \bar{h}, 0)$, for the vacuum Einstein equations. Since $\pi_1(\bar{S}) \approx \mathbb{Z}$ and $\pi_1(\bar{E}_1) \approx \mathbb{Z} \times \mathbb{Z}$, i_{1^*} is not onto. Thus, by Theorem 17, there is an incomplete future directed null geodesic emanating inward from $\bar{S}$.

C. Exterior foliations by spacetimes-with-timelike-boundary

In this section, we will obtain a version of Theorem 17 for a class of spacetimes that are not, in general, globally hyperbolic, but which admit, in a certain sense, a foliation by globally hyperbolic spacetimes-with-timelike-boundary. The Birmingham–Kottler spacetimes[3] and the Horowitz–Myers soliton[12] are examples that admit such a foliation. The motivation here is to extend our results from Sec. III B to a setting applicable to asymptotically locally anti-de sitter (AdS) spacetimes.

Let $(\mathcal{M}, \mathfrak{g})$ be a spacetime-with-timelike-boundary; see Ref. 1 for basic definitions and properties. Then,

$$\mathcal{M} = M \cup \partial\mathcal{M},$$

where $M = \text{int}(\mathcal{M})$. Set $g = \mathfrak{g}|_M$. Note that (M, g) is a spacetime.

Lemma 18. Let $(\mathcal{M}, \mathfrak{g})$ be a spacetime-with-timelike-boundary. If $(\mathcal{M}, \mathfrak{g})$ is past reflecting, then (M, g) is past reflecting.

Proof. We abbreviate $I^+(p, M)$ with $I^+_M(p)$, likewise with $I^+_{\mathcal{M}}(p)$.

Fix $p, q \in M$ such that $I^+_M(q) \subset I^+_M(p)$. We want to show $I^-_M(p) \subset I^-_M(q)$. We first show $I^+_{\mathcal{M}}(q) \subset I^+_{\mathcal{M}}(p)$. Fix $x \in I^+_{\mathcal{M}}(q)$. Either $x \in M$ or $x \in \partial\mathcal{M}$. If $x \in M$, then

$$x \in I^+_{\mathcal{M}}(q) \cap M = I^+_M(q) \subset I^+_M(p) \subset I^+_{\mathcal{M}}(p).$$

The equality follows from Proposition 2.6(d) of Ref. 1. Suppose $x \in \partial\mathcal{M}$. Let $\lambda \subset \mathcal{M}$ be a timelike curve from q to x. Since $q \in M$ and M is open in $\mathcal{M}$, λ intersects M at some point r. Then, $r \in I^+_{\mathcal{M}}(q) \cap M$, and reasoning as above, we have $r \in I^+_{\mathcal{M}}(p)$, and hence, $x \in I^+_{\mathcal{M}}(p)$.

By past reflecting of $\mathcal{M}$, we have $I^-_{\mathcal{M}}(p) \subset I^-_{\mathcal{M}}(q)$. Intersecting both sides with M and applying Proposition 2.6(d) of Ref. 1 again, we have $I^-_M(p) \subset I^-_M(q)$. □

Theorem 19. *Suppose (M, g) is a spacetime with V, S, E_1, and E_2 given as in the beginning of Sec. II B. We make the following assumptions:*

(a) *X is a piercing vector field for V.*
(b) *$E_2 \approx S \times [0, \infty)$.*

(c) *Set* $E_{2,t} \approx S \times [0,t]$ *and* $V_t = E_1 \cup E_{2,t}$. *Let* M_t *denote the union of the images of the maximal integral curves of* X *starting on* V_t. *We assume that* M_t *is a globally hyperbolic spacetime-with-timelike-boundary with Cauchy surface* V_t.

(d) *There is a* $T > 0$ *such that for all* $x \in S$, *there is an* S*-focal point along* γ_x *within* M'_T, *where* M'_T *is the interior of* M_T.

Then, $i_{1*} \colon \pi_1(S,x) \to \pi_1(E_1,x)$ *is onto. In particular,* E_1 *is simply connected whenever* S *is simply connected.*

Remarks.

- By "there is an S-focal point along γ_x within M'_T," we mean that if $\gamma_x(b)$ is an S-focal point of γ_x, then $\gamma_x|_{[0,b]}$ lies within M'_T. This terminology will also be used in Theorem 22.
- It is not hard to see that in the case S is inner trapped, M is future null complete and satisfies the null energy condition, then assumption (d) holds. One may ask if assumption (d) holds when one merely assumes that there is an S-focal point along γ_x for each $x \in S$.
- Anti-de Sitter space is an obvious example of the theorem. In this case, S is topologically the $(n-1)$-sphere and E_1 is topologically the n-ball. Hence, both S and E_1 are simply connected.
- A less trivial example of the theorem is the Horowitz–Myers soliton.[12] This class of spacetimes consists of the Einstein equations with a negative cosmological constant. In this case, S is topologically an $(n-1)$-dimensional torus with positive mean curvature, while the topology of E_1 is the product of a disk and an $(n-2)$-dimensional torus. Therefore, i_{1^*} is onto but not an isomorphism.

Proof. Let $p : \widetilde{E}_1 \to E_1$ be the covering of E_1 such that $p_*\pi_1(\widetilde{E}_1, \hat{x}) = i_{1*}\pi_1(S,x)$ with $\hat{x} \in \widetilde{S} := p^{-1}(S)$. Slightly abusing notation, let $p : \widetilde{V} \to V$ denote the covering constructed at the end of Sec. III A. Let $P : \widetilde{M} \to M$ denote the spacetime covering constructed in the beginning of Sec. III B. Let $\hat{S}, \widetilde{E}, D_1$, and D_2 be given as in Theorem 17. Let $\widetilde{M}_T = P^{-1}(M_T)$ and $\widetilde{M}'_T$ be its interior. Note that $\widetilde{M}_T$ is the union of the lift of the maximal integral curves that intersect V_T. Since the focal points lift, assumption (d) holds in $\widetilde{M}'_T$ with respect to $\hat{S}$. Hence, for each $x \in \hat{S}$, there is an S-cut point along γ_x with respect to the spacetime $\widetilde{M}'_T$.

Since $\widetilde{V}_T = P^{-1}(V_T)$ is a Cauchy surface for $\widetilde{M}_T$, consequently, $\widetilde{M}_T$ is a globally hyperbolic spacetime-with-timelike-boundary, and hence, $\widetilde{M}_T$ is past reflecting.[1] By Lemma 18, the interior $\widetilde{M}'_T$ of $\widetilde{M}_T$ is past reflecting. Since X lifts to a piercing $\widetilde{X}$ on $\widetilde{M}$ for $\widetilde{V}$, there is a piercing for the interior $\widetilde{V}'_T$ of $\widetilde{V}_T$. Thus, $D_1 \cap \widetilde{M}'_T$ is compact by Corollary 8.

If i_{1*} is not onto, then $\widetilde{S}$ contains a component $\overline{S} \neq \hat{S}$ by Proposition 15(2). Then, $D_1 \cap \widetilde{M}'_T$ contains the noncompact end $\overline{S} \times [0,T)$, which is a contradiction. □

The spacetimes in Theorem 19 are not, in general, causally simple. It would be of interest to determine whether or not these spacetimes and their covers are past reflecting. If a spacetime admits a conformal completion, which is a globally hyperbolic spacetime-with-timelike-boundary, then the original spacetime is past reflecting via Lemma 18.

IV. APPLICATIONS TO SPACETIME TOPOLOGY—THE HORIZON CASE

The topological censorship results in this section generalize aspects of Refs. 4 and 18. For more on topological censorship and references to other related results, see Ref. 5.

A. Gluing constructions

We consider the setting of a black hole with a horizon in the spacetime. Let (M,g) satisfy properties (1)–(4) in Sec. II C. Define $F_1 = B \cup E_1$ and consider the inclusion maps,

$$i : S \to V \quad i_1 : S \to F_1 \quad i_2 : S \to E_2.$$

Let $p : \widetilde{F}_1 \to F_1$ be a covering of F_1 such that $p_*\pi_1(\widetilde{F}_1, \hat{x}) = i_{1*}\pi_1(S,x)$ with $x \in S$ and $\hat{x} \in \widetilde{S} := p^{-1}(S)$. Analogous to Proposition 15, we have the following proposition:

Proposition 20.

(1) *If* $\hat{S}$ *is the connected component of* $\widetilde{S}$ *containing* $\hat{x}$, *then* $\hat{S}$ *is isometric to* S.

(2) *If* $\widetilde{S}$ *is connected, then* $i_{1^*} : \pi_1(S,x) \to \pi_1(F_1,x)$ *is onto.*

Proposition 16 also holds in this setting: if i_{2*} is an isomorphism, then for each connected component $\overline{S} \subset \widetilde{S}$, there is a covering $q : \overline{E}_2 \to E_2$ (defined in the same way as before) such that $\overline{S}$ is isometric to $\partial\overline{E}_2$.

Using the same construction at the end of Sec. III A (and slightly abusing notation again), we have a covering $p : \widetilde{V} \to V$ such that $p|_{\widetilde{E}_2} = q$ and $p|_{\widetilde{F}_1}$ is just the covering we started with. As in the beginning of Sec. III B, there is a corresponding spacetime covering $P : \widetilde{M} \to M$ such that $\widetilde{V}$ embeds into $\widetilde{M}$ and $P|_{\widetilde{V}} = p$.

B. The globally hyperbolic setting

Let (M, g) be a spacetime satisfying properties (1)–(4) in Sec. II C. In this section, we assume that V is a Cauchy surface. Analogous to Theorem 17, we have the following theorem:

Theorem 21. *Let (M, g) satisfy properties (1)–(4) in Sec. II C and assume that V is a Cauchy surface and Σ is compact. Suppose that for each $x \in S$, either γ_x crosses H or there exists an S-focal point along γ_x. In the case that S is not simply connected, assume that $i_{2^*} : \pi_1(S, x) \to \pi_1(E_2, x)$ is an isomorphism. If E_2 is noncompact, then $i_{1*}: \pi_1(S, x) \to \pi_1(F_1, x)$ is onto. In particular, F_1 (and hence E_1) is simply connected whenever S is simply connected and E_2 is noncompact.*

Proof. Let $p : \widetilde{F}_1 \to F_1$ be the covering of F_1 such that $p_* \pi_1(\widetilde{F}_1, \hat{x}) = i_{1*}\pi_1(S, x)$ with $\hat{x} \in \widetilde{S} := p^{-1}(S)$. Let $\widetilde{\Sigma} = p^{-1}(\Sigma)$ and likewise with $\widetilde{B}$. Since $B = \Sigma \times [0, \epsilon)$, we have $\widetilde{B} = \widetilde{\Sigma} \times [0, \epsilon)$.

Extend this covering to $p : \widetilde{V} \to V$ as constructed at the end of Sec. IV A, and let $P : \widetilde{M} \to M$ denote the corresponding spacetime covering. Since V is a Cauchy surface for M, $\widetilde{V}$ is a Cauchy surface for $\widetilde{M}$.

Seeking a contradiction, suppose i_{1^*} is not onto. Then, $\widetilde{S}$ is disconnected by Proposition 20(2). Let $\hat{S}$ denote the connected component of $\widetilde{S}$ containing $\hat{x}$, and let $\overline{S}$ denote some other connected component. Let $D_2 \subset \widetilde{V}$ denote the isometric copy of E_2 attached to $\hat{S}$. Let $D_1 = (\widetilde{E} \backslash D_2) \sqcup \hat{S}$, where $\widetilde{E} = p^{-1}(E)$. Note that D_1 is noncompact since it contains a copy of $\overline{E}_2$ attached to some $\overline{S}$.

Set $\mathcal{H} = \partial I^+(\widetilde{B}) \backslash \operatorname{int}_{\widetilde{V}} \widetilde{B}$. That the null generators of $\mathcal{H}$ never leave $\mathcal{H}$ follows from the same property holding for H. Next, we show that $\mathcal{H} \subset J^+(\widetilde{\Sigma}) \backslash I^+(\widetilde{\Sigma})$. Note that this step does not follow from Proposition 14 since $\widetilde{\Sigma}$ is not necessarily compact. Fix $\widetilde{y} \in \mathcal{H}$, and let $y = P(\widetilde{y})$. By lifting of curves, it suffices to show that $y \in J^+(\Sigma) \backslash I^+(\Sigma)$. Note that $\widetilde{y} \in \mathcal{H}$ implies that $y \in \overline{J^+(B)} \backslash I^+(B)$ and $y \notin \operatorname{int}_V B$. Therefore, $y \in H$.[25] Since Σ is compact, we have $y \in J^+(\Sigma) \backslash I^+(\Sigma)$ by Proposition 14.

We wish to apply Corollary 13 to the covering spacetime $\widetilde{M}$ with D_1 and $\mathcal{H}$ playing the roles of E_1 and H in the statement of Corollary 13. Since $\widetilde{M}$ is globally hyperbolic, it is past reflecting and admits a piercing for $\widetilde{V}$.

Hence, it remains to show that for each $y \in \hat{S}$, either γ_y crosses $\mathcal{H}$ or there exists an $\hat{S}$-focal point along γ_y. Suppose there does not exist an $\hat{S}$-focal point along γ_y. Since the focal points lift, there does not exist an S-focal point along $P \circ \gamma_y$. Let $x = p(y) \in S$. Then, $\gamma_x = P \circ \gamma_y$. By assumption, γ_x must cross $H = \partial I^+(B) \backslash \operatorname{int}_V B$ and hence enters $I^+(B)$. Since $I^+(\widetilde{B}) = P^{-1}\big(I^+(B)\big)$, γ_y must cross $\mathcal{H}$. □

C. Exterior foliations by spacetimes-with-timelike-boundary

In this section, we obtain a version of Theorem 19 in the setting of black holes. The Birmingham–Kottler spacetimes[3] are the main examples of Theorem 22; for these class of spacetimes, i_{1^*} is an isomorphism.

Theorem 22. *Suppose (M, g) is a spacetime satisfying properties (1)–(4) in Sec. II C and Σ is compact. We make the following assumptions:*

(a) *(M, g) admits a piercing for V.*
(b) *$E_2 \approx S \times [0, \infty)$.*
(c) *Set $E_{2,t} = S \times [0, t]$ and $V_t = B \cup E_1 \cup E_{2,t}$. Let M_t be the union of the images of the maximal integral curves of X starting on V_t. We assume that M_t is a globally hyperbolic spacetime-with-timelike-boundary with Cauchy surface V_t.*
(d) *There is a $T > 0$ such that the following hold:*
    (i) *$I^+(B) \cap M'_T \subset I^+(B, M'_T)$, where M'_T is the interior of M_T.*
    (ii) *For each $x \in S$, either γ_x crosses H within M'_T or there is an S-focal point along γ_x within M'_T.*

Then, $i_{1*}: \pi_1(S, x) \to \pi_1(F_1, x)$ is onto. In particular, F_1 (and hence E_1) is simply connected whenever S is simply connected.

Remark. By "γ_x crosses H within M'_T," we mean that if γ_x crosses H at $\gamma_x(b)$, then $\gamma_x|_{[0,b]}$ lies within M'_T. See also the remark after Theorem 19.

Proof. Let $p : \widetilde{F}_1 \to F_1$ be the covering of F_1 such that $p_* \pi_1(\widetilde{F}_1, \hat{x}) = i_{1*}\pi_1(S, x)$ with $\hat{x} \in \widetilde{S} := p^{-1}(S)$. Extend this covering to $p : \widetilde{V} \to V$ as in the end of Sec. IV A. Let $P : \widetilde{M} \to M$ denote the corresponding spacetime covering. Let $\hat{S}$, $\widetilde{E}$, $\widetilde{B}$, D_1, D_2, and $\mathcal{H}$ be given as in the Proof of Theorem 21. Likewise, the null generators of $\mathcal{H}$ do not leave $\mathcal{H}$ to the future since this property holds for H.

Let X be the piercing vector field for V. Let $\widetilde{X}$ denote the lift of X to $\widetilde{M}$. Since $\widetilde{X}$ is a piercing vector field for $\widetilde{V}$, its integral curves produce a retraction $r : \widetilde{M} \to \widetilde{V}$. Consider the globally hyperbolic spacetime-with-timelike-boundary M_T. Since the maximal integral curves lift to maximal integral curves, $\widetilde{M}_T := P^{-1}(M_T)$ is the union of the maximal integral curves of $\widetilde{X}$ starting on $\widetilde{V}_T := P^{-1}(V_T)$. Consequently, properties (i) and (ii) from assumption (d) lift to the cover:

(i) $I^+(\widetilde{B}) \cap \widetilde{M}'_T \subset I^+(\widetilde{B}, \widetilde{M}'_T)$, where $\widetilde{M}'_T$ is the interior of $\widetilde{M}_T$.
(ii) For each $\hat{x} \in \widehat{S}$, either $\gamma_{\hat{x}}$ crosses $\mathcal{H}$ within $\widetilde{M}'_T$ or there is an $\widehat{S}$-focal point along $\gamma_{\hat{x}}$ within $\widetilde{M}'_T$.

Set $\mathcal{H}_T = \mathcal{H} \cap \widetilde{M}'_T$. From (i), $\mathcal{H}_T = \partial I^+(\widetilde{B}, \widetilde{M}'_T) \setminus \mathrm{int}_{\widetilde{V}'_T} \widetilde{B}$. In addition, we have $\mathcal{H}_T \subset J^+(\widetilde{\Sigma}, \widetilde{M}'_T) \setminus I^+(\widetilde{\Sigma}, \widetilde{M}'_T)$; this follows since the same property holds for $H_T = H \cap M'_T$, which lifts to the cover via the same argument used in Theorem 21. Moreover, the null generators of $\mathcal{H}_T$ never leave $\mathcal{H}_T$ within $\widetilde{M}'_T$. Noting also that $\widetilde{M}'_T$ is past reflecting by Lemma 18, it now follows from Corollary 13 that $D_1 \cap \widetilde{M}'_T$ is compact.

If i_{1*} is not onto, then $\widetilde{S}$ contains a component $\overline{S} \neq \hat{S}$ by Proposition 20(2). Then, $D_1 \cap \widetilde{M}'_T$ contains the noncompact end $\overline{S} \times [0, T)$, which is a contradiction. □

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Gregory J. Galloway: Writing – original draft (equal). **Eric Ling**: Writing – original draft (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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[24] ψ is an immersion; hence, $\overline{\psi}$ is an immersion. Then, that $\overline{\psi}$ is an embedding follows from $\overline{\psi}$ being an injective open map onto its image. Injectivity of $\overline{\psi}$ follows from the homotopy lifting property applied to the commutative diagram along with the fact that ι_* is injective, which follows from V being a retract of M. Then, that $\overline{\psi}$ is an open map onto its image follows from p and P being local diffeomorphisms along with ι being a topological embedding.

[25] We just observed that $P(\mathcal{H}) \subset H$. Similarly, it is not hard to see that $\mathcal{H} = P^{-1}(H)$; this holds in Sec. IV C as well.