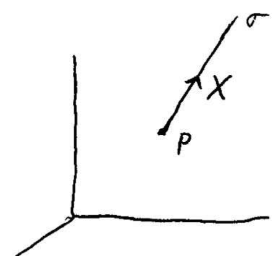
# Chapter 5. The Second Fundamental Form

## Directional Derivatives in $I\!R^3$.

Let $f : U \subset I\!R^3 \to I\!R$ be a smooth function defined on an open subset of $I\!R^3$. Fix $p \in U$ and $X \in T_p I\!R^3$. The *directional derivative* of $f$ at $p$ in the direction $X$, denoted $D_X f$ is defined as follows. Let $\sigma : I\!R \to I\!R^3$ be the parameterized straight line, $\sigma(t) = p + tX$. Note $\sigma(0) = p$ and $\sigma'(0) = X$. Then,

$$\begin{aligned} D_X f &= \frac{d}{dt} f \circ \sigma(t)|_{t=0} \\ &= \frac{d}{dt} f(p + tX)|_{t=0} \\ &\left( = \lim_{t\to 0} \frac{f(p+tX) - f(p)}{t} \right). \end{aligned}$$



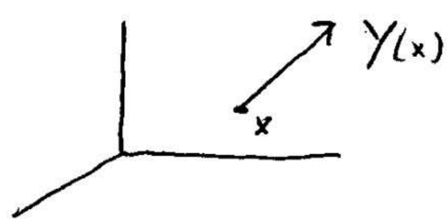

<u>Fact</u>: The directional derivative is given by the following formula,

$$\begin{aligned} D_X f &= X \cdot \nabla f(p) \\ &= (X^1, X^2, X^3) \cdot \left( \frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p) \right) \\ &= X^1 \frac{\partial f}{\partial x^1}(p) + X^2 \frac{\partial f}{\partial x^2}(p) + X^3 \frac{\partial f}{\partial x^3}(p) \\ &= \sum_{i=1}^{3} X^i \frac{\partial f}{\partial x^i}(p). \end{aligned}$$

**Proof** Chain rule!

<u>Vector Fields on $I\!R^3$</u>. A vector field on $I\!R^3$ is a rule which assigns to each point of $I\!R^3$ a vector at the point,

$$x \in I\!R^3 \to Y(x) \in T_x I\!R^3$$



Analytically, a vector field is described by a mapping of the form,

$$\begin{aligned} Y : U &\subset I\!R^3 \to I\!R^3, \\ Y(x) &= (Y^1(x), Y^2(x), Y^3(x)) \in T_x I\!R^3 . \end{aligned}$$

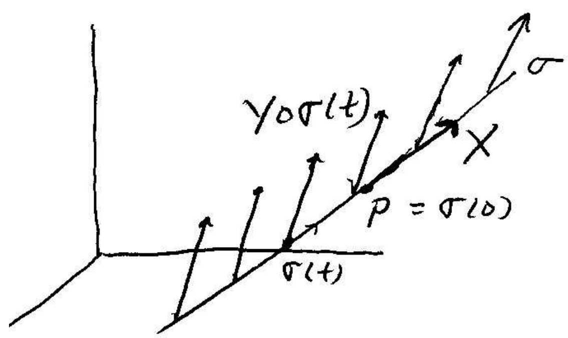
Components of $Y$ : $\quad Y^i : U \to I\!R$, $i = 1, 2, 3$.

**Ex.** $Y : I\!R^3 \to I\!R^3$, $Y(x, y, z) = (y + z, z + x, x + y)$. E.g., $Y(1, 2, 3) = (5, 4, 3)$, etc. $Y^1 = y + z$, $Y^2 = z + x$, and $Y^3 = x + y$.

The directional derivative of a vector field is defined in a manner similar to the directional derivative of a function: Fix $p \in U$, $X \in T_p I\!R^3$. Let $\sigma : I\!R \to I\!R^3$ be the parameterized line $\sigma(t) = p + tX$ $(\sigma(0) = p, \sigma'(0) = X)$. Then $t \to Y \circ \sigma(t)$ is a vector field along $\sigma$ in the sense of the definition in Chapter 2. Then, the directional derivative of $Y$ in the direction $X$ at $p$, is defined as,

$$D_X Y = \frac{d}{dt} Y \circ \sigma(t)|_{t=0}$$

I.e., to compute $D_X Y$, restrict $Y$ to $\sigma$ to obtain a vector valued function of $t$, and differentiate with respect to $t$.




In terms of components, $Y = (Y^1, Y^2, Y^3)$,

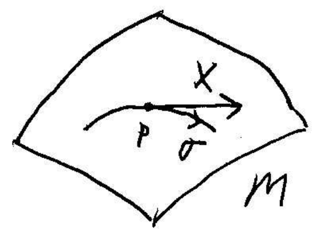
$$\begin{aligned} D_X Y &= \frac{d}{dt}(Y^1 \circ \sigma(t), Y^2 \circ \sigma(t), Y^3 \circ \sigma(t))|_{t=0} \\ &= \left( \frac{d}{dt} Y^1 \circ \sigma(t)|_{t=0} , \frac{d}{dt} Y^2 \circ \sigma(t)|_{t=0} , \frac{d}{dt} Y^3 \circ \sigma(t)|_{t=0} \right) \\ &= (D_X Y^1, D_X Y^2, D_X Y^3). \end{aligned}$$

**Directional derivatives on surfaces**.

Let $M$ be a surface, and let $f : M \to I\!R$ be a smooth function on $M$. Recall, this means that $\hat{f} = f \circ \mathbf{x}$ is smooth for all proper patches $\mathbf{x} : U \to M$ in $M$.

**Def**. For $p \in M$, $X \in T_pM$, the directional derivative of $f$ at $p$ in the direction $X$, denoted $\nabla_X f$, is defined as follows. Let $\sigma : (-\epsilon, \epsilon) \to M \subset I\!R^3$ be any smooth curve in $M$ such that $\sigma(0) = p$ and $\sigma'(0) = X$. Then,

$$\nabla_X f = \frac{d}{dt} f \circ \sigma(t)|_{t=0}$$




I.e., to compute $\nabla_X f$, restrict $f$ to $\sigma$ and differentiate with respect to parameter $t$.

**Proposition**. *The directional derivative is well-defined, i.e. independent of the particular choice of* $\sigma$.

*Proof.* Let $\mathbf{x} : U \to M$ be a proper patch containing $p$. Express $\sigma$ in terms of coordinates in the usual manner,

$$\sigma(t) = \mathbf{x}(u^1(t), u^2(t)) \,.$$

By the chain rule,

$$\frac{d\sigma}{dt} = \sum \frac{du_i}{dt} \mathbf{x}_i \qquad \left( \mathbf{x}_i = \frac{\partial \mathbf{x}}{\partial u_i} \right)$$

$X \in T_pM \Rightarrow X = \sum X^i \mathbf{x}_i$. The initial condition, $\frac{d\sigma}{dt}(0) = X$ then implies

$$\frac{du^i}{dt}(0) = X^i \,, \quad i = 1, 2.$$

Now,

$$\begin{aligned} f \circ \sigma(t) &= f(\sigma(t)) = f(\mathbf{x}(u^1(t), u^2(t)) \\ &= f \circ \mathbf{x}(u^1(t), u^2(t)) \\ &= \hat{f}(u^1(t), u^2(t)). \end{aligned}$$

Hence, by the chain rule,

$$\begin{aligned}\frac{d}{dt} f\circ\sigma(t) &= \frac{\partial \hat{f}}{\partial u^1}\frac{du^1}{dt} + \frac{\partial \hat{f}}{\partial u^2}\frac{du^2}{dt} \\ &= \sum_i \frac{\partial \hat{f}}{\partial u^i}\frac{du^i}{dt} = \sum_i \frac{du^i}{dt}\frac{\partial \hat{f}}{\partial u^i}.\end{aligned}$$

Therefore,

$$\begin{aligned}\nabla_X f &= \frac{d}{dt} f\circ\sigma(t)|_{t=0} \\ &= \sum_i \frac{du^i}{dt}(0)\frac{\partial \hat{f}}{\partial u^i}(u^1,u^2)\,, \qquad (p = \mathbf{x}(u^1,u^2)) \\ \nabla_X f &= \sum_i X^i \frac{\partial \hat{f}}{\partial u^i}(u^1,u^2)\,,\end{aligned}$$

or simply,

$$\begin{aligned}\nabla_X f &= \sum X^i \frac{\partial \hat{f}}{\partial u^i} \\ &= X^1\frac{\partial \hat{f}}{\partial u^1} + X^2\frac{\partial \hat{f}}{\partial u^2}\,. \qquad (*)\end{aligned}$$

**Ex**. Let $X = \mathbf{x}_1$. Since $\mathbf{x}_1 = 1\cdot\mathbf{x}_1 + 0\cdot\mathbf{x}_2$, $X^1 = 1$ and $X^2 = 0$. Hence the above equation implies, $\nabla_{\mathbf{x}_1} f = \frac{\partial \hat{f}}{\partial u^1}$. Similarly, $\nabla_{\mathbf{x}_2} f = \frac{\partial \hat{f}}{\partial u^2}$. I.e.,

$$\nabla_{\mathbf{x}_i} f = \frac{\partial \hat{f}}{\partial u^i}, \quad i = 1,2.$$

The following proposition summarizes some basic properties of directional derivatives in surfaces.

**Proposition**

(1) $\nabla_{(aX+bY)} f = a\nabla_X f + b\nabla_Y f$

(2) $\nabla_X(f+g) = \nabla_X f + \nabla_X g$

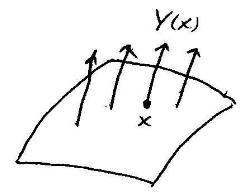
(3) $\nabla_X fg = (\nabla_X f)g + f(\nabla_X g)$

**Exercise 5.1**. Prove this proposition.

Vector fields along a surface.

A vector field along a surface $M$ is a rule which assigns to each point of $M$ a vector at that point,

$$x \in M \to Y(x) \in T_x\mathbb{R}^3$$




N.B. $Y(x)$ need not be tangent to $M$.

Analytically vector fields along a surface $M$ are described by mappings.

$$Y : M \to \mathbb{R}^3$$
$$Y(x) = (Y^1(x), Y^2(x), Y^3(x)) \in T_x\mathbb{R}^3$$

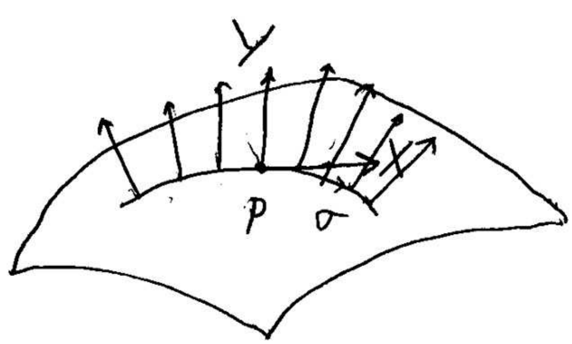
Components of $Y$: $Y^i : M \to \mathbb{R}$, $i = 1, 2, 3$. We say $Y$ is *smooth* if its component functions are smooth.

The directional derivative of a vector field along $M$ is defined in a manner similar to the directional derivative of a function defined on $M$.

Given a vector field along $M$, $Y : M \to \mathbb{R}^3$, for $p \in M, X \in T_pM$, the directional derivative of $Y$ in the direction $X$, denoted $\nabla_X Y$, is defined as,

$$\nabla_X Y = \frac{d}{dt} Y \circ \sigma(t)|_{t=0}$$

where $\sigma : (-\epsilon, \epsilon) \to M$ is a smooth curve in $M$ such that $\sigma(0) = p$ and $\dfrac{d\sigma}{dt}(0) = X$.



I.e. to compute $\nabla_X Y$, restrict $Y$ to $\sigma$ to obtain a vector valued function of $t$ - then differentiate with respect to $t$.

**Fact** If $Y(x) = (Y^1(x), Y^2(x), Y^3(x))$ then,

$$\nabla_X Y = (\nabla_X Y^1, \nabla_X Y^2, \nabla_X Y^3)$$

*Proof*: Exercise.

Surface Coordinate Expression. Let $\mathbf{x} : U \to M$ be a proper patch in $M$ containing $p$. Let $X \in T_pM$, $X = \sum_i x^i \mathbf{x}_i$. An argument like that for functions on $M$ shows,

$$\nabla_X Y = \sum_{i=1}^{2} X^i \frac{\partial \hat{Y}}{\partial u^i}(u^1, u^2), \qquad (p = \mathbf{x}(u^1, u^2))$$

where $\hat{Y} = Y \circ \mathbf{x} : U \to I\!R^3$ is $Y$ expressed in terms of coordinates.

**Exercise 5.2**. Derive the expression above for $\nabla_X Y$. In particular, show

$$\nabla_{\mathbf{x}_i} Y = \frac{\partial \hat{Y}}{\partial u^i}, \quad i = 1, 2.$$

Some basic properties are described in the following proposition.

**Proposition**

(1) $\nabla_{aX+bY} Z = a\nabla_X Z + b\nabla_Y Z$

(2) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$

(3) $\nabla_X (fY) = (\nabla_X f)Y + f\nabla_X Y$

(4) $\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

## The Weingarten Map and the 2nd Fundamental Form.

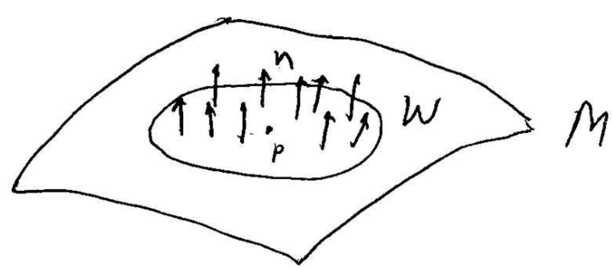
We are interested in studying the *shape* of surfaces in $\mathbb{R}^3$. Our approach (essentially due to Gauss) is to study how the unit normal to the surface "wiggles" along the surface.

The objects which describe the shape of $M$ are:

1. The *Weingarten Map*, or *shape* operator. For each $p \in M$ this is a certain linear transformation $L : T_pM \to T_pM$.

2. The *second fundamental form*. This is a certain bilinear form $\mathcal{L} : T_pM \times T_pM \to \mathbb{R}$ associated in a natural way with the Weingarten map.

We now describe the Weingarten map. Fix $p \in M$. Let $n : W \to \mathbb{R}^3$, $p \in W \to n(p) \in T_p\mathbb{R}^3$, be a smooth *unit normal* vector field defined along a neighborhood $W$ of $p$.




**Remarks**

1. $n$ can always be constructed by introducing a proper patch $\mathbf{x} : U \to M$, $\mathbf{x} = \mathbf{x}(u^1, u^2)$ containing $p$:

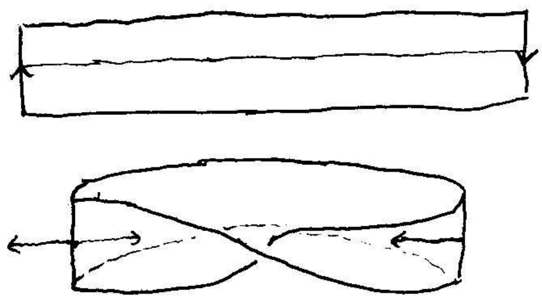
$$\hat{n} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|},$$

$\hat{n} : U \to \mathbb{R}^3$, $\hat{n} = \hat{n}(u^1, u^2)$. Then, $n = \hat{n} \circ \mathbf{x}^{-1} : \mathbf{x}(U) \to \mathbb{R}^3$ is a smooth unit normal v.f. along $\mathbf{x}(U)$.

2. The choice of $n$ is not quite unique: $n \to -n$; choice of $n$ is unique "up to sign"

3. A smooth unit normal field $n$ always exists in a neighborhood of any given point $p$, but it may not be possible to extend $n$ to all of $M$. This depends on whether or not $M$ is an *orientable* surface.

**Ex.** Möbius band.



**Lemma**. Let $M$ be a surface, $p \in M$, and $n$ be a smooth unit normal vector field defined along a neighborhood $W \subset M$ of $p$. Then for any $X \in T_pM$, $\nabla_X n \in T_pM$.

*Proof.* It suffices to show that $\nabla_X n$ is perpendicular to $n$. $|n| = 1 \Rightarrow \langle n, n\rangle = 1 \Rightarrow$

$$\begin{aligned} \nabla_X \langle n, n\rangle &= \nabla_X 1 \\ \langle \nabla_X n, n\rangle + \langle n, \nabla_X n\rangle &= 0 \\ 2\langle \nabla_X n, n\rangle &= 0 \end{aligned}$$

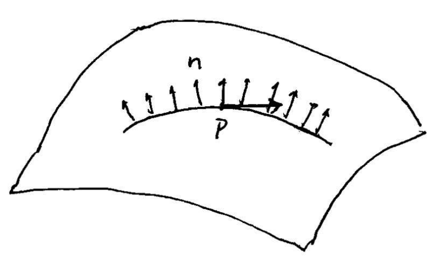
and hence $\nabla_X n \perp n$.

**Def**. Let $M$ be a surface, $p \in M$, and $n$ be a smooth unit normal v.f. defined along a nbd $W \subset M$ of $p$. The *Weingarten Map* (or *shape operator*) is the map $L : T_pM \to T_pM$ defined by,

$$L(X) = -\nabla_X n\,.$$

**Remarks**

1. The minus sign is a convention – will explain later.
2. $L(X) = -\nabla_X n = -\dfrac{d}{dt} n \circ \sigma(t)|_{t=0}$



**Lemma**: $L : T_pM \to T_pM$ is a *linear* map, i.e.,

$$L(aX + bY) = aL(X) + bL(Y)\,.$$

for all $X, Y, \in T_pM,\ \ a, b \in I\!R$.

*Proof.* Follows from properties of directional derivative,

$$\begin{aligned} L(aX + bY) &= -\nabla_{aX+bY} n \\ &= -[a\nabla_X n + b\nabla_Y n] \\ &= a(-\nabla_X n) + b(-\nabla_Y n) \\ &= aL(X) + bL(Y). \end{aligned}$$

**Ex.** Let $M$ be a *plane* in $I\!R^3$:

$$M : ax + by + cz = d$$

Determine the Weingarten Map at each point of $M$. Well,

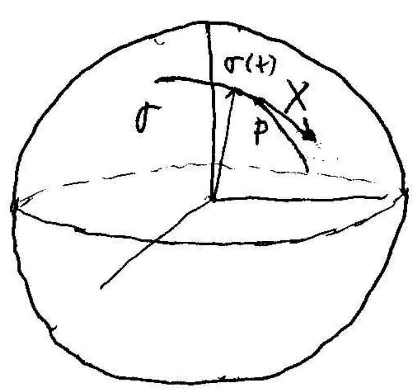
$$n = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}} = \left(\frac{a}{\lambda}, \frac{b}{\lambda}, \frac{c}{\lambda}\right),$$

where $\lambda = \sqrt{a^2 + b^2 + c^2}$. Hence,

$$\begin{aligned} L(X) &= -\nabla_X n = -\nabla_X \left(\frac{a}{\lambda}, \frac{b}{\lambda}, \frac{c}{\lambda}\right) \\ &= -\left(\nabla_X \frac{a}{\lambda}, \nabla_X \frac{b}{\lambda}, \nabla_X \frac{c}{\lambda}\right) = \mathbf{0}. \end{aligned}$$

Therefore $L(X) = 0\ \forall\ \ X \in T_pM$, i.e. $L \equiv 0$.

**Ex.** Let $M = S_r^2$ be the sphere of radius $r$, let $n$ be the *outward* pointing unit normal. Determine the Weingarten map at each point of $M$.



Fix $p \in S_r^2$, and let $X \in T_pS_r^2$. Let $\sigma : (-\varepsilon, \varepsilon) \to S_r^2$ be a curve in $S_r^2$ such that $\sigma(0) = p, \dfrac{d\sigma}{dt}(0) = X$

Then,

$$L(X) = -\nabla_X n = -\frac{d}{dt} n \circ \sigma(t)|_{t=0} \,.$$

But note, $n \circ \sigma(t) = n(\sigma(t)) = \dfrac{\sigma(t)}{|\sigma(t)|} = \dfrac{\sigma(t)}{r}$. Hence,

$$\begin{aligned} L(X) &= -\frac{d}{dt}\frac{\sigma(t)}{r}|_{t=0} = -\frac{1}{r}\frac{d\sigma}{dt}|_{t=0} \\ L(X) &= -\frac{1}{r}X \,. \end{aligned}$$

for all $X \in T_pM$. Hence, $L = -\dfrac{1}{r}id$, where $id : T_pM \to T_pM$ is the identity map, $id(X) = X$.

**Remark**. If we had taken the inward pointing normal then $L = \dfrac{1}{r}id$.

**Def.** For each $p \in M$, the *second fundamental form* is the bilinear form $\mathcal{L} = T_pM \times T_pM \to I\!R$ defined by,

$$\begin{aligned} \mathcal{L}(X, Y) &= \langle L(X), Y\rangle \\ &= -\langle \nabla_X n, Y\rangle. \end{aligned}$$

$\mathcal{L}$ is indeed *bilinear*, e.g.,

$$\begin{aligned} \mathcal{L}(aX + bY, Z) &= \langle L(aX + bY), Z\rangle \\ &= \langle aL(X) + bL(Y), Z\rangle \\ &= a\langle L(X), Z\rangle + b\langle L(Y), Z\rangle \\ &= a\mathcal{L}(X, Z) + b\mathcal{L}(Y, Z). \end{aligned}$$

**Ex.** $M =$ plane, $\mathcal{L} \equiv 0$:

$$\mathcal{L}(X, Y) = \langle L(X), Y\rangle = \langle 0, Y\rangle = 0.$$

**Ex.** The sphere $S_r^2$ of radius $r$, $\mathcal{L} : T_pS_r^2 \times T_pS_r^2 \to I\!R$,

$$\begin{aligned} \mathcal{L}(X, Y) &= \langle L(X), Y\rangle \\ &= \langle -\frac{1}{r}X, Y\rangle \\ &= -\frac{1}{r}\langle X, Y\rangle \end{aligned}$$

Hence, $\mathcal{L} = -\frac{1}{r}\langle\ ,\ \rangle$. Multiple of the first fundamental form!

Coordinate expressions

Let $\mathbf{x} : U \to M$ be a patch containing $p \in M$. Then $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a basis for $T_pM$. We express $L : T_pM \to T_pM$ and $\mathcal{L} : T_pM \times T_pM \to \mathbb{R}$ with respect to this basis. Since $L(\mathbf{x}_j) \in T_pM$, we have,

$$\begin{aligned} L(\mathbf{x}_j) &= L^1{}_j\mathbf{x}_1 + L^2{}_j\mathbf{x}_2\,, \qquad j = 1, 2 \\ &= \sum_{i=1}^{2} L^i{}_j\mathbf{x}_i\,. \end{aligned}$$

The numbers $L^i{}_j$, $1 \le i, j \le 2$, are called the components of $L$ with respect to the coordinate basis $\{\mathbf{x}_1, \mathbf{x}_2\}$. The $2 \times 2$ matrix $[L^i{}_j]$ is the matrix representing the linear map $L$ with respect to the basis $\{\mathbf{x}_1, \mathbf{x}_2\}$.

**Exercise 5.3** Let $X \in T_pM$ and let $Y = L(X)$. In terms of components, $X = \sum_j X^j\mathbf{x}_j$ and $Y = \sum_i Y^i\mathbf{x}_i$. Show that

$$Y^i = \sum_j L^i{}_j X^j\,, \qquad i = 1, 2\,,$$

which in turn implies the matrix equation,

$$\begin{bmatrix} Y^1 \\ Y^2 \end{bmatrix} = \left[L^i{}_j\right] \begin{bmatrix} X^1 \\ X^2 \end{bmatrix}.$$

This is the Weingarten map expressed as a matrix equation.

Introduce the unit normal field along $W = \mathbf{x}(U)$ with respect to the patch $\mathbf{x} : U \to M$,

$$\begin{aligned} \hat{n} &= \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|}\,, \qquad \hat{n} = \hat{n}(u^1, u^2)\,, \\ n &= \hat{n} \circ \mathbf{x}^{-1} : W \to \mathbb{R}\,. \end{aligned}$$

Then by Exercise 5.2,

$$L(\mathbf{x}_j) = -\nabla_{\mathbf{x}_j} n = -\frac{\partial \hat{n}}{\partial u^j}\,.$$

Setting $n_j = \frac{\partial \hat{n}}{\partial u^j}$ we have

$$n_j = -L(\mathbf{x}_j)$$

$$n_j = -\sum_i L^i{}_j\mathbf{x}_i\,, \quad j = 1, 2 \quad \text{(The Weingarten equations.)}$$

These equations can be used to compute the components of the Weingarten map. However, in practice it turns out to be more useful to have a formula for computing the components of the second fundamental form.

Components of $\mathcal{L}$:

The components of $\mathcal{L}$ with respect to $\{\mathbf{x}_1, \mathbf{x}_2\}$ are defined as,

$$L_{ij} = \mathcal{L}(\mathbf{x}_i, \mathbf{x}_j)\,, \qquad 1 \le i, j \le 2\,.$$

By bilinearity, the components completely determine $\mathcal{L}$,

$$\begin{aligned}\mathcal{L}(X,Y) &= \mathcal{L}(\sum_i X^i\mathbf{x}_i, \sum_j Y^j\mathbf{x}_j) \\ &= \sum_{i,j} X^iY^j\mathcal{L}(\mathbf{x}_i, \mathbf{x}_j) \\ &= \sum_{i,j} L_{ij}X^iY^j\,.\end{aligned}$$

The following proposition provides a useful formula for computing the $L_{ij}$'s.

**Proposition.** The components $L_{ij}$ of $\mathcal{L}$ are given by,

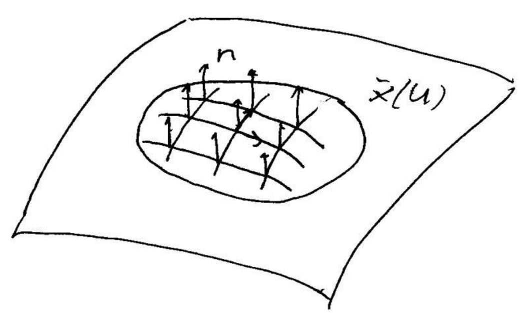
$$L_{ij} = \langle \hat{n}, \mathbf{x}_{ij} \rangle\,,$$

where $\mathbf{x}_{ij} = \dfrac{\partial^2 \mathbf{x}}{\partial u^j \partial u^i}$.

**Remark.** Henceforth we no longer distinguish between $n$ and $\hat{n}$, i.e., lets agree to drop the " ˆ ", then,

$$L_{ij} = \langle n, \mathbf{x}_{ij} \rangle\,,$$

*Proof*:



Along $\mathbf{x}(U)$ we have, $\langle n, \frac{\partial \mathbf{x}}{\partial u^j}\rangle = 0$, and hence,

$$\begin{aligned} \frac{\partial}{\partial u^i}\langle n, \frac{\partial \mathbf{x}}{\partial u^j}\rangle &= 0 \\ \langle \frac{\partial n}{\partial u^i}, \frac{\partial \mathbf{x}}{\partial u^j}\rangle + \langle n, \frac{\partial^2 \mathbf{x}}{\partial u^i \partial u^j}\rangle &= 0 \\ \langle \frac{\partial n}{\partial u^i}, \frac{\partial \mathbf{x}}{\partial u^j}\rangle &= -\langle n, \frac{\partial^2 \mathbf{x}}{\partial u^i \partial u^j}\rangle = -\langle n, \frac{\partial^2 \mathbf{x}}{\partial u^j \partial u^i}\rangle\,, \end{aligned}$$

or, using shorthand notation,

$$\langle n_i, \mathbf{x}_j\rangle = -\langle n, \mathbf{x}_{ij}\rangle\,.$$

But,

$$\begin{aligned} L_{ij} &= \mathcal{L}(\mathbf{x}_i, \mathbf{x}_j) = \langle L(\mathbf{x}_i), \mathbf{x}_j\rangle \\ &= -\langle n_i, \mathbf{x}_j\rangle\,, \end{aligned}$$

and hence $L_{ij} = \langle n, \mathbf{x}_{ij}\rangle$. □

Observe,

$$\begin{aligned} L_{ij} &= \langle n, \mathbf{x}_{ij}\rangle \\ &= \langle n, \mathbf{x}_{ji}\rangle \quad \text{(mixed partials equal!)} \\ L_{ij} &= L_{ji}\,, \qquad 1 \le i, j \le 2\,. \end{aligned}$$

In other words, $\mathcal{L}(\mathbf{x}_i, \mathbf{x}_j) = \mathcal{L}(\mathbf{x}_j, \mathbf{x}_i)$.

**Proposition**. The second fundamental form $\mathcal{L} : T_pM \times T_pM \to I\!R$ is symmetric, i.e.

$$\mathcal{L}(X, Y) = \mathcal{L}(Y, X) \ \ \forall \ \ X, Y \in T_pM\,.$$

**Exercise 5.4** Prove this proposition by showing $\mathcal{L}$ is symmetric iff $L_{ij} = L_{ji}$ for all $1 \le i, j \le 2$.

Relationship between ${L^i}_j$ and $L_{ij}$

$$\begin{aligned} L_{ij} &= \mathcal{L}(\mathbf{x}_i, \mathbf{x}_j) = \mathcal{L}(\mathbf{x}_j, \mathbf{x}_i) \\ &= \langle L(\mathbf{x}_j), \mathbf{x}_i\rangle = \langle \sum_k {L^k}_j \mathbf{x}_k, \mathbf{x}_i, \rangle \\ &= \sum_k {L^k}_j \langle \mathbf{x}_k, \mathbf{x}_i\rangle \\ L_{ij} &= \sum_k g_{ik} {L^k}_j, \quad 1 \le i, j \le 2 \end{aligned}$$

Classical tensor jargon: $L_{ij}$ obtained from $L^k{}_j$ by "lowering the index $k$ with the metric". The equation above implies the matrix equation

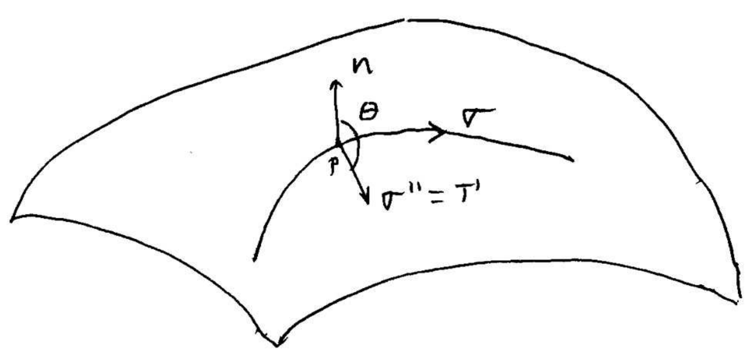
$$[L_{ij}] = [g_{ij}][L^i{}_j]\,.$$

## Geometric Interpretation of the 2nd Fundamental Form

Normal Curvature. Let $s \to \sigma(s)$ be a unit speed curve lying in a surface $M$. Let $p$ be a point on $\sigma$, and let $n$ be a smooth unit normal v.f. defined in a nbd $W$ of $p$. The *normal curvature* of $\sigma$ at $p$, denoted $\kappa_n$, is defined to be the component of the curvature vector $\sigma'' = T'$ along $n$, i.e.,

$$\begin{aligned}\kappa_n &= \text{normal component of the curvature vector}\\ &= \langle \sigma'', n\rangle\\ &= \langle T', n\rangle\\ &= |T'|\,|n|\,\cos\theta\\ &= \kappa\cos\theta\,,\end{aligned}$$

where $\theta$ is the angle between the curvature vector $T'$ and the surface normal $n$. If $\kappa \neq 0$ then, recall, we can introduce the principal normal $N$ to $\sigma$, by the equation, $T' = \kappa N$; in this case $\theta$ is the angle beween $N$ and $n$.




**Remark**: $\kappa_n$ gives a measure of how much $\sigma$ is bending in the direction perpendicular to the surface; it neglects the amount of bending tangent to the surface.

**Proposition**. *Let $M$ be a surface, $p \in M$. Let $X \in T_pM$, $|X| = 1$ (i.e. $X$ is a* **unit** *tangent vector). Let $s \to \sigma(s)$ be any unit speed curve in $M$ such that $\sigma(0) = p$ and $\sigma'(0) = X$. Then*

$$\begin{aligned}\mathcal{L}(X,X) &= \text{normal curvature of } \sigma \text{ at } p\\ &= \langle \sigma'', n\rangle.\end{aligned}$$

*Proof.* Along $\sigma$,

$$\begin{aligned} \langle \sigma'(s), n \circ \sigma(s) \rangle &= 0\,, \qquad \text{for all } s \\ \frac{d}{ds}\langle \sigma', n \circ \sigma \rangle &= 0 \\ \langle \sigma'', n \circ \sigma \rangle + \langle \sigma', \frac{d}{ds} n \circ \sigma \rangle &= 0\,. \end{aligned}$$

At $s = 0$,

$$\begin{aligned} \langle \sigma'', n \rangle + \langle X, \nabla_X n \rangle &= 0 \\ \langle \sigma'', n \rangle &= -\langle X, \nabla_X n \rangle \\ \kappa_n &= \langle X, L(X) \rangle \\ \kappa_n &= \langle L(X), X \rangle \\ &= \mathcal{L}(X, X). \end{aligned}$$

**Remark:** the sign convention used in the definition of the Weingarten map ensures that $\mathcal{L}(X, X) = +\kappa_n$ (rather than $-\kappa_n$).

**Corollary**. *All unit speed curves lying in a surface $M$ which pass through $p \in M$ and have the same unit tangent vector $X$ at $p$, have the same normal curvature at $p$. That is, the normal curvature depends only on the tangent direction $X$.*

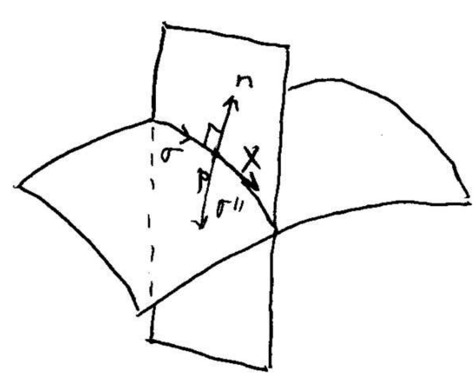
Thus it makes sense to say:

$\mathcal{L}(X, X)$ is the normal curvature in the direction $X$.

Given a unit tangent vector $X \in T_pM$, there is a distinguished curve in $M$, called the *normal section* at $p$ in the direction $X$. Let,

$$\Pi = \text{plane through } p \text{ spanned by } n \text{ and } X.$$

$\Pi$ cuts $M$ in a curve $\sigma$. Parameterize $\sigma$ wrt arc length, $s \to \sigma(s)$, such that $\sigma(0) = p$ and $\frac{d\sigma}{ds}(0) = X$ :



By definition, $\sigma$ is the *normal section* at $p$ in the direction $X$. By the previous proposition,

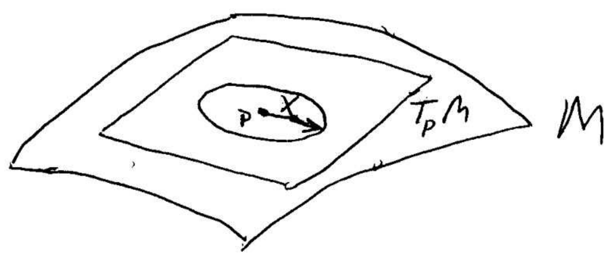
$$\begin{aligned}\mathcal{L}(X,X) &= \text{normal curvature of the normal section } \sigma \\ &= \langle \sigma'', n\rangle = \langle T', n\rangle \\ &= \kappa\cos\theta\,,\end{aligned}$$

where $\theta$ is the angle between $n$ and $T'$. Since $\sigma$ lies in $\Pi$, $T'$ is tangent to $\Pi$, and since $T'$ is also perpendicular to $X$, it follows that $T'$ is a multiple of $n$. Hence, $\theta = 0$ or $\pi$, which implies that $\mathcal{L}(X,X) = \pm\kappa$.

Thus we conclude that,

$$\mathcal{L}(X,X) = \textit{signed} \text{ curvature of the normal section at } p \text{ in the direction } X.$$

<u>Principal Curvatures.</u>




The set of unit tangent vectors at $p$, $X \in T_pM$, $|X| = 1$, forms a circle in the tangent plane to $M$ at $p$. Consider the function from this circle into the reals,

$$\begin{aligned} X &\to \text{normal curvature in direction } X \\ X &\to \mathcal{L}(X,X). \end{aligned}$$

The *principal curvatures* of $M$ at $p$, $\kappa_1 = \kappa_1(p)$ and $\kappa_2 = \kappa_2(p)$, are defined as follows,

$$\begin{aligned} \kappa_1 &= \textit{the maximum normal curvature at } p \\ &= \max_{|X|=1} \mathcal{L}(X,X) \end{aligned}$$

$$\begin{aligned} \kappa_2 &= \textit{the minimum normal curvature at } p \\ &= \min_{|X|=1} \mathcal{L}(X,X) \end{aligned}$$

This is the *geometric* characterization of principal curvatures. There is also an important *algebraic* characterization.

## Some Linear Algebra

Let $V$ be a vector space over the reals, and let $\langle \ , \ \rangle : V \times V \to I\!R$ be an inner product on $V$; hence $V$ is an inner product space. Let $L : V \to V$ be a linear transformation. Our main application will be to the case: $V = T_pM$, $\langle \ , \ \rangle =$ induced metric, and $L =$ Weingarten map.

$L$ is said to be *self adjoint* provided

$$\langle L(v), w\rangle = \langle v, L(w)\rangle \quad \forall \ v, w \in V\,.$$

*Remark.* Let $V = I\!R^n$, with the usual dot product, and let $L : I\!R^n \to I\!R^n$ be a linear map. Let $[L^i{}_j] =$ matrix representing $L$ with respect to the *standard* basis, $e_1 : (1, 0, ..., 0)$, etc. Then $L$ is self-adjoint if and only if $[L^i{}_j]$ is *symmetric* $[L^i{}_j] = [L^j{}_i]$.

**Proposition**. The Weingarten map $L : T_pM \to T_pM$ is self adjoint, i.e.

$$\langle L(X), Y\rangle = \langle X, L(Y)\rangle \ \ \forall \ \ X, Y \in T_pM,$$

where $\langle \ , \ \rangle =$ 1st fundamental form.

**Proof**. We have,

$$\begin{aligned}\langle L(X), Y\rangle &= \mathcal{L}(X, Y) = \mathcal{L}(Y, X)\\ &= \langle L(Y), X\rangle = \langle X, L(Y)\rangle.\end{aligned}$$

Self adjoint linear transformations have very nice properties, as we now discuss. For this discussion, we restrict attention to 2-dimensional vector spaces, $\dim V = 2$.

A vector $v \in V$, $v \neq 0$, is called an *eigenvector* of $L$ if there is a real number $\lambda$ such that,

$$L(v) = \lambda v\,.$$

$\lambda$ is called an *eigenvalue* of $L$. The eigenvalues of $L$ can be determined by solving

$$\det(A - \lambda I) = 0 \qquad (*)$$

where $A$ is a matrix representing $L$ and $I =$ identity matrix. The equation $(*)$ is a quadratic equation in $\lambda$, and hence has at most 2 real roots; it may have no real roots.

**Theorem** (Fundamental Theorem of Self Adjoint Operators) *Let $V$ be a 2-dimensional inner product space. Let $L : V \to V$ be a self-adjoint linear map. Then $V$ admits an orthonormal basis consisting of eigenvectors of $L$. That is, there exists an orthonormal basis $\{e_1, e_2\}$ of $V$ and real numbers $\lambda_1, \lambda_2$, $\lambda_1 \geq \lambda_2$ such that*

$$L(e_1) = \lambda_1 e_1, \quad L(e_2) = \lambda_2 e_2,$$

*i.e., $e_1$ and $e_2$ are eigenvectors of $L$ and $\lambda_1, \lambda_2$ are the corresponding eigenvalues. Moreover the eigenvalues are given by*

$$\begin{aligned} \lambda_1 &= \max_{|v|=1} \langle L(v), v\rangle \\ \lambda_2 &= \min_{|v|=1} \langle L(v), v\rangle \,. \end{aligned}$$

*Proof.* See handout from Do Carmo.

*Remark on orthogonality of eigenvectors.* Let $e_1, e_2$ be eigenvectors with eigenvalues $\lambda_1, \lambda_2$. If $\lambda_1 \neq \lambda_2$, then $e_1$ and $e_2$ are necessarily orthogonal, as seen by the following,

$$\lambda_1 \langle e_1, e_2\rangle = \langle L(e_1), e_2\rangle = \langle e_1, L(e_2)\rangle = \lambda_2 \langle e_1, e_2\rangle \,,$$

$\Rightarrow \quad (\lambda_1 - \lambda_2)\langle e_1, e_2\rangle = 0 \Rightarrow \langle e_1, e_2\rangle = 0$. On the other hand, if $\lambda_1 = \lambda_2 = \lambda$ then $L(v) = \lambda v$ for all $v$. Hence *any* o.n. basis is a basis of eigenvectors.

We now apply these facts to the Weingarten map,

$$\begin{aligned} &L : T_pM \to T_pM \,, \\ &\mathcal{L} : T_pM \times T_pM \to I\!R, \quad \mathcal{L}(X, Y) = \langle L(X), Y\rangle \,. \end{aligned}$$

Since $L$ is self adjoint, and, by definition,

$$\kappa_1 = \max_{|X|=1} \mathcal{L}(X, X) = \max_{|X|=1} \langle L(X), X\rangle$$

$$\kappa_2 = \min_{|X|=1} \mathcal{L}(X, X) = \min_{|X|=1} \langle L(X), X\rangle \,,$$

we obtain the following.

**Theorem.** *The principal curvatures $\kappa_1, \kappa_2$ of $M$ at $p$ are the eigenvalues of the Weingarten map $L : T_pM \to T_pM$. There exists an orthonormal basis $\{e_1, e_2\}$ of $T_pM$ such that*

$$L(e_1) = \kappa_1 e_1, \quad L(e_2) = \kappa_2 e_2 \,,$$

*i.e., $e_1, e_2$ are eigenvectors of $L$ associated with the eigenvalues $\kappa_1, \kappa_2$, respectively. The eigenvectors $e_1$ and $e_2$ are called* **principal directions**.

Observe that,

$$\begin{aligned} \kappa_1 &= \kappa_1 \langle e_1, e_1\rangle = \langle L(e_1), e_1\rangle = \mathcal{L}(e_1, e_1) \\ \kappa_2 &= \kappa_2 \langle e_2, e_2\rangle = \langle L(e_2), e_2\rangle = \mathcal{L}(e_2, e_2) \,, \end{aligned}$$

i.e., the principal curvature $\kappa_1$ is the normal curvature in the principal direction $e_1$, and similarly for $\kappa_2$.

Now, let $A$ be the matrix associated to the Weingarten map $L$ with respect to the orthonormal basis $\{e_1, e_2\}$; thus,

$$\begin{aligned} L(e_1) &= \kappa_1 e_1 + 0 e_2 \\ L(e_2) &= 0 e_1 + \kappa_2 e_2 \end{aligned}$$

which implies,

$$A = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} .$$

Then,

$$\begin{aligned} \det L &= \det A = \kappa_1 \kappa_2 \\ \operatorname{tr} L &= \operatorname{tr} A = \kappa_1 + \kappa_2 \,. \end{aligned}$$

**Definition.** The *Gaussian curvature* of $M$ at $p$, $K = K(p)$, and the *mean curvature* of $M$ at $p$, $H = H(p)$ are defined as follows,

$$\begin{aligned} K &= \det L = \kappa_1 \kappa_2 \\ H &= \operatorname{tr} L = \kappa_1 + \kappa_2 \,. \end{aligned}$$

**Remarks**. The Gaussian curvature is the more important of the two curvatures; it is what is meant by the *curvature* of a surface. A famous discovery by Gauss is that it is intrinsic – in fact can be computed in terms of the $g_{ij}$'s (This is not obvious!). The *mean curvature* (which has to do with minimal surface theory) is *not* intrinsic. This can be easily seen as follows. Changing the normal $n \to -n$ changes the sign of the Weingarten map,

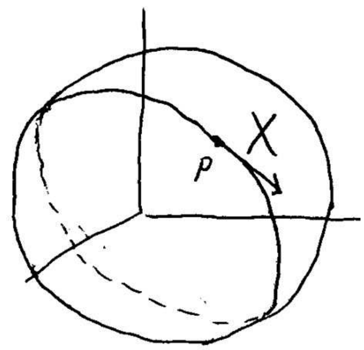
$$L_{-n} = -L_n \,.$$

This in turn changes the sign of the principal curvatures, hence $H = \kappa_1 + \kappa_2$ changes sign, but $K = \kappa_1 \kappa_2$ does not change sign.

Some Examples

**Ex.** For $S_r^2$ = sphere of radius $r$, compute $\kappa_1, \kappa_2, K, H$ (Use outward normal).

Geometrically: $p \in S_r^2$, $X \in T_pM$, $|X| = 1$,




$$\begin{aligned} \mathcal{L}(X,X) &= \pm \text{ curvature of normal section in direction } X \\ &= -\text{curvature of great circle} \\ &= -\frac{1}{r}. \end{aligned}$$

Therefore

$$\kappa_1 = \max_{|X|=1} \mathcal{L}(X,X) = -\frac{1}{r}\,,$$

$$\kappa_2 = \min_{|X|=1} \mathcal{L}(X,X) = -\frac{1}{r}\,,$$

$$K = \kappa_1\kappa_2 = \frac{1}{r^2} > 0, \quad H = \kappa_1 + \kappa_2 = -\frac{2}{r}\,.$$

Algebraically: Find eigenvalues of Weingarten map: $L : T_pM \to T_pM$. We showed previously,

$$\begin{aligned} L &= -\frac{1}{r} id, \quad \text{i.e.,} \\ L(X) &= -\frac{1}{r} X \quad \text{for all } X \in T_pM. \end{aligned}$$

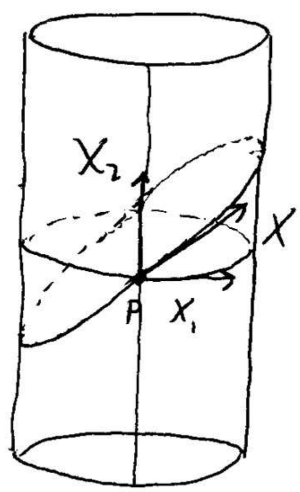
Thus, with respect to *any* orthonormal basis $\{e_1, e_2\}$ of $T_pM$,

$$L(e_i) = -\frac{1}{r} e_i \quad i = 1, 2.$$

Therefore, $\kappa_1 = \kappa_2 = -\frac{1}{r}, \quad K = \frac{1}{r^2}, \quad H = -\frac{2}{r}.$

**Ex.** Let $M$ be the cylinder of radius $a$: $x^2 + y^2 = a^2$. Compute $\kappa_1, \kappa_2, K, H$. (Use the inward pointing normal)

<u>Geometrically:</u>




$$\begin{aligned}
\mathcal{L}(X_1, X_1) &= \pm \text{ curvature of normal section in direction } X_1 \\
&= + \text{ curvature of circle of radius } a \\
&= \frac{1}{a},
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}(X_2, X_2) &= \pm \text{ curvature of normal section in direction } X_2 \\
&= \text{ curvature of line} \\
&= 0.
\end{aligned}$$

In general, for $X \neq X_1, X_2$,

$$\mathcal{L}(X, X) = \text{ curvature of ellipse through } p.$$

The curvature is between 0 and $\frac{1}{a}$, and thus,

$$0 \leq \mathcal{L}(X, X) \leq \frac{1}{a}.$$

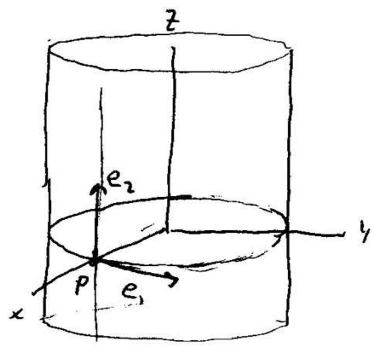
We conclude that,

$$\kappa_1 = \max_{|X|=1} \mathcal{L}(X,X) = \mathcal{L}(X_1,X_1) = \frac{1}{a}\,,$$

$$\kappa_2 = \min_{|X|=1} \mathcal{L}(X,X) = \mathcal{L}(X_2,X_2) = 0\,.$$

Thus, $K = 0$ (cylinder is flat!) and $H = \frac{1}{a}$.

Algebraically: Determine the eigenvalues of the Weingarten map. By a rotation and translation we may take $p$ to be the point $p = (a,0,0)$. Let $e_1, e_2 \in T_pM$ be the tangent vectors $e_1 = (0,1,0)$ and $e_2 = (0,0,1)$.




To compute $L(e_1)$, consider the circle,

$$\sigma(s) = (a\cos(\frac{s}{a}), a\sin(\frac{s}{a}), 0)$$

Note that $\sigma(0) = p$ and $\sigma'(0) = e_1$. Thus,

$$\begin{aligned} L(e_1) &= -\nabla_{e_1} n \\ &= -\frac{d}{ds} n(\sigma(s))|_{s=0} \end{aligned}$$

. But,

$$\begin{aligned} n(\sigma(s)) &= n(\sigma(s)) = \frac{\sigma(s)}{|\sigma(s)|} = \frac{\sigma(s)}{a} \\ &= -\left(\cos\left(\frac{s}{a}\right), \sin\left(\frac{s}{a}\right), 0\right) \end{aligned}$$

Therefore,

$$\begin{aligned} L(e_1) &= \frac{d}{ds}\left(\cos\left(\frac{s}{a}\right), \sin\left(\frac{s}{a}\right), 0\right)|_{s=0} \\ &= \frac{1}{a}\left(-\sin\left(\frac{s}{a}\right), \cos\left(\frac{s}{a}\right), 0\right)|_{s=0} \\ &= \frac{1}{a}(0,1,0) \\ L(e_1) &= \frac{1}{a}e_1 \end{aligned}$$

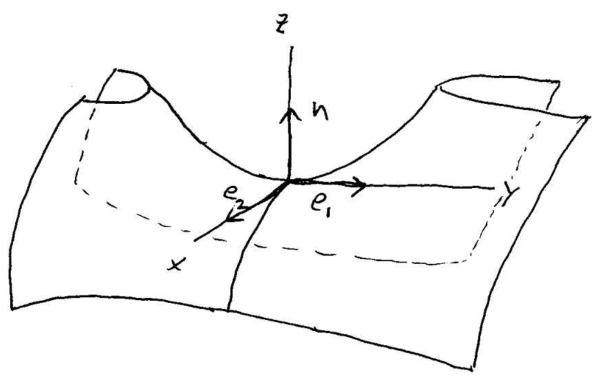
Thus, $e_1$ is an eigenvector with eigenvalue $\frac{1}{a}$. Similarly (exercise!),

$$L(e_2) = \mathbf{0} = 0 \cdot e_2$$

i.e., $e_2$ is an eigenvector with eigenvalue 0. (Note; $e_2$ is tangent to a vertical line in the surface, along which $n$ is constant.)

We conclude that, $\kappa_1 = \dfrac{1}{a}$, $\kappa_2 = 0$, $K = 0$, $H = \dfrac{1}{a}$.

**Ex.** Consider the saddle surface, $M$: $z = y^2 - x^2$, Compute $\kappa_1$, $\kappa_2$, $K$, $H$ at $p = (0,0,0)$.




$$\begin{aligned} \mathcal{L}(e_1, e_1) &= \pm \text{ curvature of normal section in direction of } e_1 \\ &= + \text{ curvature of } z = y^2 \end{aligned}$$

The curvature is given by,

$$\kappa = \frac{\left|\dfrac{d^2z}{dy^2}\right|}{\left[1 + \left(\dfrac{dz}{dy}\right)^2\right]^{3/2}} = 2$$

and so, $\mathcal{L}(e_1, e_1) = 2$. Similarly, $\mathcal{L}(e_2, e_2) = -2$. Observe,

$$\mathcal{L}(e_2, e_2) \leq \mathcal{L}(X, X) \leq \mathcal{L}(e_1, e_1)$$

Therefore, $\kappa_1 = 2$, $\kappa_2 = -2$, $K = -4$, and $H = 0$ at $(0, 0, 0)$.

**Exercise 5.5**. For the saddle surface $M$ above, consider the Weingarten map $L : T_pM \to T_pM$ at $p = (0, 0, 0)$. Compute $L(e_1)$ and $L(e_2)$ directly from the definition of the Weingarten map to show,

$$L(e_1) = 2e_1 \text{ and } L(e_2) = -2e_2\,.$$

Hence, $-2$ and $2$ are the eigenvalues of $L$, which means $\kappa_1 = 2$ and $\kappa_2 = -2$.

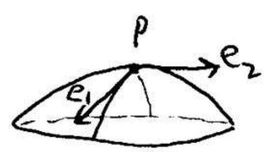
*Remark.* We have computed the quantities $\kappa_1$, $\kappa_2$, $K$, and $H$ of the saddle surface only at a single point. To compute these quantities at all points, we will need to develop better computational tools.

Significance of the sign of Gaussian Curvature

We have,

$$K = \det L = \kappa_1 \kappa_2.$$

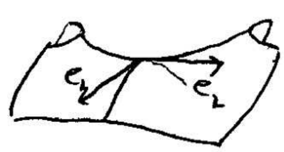
1. $K > 0 \iff \kappa_1$ and $\kappa_2$ have the same sign $\iff$ the normal sections in the principal directions $e_1, e_2$ both bend in the same direction,




$K > 0$ at $p$.

**Ex.** $z = ax^2 + by^2$, $a, b$ have the same sign (elliptic paraboloid). At $p = (0, 0, 0)$, $K = 4ab > 0$.

2. $K < 0 \iff \kappa_1$ and $\kappa_2$ have opposite signs $\iff$ normal sections in principle directions $e_1$ and $e_2$ bend in opposite directions,



$K < 0$ at $p$.

**Ex.** $z = ax^2 + by^2$, $a, b$ have opposite sign (hyperbolic paraboloid). At $p = (0, 0, 0)$, $K = 4ab < 0$.

Thus, roughly speaking,

$$K > 0 \text{ at } p \Rightarrow \text{surface is "bowl-shaped" near } p$$
$$K < 0 \text{ at } p \Rightarrow \text{surface is "saddle-shaped" near } p$$

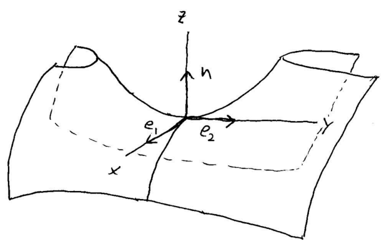
This rough observation can be made more precise, as we now show. Let $M$ be a surface, $p \in M$. Let $e_1, e_2$ be principal directions at $p$. Choose $e_1, e_2$ so that $\{e_1, e_2, n\}$ is a positively oriented orthonornal basis.

By a translation and rotation of the surface, we can assume, (see the figure),

(1) $p = (0,0,0)$

(2) $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $n = (0,0,1)$ at $p$

(3) Near $p = (0,0,0)$, the surface can be described by an equation of form, $z = f(x,y)$, where $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is smooth and $f(0,0) = 0$.




**Claim:**

$$z = \frac{1}{2}\kappa_1 x^2 + \frac{1}{2}\kappa_2 y^2 + \text{ higher order terms}$$

*Proof.* Consider the Taylor series about $(0,0)$ for functions of two variables,

$$z = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2}f_{yy}(0,0)y^2 + \text{ higher order terms}.$$

We must compute 1st and 2nd order partial derivatives of $f$ at $(0,0)$. Introduce the Monge patch,

$$\mathbf{x}: \quad \begin{aligned} x &= u \\ y &= v \\ z &= f(u,v) \end{aligned}$$

i.e. $\mathbf{x}(u,v) = (u, v, f(u,v))$.

We have,

$$\begin{aligned}
\mathbf{x}_1 &= \mathbf{x}_u = (1,0,f_u)\,, \\
\mathbf{x}_2 &= \mathbf{x}_v = (0,1,f_v)\,, \\
n &= \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} \\
&= \frac{(-f_u,-f_v,1)}{\sqrt{1+f_u^2+f_v^2}}
\end{aligned}$$

At $(u,v)=(0,0)$: $n=(0,0,1) \Rightarrow f_u = f_v = 0, \quad \Rightarrow \quad \mathbf{x}_1 = (1,0,0) = e_1$ and $\mathbf{x}_2 = (0,1,0) = e_2$.

Recall, the components of the 2nd fundamental form $L_{ij} = \mathcal{L}(\mathbf{x}_i,\mathbf{x}_j)$ may be computed from the formula,

$$L_{ij} = \langle n, \mathbf{x}_{ij}\rangle, \quad \mathbf{x}_{ij} = \frac{\partial^2 \mathbf{x}}{\partial u^j \partial u^i}\,.$$

In particular, $L_{11} = \langle n, \mathbf{x}_{11}\rangle$, where $\mathbf{x}_{11} = \mathbf{x}_{uu} = (0,0,f_{uu})$.

At $(u,v)=(0,0)$: $\quad L_{11} = \langle n, \mathbf{x}_{11}\rangle = (0,0,1)\cdot(0,0,f_{uu}(0,0)) = f_{uu}(0,0)\,.$
Therefore, $\quad f_{uu}(0,0) = L_{11} = \mathcal{L}(\mathbf{x}_1,\mathbf{x}_1) = \mathcal{L}(e_1,e_1) = \kappa_1$. Similarly,

$$\begin{aligned}
f_{vv}(0,0) &= \mathcal{L}(e_2,e_2) = \kappa_2 \\
f_{uv}(0,0) &= \mathcal{L}(e_1,e_2) = \langle L(e_1), e_2\rangle = \lambda_1(e_1,e_2) = 0.
\end{aligned}$$

Thus, setting $x=u, \;\; y=v$, we have shown,

$$\begin{aligned}
f_x(0,0) &= f_y(0,0) = 0 \\
f_{xx}(0,0) &= \kappa_1, \;\; f_{yy}(0,0) = \kappa_2, \;\; f_{xy}(0,0) = 0,
\end{aligned}$$

which, substituting in the Taylor expansion, implies,

$$z = \frac{1}{2}\kappa_1 x^2 + \frac{1}{2}\kappa_2 y^2 + \text{ higher order terms.}$$

**Computational Formula for Gaussian Curvature.**

We have,

$$\begin{aligned}
K &= \text{Gaussian curvature} \\
&= \det L = \det[L^i{}_j].
\end{aligned}$$

From the equation at the top of p. 14,

$$\begin{aligned}[L_{ij}] &= [g_{ij}][L^i{}_j], \\ \det[L_{ij}] &= \det[g_{ij}]\det[L^i{}_j] \\ &= \det[g_{ij}]\cdot K\end{aligned}$$

Hence,

$$K = \frac{\det[L_{ij}]}{\det[g_{ij}]}\,, \qquad g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j\rangle, \quad L_{ij} = \langle n, \mathbf{x}_{ij}\rangle\,.$$

Further,

$$\begin{aligned}\det[L_{ij}] &= \det\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22}\end{bmatrix} \\ &= L_{11}L_{22} - L_{12}^2\,,\end{aligned}$$

since $L_{12} = L_{21}$, and similarly,

$$\det[g_{ij}] = g_{11}, g_{22} - g_{12}^2\,.$$

Thus,

$$K = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2}\,.$$

**Ex.** Compute the Gaussian curvature of the saddle surface $z = y^2 - x^2$. Introduce the Monge patch, $\mathbf{x}(u,v) = (u, v, v^2 - u^2)$.

Compute metric components $g_{ij}$:

$$\begin{aligned}\mathbf{x}_u &= (1,0,-2u), \quad \mathbf{x}_v = (0,1,2v)\,, \\ g_{uu} &= \langle \mathbf{x}_u, \mathbf{x}_u\rangle = (1,0,-2u)\cdot(1,0,-2u) \\ &= 1 + 4u^2.\end{aligned}$$

Similarly,

$$\begin{aligned}g_{vv} &= \langle \mathbf{x}_v, \mathbf{x}_v\rangle = 1 + 4v^2\,, \\ g_{uv} &= \langle \mathbf{x}_u, \mathbf{x}_v\rangle = -4uv\,.\end{aligned}$$

Thus,

$$\begin{aligned}\det[g_{ij}] &= g_{uu}g_{vv} - g_{uv}^2 \\ &= (1+4u^2)(1+4v^2) - 16u^2v^2 \\ &= 1 + 4u^2 + 4v^2.\end{aligned}$$

Compute the second fundamental form components $L_{ij}$:

We use, $L_{ij} = \langle n, \mathbf{x}_{ij} \rangle$. We have,

$$n = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = \frac{(2u, -2v, 1)}{\sqrt{1 + 4u^2 + 4v^2}},$$

and,

$$\mathbf{x}_{uu} = (0, 0, -2), \quad \mathbf{x}_{vv} = (0, 0, 2), \quad \mathbf{x}_{uv} = (0, 0, 0).$$

Then,

$$\begin{aligned} L_{uu} &= \langle n, \mathbf{x}_{uu} \rangle = \frac{-2}{\sqrt{1 + 4u^2 + 4v^2}} \\ L_{vv} &= \langle n, \mathbf{x}_{vv} \rangle = \frac{2}{\sqrt{1 + 4u^2 + 4v^2}} \\ L_{uv} &= \langle n, \mathbf{x}_{uv} \rangle = 0. \end{aligned}$$

Thus,

$$\det[L_{ij}] = L_{uu}L_{vv} - L_{uv}^2 = \frac{-4}{1 + 4u^2 + 4v^2},$$

and therefore,

$$\begin{aligned} K(u, v) &= \frac{\det[L_{ij}]}{\det[g_{ij}]} = \frac{-4}{1 + 4u^2 + 4v^2} \cdot \frac{1}{1 + 4u^2 + 4v^2} \\ K(u, v) &= \frac{-4}{(1 + 4u^2 + 4v^2)^2} \end{aligned}.$$

Hence the saddle surface $z = y^2 - x^2$ has Gaussian curvature function,

$$K(x, y) = \frac{-4}{(1 + 4x^2 + 4y^2)^2}.$$

Observe that $K < 0$ and, $K = \dfrac{-4}{(1 + 4r^2)^2} \sim \dfrac{1}{r^4}$, where $r = \sqrt{x^2 + y^2}$ is the distance from the $z$-axis. As $r \to \infty$, $K \to 0$ rapidly.

**Exercise 5.6**. Consider the surface $M$ which is the graph of $z = f(x, y)$. Show that the Gaussian curvature $K = K(x, y)$ is given by,

$$K(x, y) = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}$$

where $f_x = \dfrac{\partial f}{\partial x}$, $f_{xx} = \dfrac{\partial^2 f}{\partial x^2}$, etc.

**Exercise 5.7.** Let $M$ be the torus of large radius $R$ and small radius $r$ described in Exercise 3.3. Using the parameterization,

$$\mathbf{x}(t,\theta) = ((R + r\cos t)\cos\theta, (R + r\cos t)\sin\theta, r\sin t)$$

show that the Gaussian curvature $K = K(t,\theta)$ is given by,

$$K = \frac{\cos t}{r(R + r\cos t)}.$$

Where on the torus is the Gaussian curvature negative? Where is it positive?

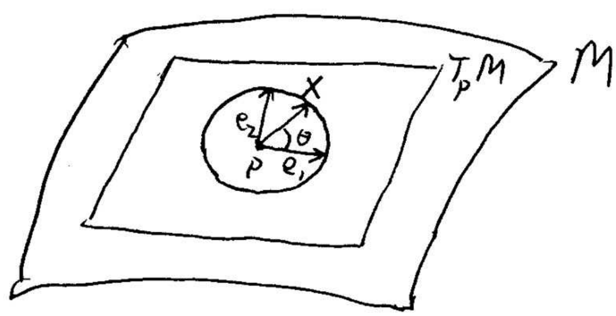
**Exercise 5.8**. Derive the following expression for the mean curvature $H$,

$$H = \frac{g_{11}L_{22} - 2g_{12}L_{12} + g_{22}L_{11}}{g_{11}g_{22} - g_{12}^2}$$

The principal curvatures $\kappa_1$ and $\kappa_2$ at a point $p \in M$ are the normal curvatures in the principal directions $e_1$ and $e_2$. The normal curvature in any direction $X$ is determined by $\kappa_1$ and $\kappa_2$ as follows.

If $X \in T_pM$, $|X| = 1$ then $X$ can be expressed as (see the figure),

$$X = \cos\theta e_1 + \sin\theta e_2 .$$




**Proposition** (Euler's formula). The normal curvature in the direction $X$ is given by,

$$\mathcal{L}(X,X) = \kappa_1\cos^2\theta + \kappa_2\sin^2\theta ,$$

where $\kappa_1, \kappa_2$ are the principal curvatures, and $\theta$ is the angle between $X$ and the principal direction $e_1$.

**Proof**. Use the shorthand, $c = \cos\theta$, $s = \sin\theta$. Then $X = ce_1 + se_2$, and

$$\begin{aligned} L(X) &= L(ce_1 + se_2) \\ &= cL(e_1) + sL(e_2) \\ &= c\kappa_1 e_1 + s\kappa_2 e_2. \end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}(X,X) &= \langle L(X), X\rangle \\ &= \langle c\kappa_1 e_1 + s\kappa_2 e_2, ce_1 + se_2\rangle \\ &= c^2\kappa_1 + s^2\kappa_2 .\end{aligned}$$

**Exercise 5.9**. Assuming $\kappa_1 > \kappa_2$, determine where (i.e., for which values of $\theta$) the function,

$$\kappa(\theta) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta, \quad 0 \le \theta \le 2\pi$$

achieves its maximum and minimum. The answer shows that the principal directions $e_1, e_2$ are unique, up to sign, in this case.

## Gauss Theorema Egregium

The Weingarten map,

$$L(X) = -\nabla_X n$$

is an *extrinsically* defined object - it involves the normal to the surface. There is no reason to suspect that the determinant of $L$, the Gaussian curvature, is *intrinsic*, i.e. can be computed from measurements taken in the surface. But Gauss carried out some courageous computations and made the extraordinary discovery that, in fact, the Gaussian curvature $K$ *is* intrinsic - i.e., can be computed from the $g_{ij}$'s. This is the most important result in the subject - albeit not the prettiest! If this result were not true then the subject of differential geometry, as we know it, would not exist.

We now embark on the same path - courageously carrying out the same computation.

*Some notation*. Introduce the "inverse" metric components, $g^{ij}$, $1 \le i, j \le 2$, by

$$[g^{ij}] = [g_{ij}]^{-1},$$

i.e. $g^{ij}$ is the $i$-$j$th entry of the inverse of the matrix $[g_{ij}]$. Using the formula,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

we can express $g^{ij}$ explicitly in terms of the $g_{ij}$, e.g.

$$g^{11} = \frac{g_{22}}{g_{11}g_{22} - g_{12}^2}, \quad \text{etc.}$$

Note, in an orthogonal coordinate system, i.e., a proper patch in which $g_{12} = \langle \mathbf{x}_1, \mathbf{x}_2\rangle = 0$, we have simply,

$$g^{11} = \frac{1}{g_{11}}, \quad g^{22} = \frac{1}{g_{22}}, \quad g^{12} = g^{21} = 0\,.$$

By definition of inverse, we have

$$[g_{ij}][g^{ij}] = I$$

where $I$ = identity matrix = $[\delta_i{}^j]$, where $\delta_i{}^j$ is the Kronecker delta (cf., Chapter 1),

$$\delta_i{}^j = \begin{cases} 0 & , i \neq j \\ 1, & i = j\,, \end{cases}$$

and so,

$$[g_{ij}][g^{ij}] = [\delta_i{}^j]\,.$$

The product formula for matrices then implies,

$$\sum_k g_{ik} g^{kj} = \delta_i{}^j$$

or, by the Einstein summation convention,

$$g_{ik} g^{kj} = \delta_i^j\,.$$

Now, let $M$ be a surface and $\mathbf{x} : U \subset I\!\!R^2 \to M \subset I\!\!R^3$ be any proper patch in $M$. Then,

$$\begin{aligned} \mathbf{x} &= \mathbf{x}(u^1, u^2) = (x(u^1, u^2), y(u^1, u^2), z(u^1, u^2))\,, \\ \mathbf{x}_i &= \frac{\partial \mathbf{x}}{\partial u^i} = \left( \frac{\partial x}{\partial u^i}, \frac{\partial y}{\partial u^i}, \frac{\partial z}{\partial u^i} \right)\,, \\ \mathbf{x}_{ij} &= \frac{\partial^2 \mathbf{x}}{\partial u^j \partial u^i} = \left( \frac{\partial^2 x}{\partial u^j \partial u^i}, \frac{\partial^2 y}{\partial u^j \partial u^i}, \frac{\partial^2 z}{\partial u^j \partial u^i} \right)\,. \end{aligned}$$

We seek useful expressions for these second derivatives. At any point $p \in \mathbf{x}(U)$, $\{\mathbf{x}_1, \mathbf{x}_2, n\}$ form a basis for $T_p I\!\!R^3$. Since at $p$, $\mathbf{x}_{ij} \in T_p I\!\!R^3$, we can write,

$$\begin{aligned} \mathbf{x}_{ij} &= \Gamma^1_{ij}\mathbf{x}_1 + \Gamma^2_{ij}\mathbf{x}_2 + \lambda_{ij} n\,, \\ \mathbf{x}_{ij} &= \sum_{\ell=1}^{2} \Gamma^\ell_{ij}\mathbf{x}_\ell + \lambda_{ij} n\,. \end{aligned}$$

or, making use of the Einstein summation convention,

$$\mathbf{x}_{ij} = \Gamma^\ell_{ij}\mathbf{x}_\ell + \lambda_{ij} n\,. \qquad (*)$$

We obtain expressions for $\lambda_{ij}, \Gamma^{\ell}_{ij}$. Dotting $(*)$ with $n$ gives,

$$\begin{aligned} \langle \mathbf{x}_{ij}, n \rangle &= \Gamma^{\ell}_{ij}\langle \mathbf{x}_{\ell}, n\rangle + \lambda_{ij}\langle n, n\rangle \\ \Rightarrow \quad \lambda_{ij} &= \langle \mathbf{x}_{ij}, n\rangle = \langle n, \mathbf{x}_{ij}\rangle \\ \lambda_{ij} &= L_{ij}\,. \end{aligned}$$

Dotting $(*)$ with $\mathbf{x}_k$ gives,

$$\begin{aligned} \langle \mathbf{x}_{ij}, \mathbf{x}_k \rangle &= \Gamma^{\ell}_{ij}\langle \mathbf{x}_{\ell}, \mathbf{x}_k\rangle + \lambda_{ij}\langle n, \mathbf{x}_k\rangle \\ \langle \mathbf{x}_{ij}, \mathbf{x}_k \rangle &= \Gamma^{\ell}_{ij} g_{\ell k}\,. \end{aligned}$$

Solving for $\Gamma^{\ell}_{ij}$,

$$\begin{aligned} \langle \mathbf{x}_{ij}, \mathbf{x}_k \rangle g^{km} &= \Gamma^{\ell}_{ij} g_{\ell k} g^{km} \\ &= \Gamma^{\ell}_{ij}\delta^{m}_{\ell} \\ \langle \mathbf{x}_{ij}, \mathbf{x}_k \rangle g^{km} &= \Gamma^{m}_{ij} \end{aligned}$$

Thus,

$$\Gamma^{\ell}_{ij} = g^{k\ell}\langle \mathbf{x}_{ij}, \mathbf{x}_k\rangle$$

**Claim**. The quantity $\langle \mathbf{x}_{ij}, \mathbf{x}_k\rangle$ is given by,

$$\begin{aligned} \langle \mathbf{x}_{ij}, \mathbf{x}_k \rangle &= \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right) \\ &= \frac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k}) \end{aligned}.$$

**Proof of Claim**. We use Gauss' trick of permuting indices.

$$\begin{aligned} g_{ij,k} &= \frac{\partial}{\partial u^k} g_{ij} = \frac{\partial}{\partial u^k}\langle \mathbf{x}_i, \mathbf{x}_j\rangle \\ &= \langle \frac{\partial \mathbf{x}_i}{\partial u^k}, \mathbf{x}_j\rangle + \langle \mathbf{x}_i, \frac{\partial \mathbf{x}_j}{\partial u^k}\rangle \end{aligned}$$

$$(1) \quad g_{ij,k} = \langle \mathbf{x}_{ik}, \mathbf{x}_j\rangle + \langle \mathbf{x}_i, \mathbf{x}_{jk}\rangle$$

$$(j \leftrightarrow k) \quad (2) \quad g_{ik,j} = \langle \mathbf{x}_{ij}, \mathbf{x}_k\rangle + \langle \mathbf{x}_i, \mathbf{x}_{kj}\rangle$$

$$(i \leftrightarrow j) \quad (3) \quad g_{jk,i} = \langle \mathbf{x}_{ji}, \mathbf{x}_k\rangle + \langle \mathbf{x}_j, \mathbf{x}_{ki}\rangle$$

Then $(2) + (3) - (1)$ gives:

$$g_{ik,j} + g_{jk,i} - g_{ij,k} = 2\langle \mathbf{x}_{ij}, \mathbf{x}_k \rangle$$

Thus,

$$\Gamma^\ell_{ij} = \frac{1}{2} g^{k\ell}(g_{ik,j} + g_{jk,i} - g_{ij,k})\,.$$

**Remark**. These are known as the *Christoffel* symbols.

Summarizing, we have,

$$\mathbf{x}_{ij} = \Gamma^\ell_{ij}\mathbf{x}_\ell + L_{ij}n, \qquad \textit{Gauss Formula}$$

where $L_{ij}$ are the components of the 2nd fundamental form and $\Gamma^\ell_{ij}$ are the Christoffel symbols as given above. Let us also recall the *Weingarten* equations (p. 11),

$$n_j = -L^i{}_j \mathbf{x}_i$$

**Remark.** The vector fields $\mathbf{x}_1, \mathbf{x}_2, n$, play a role in surface theory roughly analogous to the Frenet frame for curves. The two formulas above for the partial derivatives of $\mathbf{x}_1, \mathbf{x}_2, n$ then play a role roughly analogous to the Frenet formulas.

Now, Gauss takes things one step further and computes the 3rd derivatives, $\mathbf{x}_{ijk} = \dfrac{\partial}{\partial u^k}\mathbf{x}_{ij}$:

$$\begin{aligned}
\mathbf{x}_{ijk} &= \frac{\partial}{\partial u^k}(\Gamma^\ell_{ij}\mathbf{x}_\ell + L_{ij}n) = \frac{\partial}{\partial u^k}\Gamma^\ell_{ij}\mathbf{x}_\ell + \frac{\partial}{\partial u^k}L_{ij}n \\
&= \Gamma^\ell_{ij,k}\mathbf{x}_\ell + \Gamma^\ell_{ij}\mathbf{x}_{\ell k} + L_{ij,k}n + L_{ij}n_k \\
&= \Gamma^\ell_{ij,k}\mathbf{x}_\ell + \Gamma^\ell_{ij}(\Gamma^m_{\ell k}\mathbf{x}_m + L_{\ell k}n) + L_{ij,k}n + L_{ij}(-L^\ell{}_k\mathbf{x}_\ell) \\
&= \Gamma^\ell_{ij,k}\mathbf{x}_\ell + \underbrace{\Gamma^\ell_{ij}\Gamma^m_{\ell k}\mathbf{x}_m}_{\Gamma^m_{ij}\Gamma^\ell_{mk}\mathbf{x}_\ell} + \Gamma^\ell_{ij}L_{\ell k}n + L_{ij,k}n - L_{ij}L^\ell{}_k\mathbf{x}_\ell
\end{aligned}$$

Thus,

$$\mathbf{x}_{ijk} = (\Gamma^\ell_{ij,k} + \Gamma^m_{ij}\Gamma^\ell_{mk} - L_{ij}L^\ell{}_k)\mathbf{x}_\ell + (L_{ij,k} + \Gamma^\ell_{ij}L_{\ell k})n\,,$$

and interchanging indices $(j \leftrightarrow k)$,

$$\mathbf{x}_{ikj} = (\Gamma^\ell_{ik,j} + \Gamma^m_{ik}\Gamma^\ell_{mj} - L_{ik}L^\ell{}_j)\mathbf{x}_\ell + (L_{ik,j} + \Gamma^\ell_{ik}L_{\ell j})n\,.$$

Now, $\mathbf{x}_{ikj} = \mathbf{x}_{ijk}$ implies

$$\Gamma^\ell_{ik,j} + \Gamma^m_{ik}\Gamma^\ell_{mj} - L_{ik}L^\ell{}_j = \Gamma^\ell_{ij,k} + \Gamma^m_{ij}\Gamma^\ell_{mk} - L_{ij}L^\ell{}_k =$$

or,

$$\underbrace{\Gamma^{\ell}_{ik,j} - \Gamma^{\ell}_{ij,k} + \Gamma^{m}_{ik}\Gamma^{\ell}_{mj} - \Gamma^{m}_{ij}\Gamma^{\ell}_{mk}}_{R^{\ell}_{ijk}} = L_{ik}L^{\ell}{}_{j} - L_{ij}L^{\ell}{}_{k}\,.$$

These are the components of the famous *Riemann curvature tensor*. **Observe**: $R^{\ell}_{ijk}$ are *intrinsic*, i.e. can be computed from the $g_{ij}$'s (involve 1st and 2nd derivatives of the $g_{ij}$'s).

We arrive at,

$$R^{\ell}_{ijk} = L_{ik}L^{\ell}{}_{j} - L_{ij}L^{\ell}{}_{k} \qquad \textit{The Gauss Equations}\,.$$

**Gauss' Theorem Egregium.** *The Gaussian curvature of a surface is intrinsic, i.e. can be computed in terms of the $g_{ij}$'s.*

*Proof.* This follows from the Gauss equations. Multiply both sides by $g_{m\ell}$,

$$g_{m\ell}R^{\ell}_{ijk} = L_{ik}g_{m\ell}L^{\ell}{}_{j} - L_{ij}g_{m\ell}L^{\ell}{}_{k}.$$

But recall (see p. 13),

$$L_{mj} = g_{m\ell}L^{\ell}{}_{j}\,.$$

Hence,

$$g_{m\ell}R^{\ell}_{ijk} = L_{ik}L_{mj} - L_{ij}L_{mk}\,.$$

Setting $i = k = 1$, $m = j = 2$ we obtain,

$$\begin{aligned} g_{2\ell}R^{\ell}_{121} &= L_{11}L_{22} - L_{12}L_{21} \\ &= \det[L_{ij}]\,. \end{aligned}$$

Thus,

$$\begin{aligned} K &= \frac{\det[L_{ij}]}{\det[g_{ij}]} \\ K &= \frac{g_{2\ell}R^{\ell}_{121}}{g}, \qquad g = \det[g_{ij}] \end{aligned}$$

□

**Comment**. Gauss' Theorema Egregium can be interpreted in a slightly different way in terms of *isometries*. We discuss this point here very *briefly* and very *informally*.

Let $M$ and $N$ be two surfaces. A one-to-one, onto map $f : M \to N$ that preserves lengths and angles is called an isometry. (This may be understood at the level of tangent vectors: $f$ takes curves to curves, and hence velocity vectors to velocity vectors. $f$ is an isometry $\Leftrightarrow$ $f$ preserves angle between velocity vectors and preserve length of velocity vectors.)

**Ex.** The process of wrapping a piece of paper into a cylinder is an isometry.

**Theorem** Gaussian curvature is a *bending invariant*, i.e. is invariant under isometries, by which we mean: if $f : M \to N$ is an isometry then

$$K_N(f(p)) = K_M(p)\,,$$

i.e., the Gaussian curvature is the same at corresponding points.

**Proof** $f$ preserves lengths and angles. Hence, in appropriate coordinate systems, the metric components for $M$ and $N$ are the same. By the formula for $K$ above, the Gaussian curvature will be the same at corresponding points.

**Application 1**. The cylinder has Gaussian curvature $K = 0$ (because a plane has zero Gaussian curvature).

**Application 2**. No piece of a plane can be bent into a piece of a sphere without distorting lengths (because $K_{\text{plane}} = 0$, $K_{\text{sphere}} = \dfrac{1}{r^2}$, $r$ =radius).

**Theorem (Riemann).** Let $M$ be a surface with vanishing Gaussian curvature, $K = 0$. Then each $p \in M$ has a neighborhood which is isometric to an open set in the Euclidean plane.

**Exercise 5.10**. Although The Gaussian curvature $K$ is a "bending invariant", show that the principal curvatures $\kappa_1, \kappa_2$ are not. I.e., show that the principle curvatures are not in general invariant under an isometry. (Hint: Consider the bending of a rectangle into a cylinder).