Chapter 4. The First Fundamental Form (Induced Metric)

We begin with some definitions from linear algebra.

Def. Let V be a vector space (over $\mathbb{R}$). A *bilinear form* on V is a map of the form $B : V \times V \to \mathbb{R}$ which is *bilinear*, i.e. linear in each "slot",

$$\begin{aligned} B(aX + bY, Z) &= aB(X, Z) + bB(Y, Z), \\ B(X, cY + dZ) &= cB(X, Y) + dB(X, Z). \end{aligned}$$

A bilinear form B is *symmetric* provided $B(X, Y) = B(Y, X)$ for all $X, Y \in V$.

Def. Let V be a vector space. An *inner* product on V is a bilinear form $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ which is symmetric and positive definite.

1. <u>bilinear:</u> linear in each slot,
2. <u>symmetric:</u> $\langle X, Y\rangle = \langle Y, X\rangle$ for all X, Y.
3. <u>positive definite:</u> $\langle X, X\rangle \geq\ 0\ \ \forall\ X$, and $= 0$ iff $X = 0$.

Ex. $\langle\ ,\ \rangle : T_p\mathbb{R}^3 \times T_p\mathbb{R}^3 \to \mathbb{R}$,

$$\langle X, Y\rangle = X \cdot Y \quad \text{(usual Euclidean dot product)}\,.$$

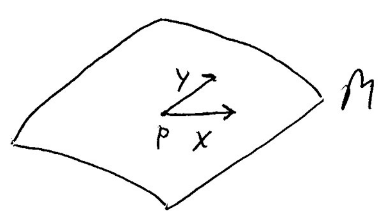
Exercise 4.1 Verify carefully that the Euclidean dot product is indeed an inner product.

Def. Let M be a surface. A *metric* on M is an assignment, to each point $p \in M$, of an inner product $\langle\ ,\ \rangle : T_pM \times T_pM \to \mathbb{R}$.

Because our surfaces sit in Euclidean space, they inherit in a natural way, a metric called the *induced metric* or *first fundamental form*.

Def. Let M be a surface. The *induced metric* (or *first fundamental form*) of M is the assignment to each $p \in M$ of the inner product,

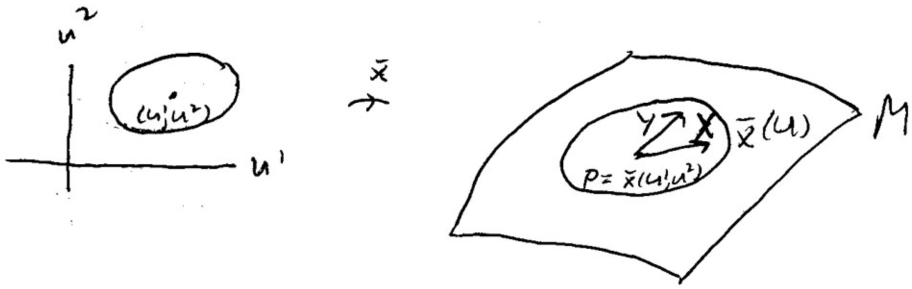
$$\begin{aligned} \langle\ ,\ \rangle : T_pM \ &\times\ T_pM \to \mathbb{R}\,, \\ \langle X, Y\rangle \ &=\ X \cdot Y \quad \text{(ordinary scalar product of } X \text{ and } Y \\ &\qquad\qquad \text{viewed as vectors in } \mathbb{R}^3 \text{ at } p) \end{aligned}$$



I.e., the induced metric is just the Euclidean dot product, restricted to the tangent spaces of M.

We will only consider surfaces in the induced metric. Just as the Euclidean dot product contains all geometric information about $\mathbb{R}^3$, the induced metric contains all geometric information about M, as we shall see.

The Metric in a Coordinate Patch.



Let $\mathbf{x} : U \subset \mathbb{R}^2 \to M \subset \mathbb{R}^3$ be a proper patch in M. Let $p \in \mathbf{x}(U)$ be any point in $\mathbf{x}(U)$, $p = \mathbf{x}(u^1, u^2)$, and let $X, Y \in T_pM$. Then,

$$\begin{aligned} X &= X^1 \frac{\partial \mathbf{x}}{\partial u^1} + X^2 \frac{\partial \mathbf{x}}{\partial u^2} = X^1 \mathbf{x}_1 + X^2 \mathbf{x}_2 \,, \\ X &= \sum_i X^i \mathbf{x}_i \,, \quad \mathbf{x}_i = \mathbf{x}_i(u^1, u^2) \,, \end{aligned}$$

and similarly,

$$Y = \sum_j Y^j \mathbf{x}_j \,.$$

Then,

$$\begin{aligned} \langle X, Y \rangle &= \langle \sum_i X^i \mathbf{x}_i, \sum_j Y^j \mathbf{x}_j \rangle \\ &= \sum_{i,j} X^i Y^j \langle \mathbf{x}_i, \mathbf{x}_j \rangle. \end{aligned}$$

The *metric components* are the functions $g_{ij} : U \to \mathbb{R}$, $1 \le i, j \le 2$, defined by

$$g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle \,, \quad g_{ij} = g_{ij}(u^1, u^2) \,.$$

Thus, in coordinates,

$$\langle X, Y \rangle = \sum_{i,j=1}^{2} g_{ij} X^i Y^j.$$

Note that the metric in $\mathbf{x}(U)$ is completely determined by the g_{ij}'s. The metric components may be displayed by a 2×2 matrix,

$$[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} \\ \\ g_{21} & g_{22} \end{bmatrix},$$

Note: $g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle = \langle \mathbf{x}_j, \mathbf{x}_i \rangle = g_{ji}$. Hence, the matrix of metric components is symmetric; and there are only three distinct components,

$$g_{11} = \langle \mathbf{x}_1, \mathbf{x}_1 \rangle, \quad g_{12} = \langle \mathbf{x}_1, \mathbf{x}_2 \rangle = \langle \mathbf{x}_2, \mathbf{x}_1 \rangle = g_{21}, \quad g_{22} = \langle \mathbf{x}_2, \mathbf{x}_2 \rangle$$

Notation:

1. Gauss: $g_{11} = E, \quad g_{12} = g_{21} = F, \quad g_{22} = G.$
2. $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$. Then one writes:

$$g_{uu} = \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \; g_{uv} = \langle \mathbf{x}_u, \mathbf{x}_v \rangle, \quad g_{vv} = \langle \mathbf{x}_v, \mathbf{x}_v \rangle \,.$$

Ex. Consider the parameterization of S^2_r in terms of geographic coordinates,

$$\mathbf{x}(\theta, \phi) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta) \;,$$

$0 < \theta < \pi, \quad 0 < \phi < 2\pi$. We compute the metric components in these coordinates. We have,

$$\begin{aligned}
\mathbf{x}_\theta \quad &= \frac{\partial \mathbf{x}}{\partial \theta} = r(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \\
\mathbf{x}_\phi \quad &= r(-\sin\theta\sin\phi, \sin\theta\cos\phi, 0)\,, \\
g_{\theta\theta} \quad &= \langle \mathbf{x}_\theta, \mathbf{x}_\theta \rangle \\
&= r^2[\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta] \\
&= r^2(\cos^2\theta + \sin^2\theta) = r^2\,, \\
g_{\theta\phi} \quad &= \langle \mathbf{x}_\theta, \mathbf{x}_\phi \rangle \\
&= r^2[-\cos\theta\cos\phi\sin\theta\sin\phi + \cos\theta\sin\phi\sin\theta\cos\phi] \\
&= 0 \qquad \text{(geometric significance?)}\,, \\
g_{\phi\phi} \quad &= r^2[\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi] \\
&= r^2\sin^2\theta\,.
\end{aligned}$$

Thus,

$$[g_{ij}] = \begin{bmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\theta\phi} & g_{\phi\phi} \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \sin^2 \theta \end{bmatrix}$$

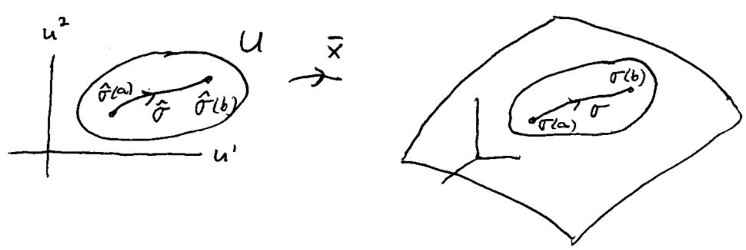
<u>Length and Angle Measurement in M.</u>

Let $\sigma : [a,b] \to M \subset I\!R^3$ be a smooth curve in a surface M. Viewed as a curve in $I\!R^3$, $\sigma(t) = (x(t), y(t), z(t))$, we can compute its length by the formula,

$$\begin{aligned} \text{Length of } \sigma &= \int_a^b \left| \frac{d\sigma}{dt} \right| dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \end{aligned}$$

But this formula does not make sense to creatures living in the surface: x, y, z are Euclidean *space* coordinates. Creatures living in the surface must use *surface* coordinates – i.e., we must express σ in terms of surface coordinates.

Let $\mathbf{x} : U \to M \subset I\!R^3$ be a proper patch in M and suppose σ is contained in this patch, $\sigma \subset \mathbf{x}(U)$:



We express σ in terms of coordinates: $\hat\sigma = x^{-1} \circ \sigma : [a,b] \to U \subset I\!R^2$, $\hat\sigma(t) = (u'(t), u^2(t))$, i.e,

$$\hat\sigma : \begin{array}{l} u^1 = u^1(t) \\ u^2 = u^2(t) \end{array}, \quad a \le t \le b.$$

Then, $\sigma = \mathbf{x} \circ \hat\sigma$, i.e., $\sigma(t) = \mathbf{x}(\hat\sigma(t))$, hence,

$$\sigma(t) = \mathbf{x}(u^1(t), u^2(t))\,.$$

By the chain rule,

$$\begin{aligned} \frac{d\sigma}{dt} &= \frac{\partial \mathbf{x}}{\partial u^1}\frac{du^1}{dt} + \frac{\partial \mathbf{x}}{\partial u^2}\frac{du^2}{dt} \\ &= \frac{du^1}{dt}\mathbf{x}_1 + \frac{du^2}{dt}\mathbf{x}_2\,, \end{aligned}$$

or,

$$\frac{d\sigma}{dt} = \sum_i \frac{du^i}{dt}\mathbf{x}_i\,.$$

This shows that $\dfrac{du^i}{dt}$, $i = 1, 2$, are the components of the velocity vector with respect to the basis $\{\mathbf{x}_1, \mathbf{x}_2\}$.

Computing the dot product,

$$\begin{aligned}
\langle\frac{d\sigma}{dt}, \frac{d\sigma}{dt}\rangle &= \langle\sum_i \frac{du^i}{dt}\mathbf{x}_i, \sum_j \frac{du^j}{dt}\mathbf{x}_j\rangle \\
&= \sum_{i,j} \frac{du^i}{dt}\frac{du^j}{dt}\langle\mathbf{x}_i, \mathbf{x}_j\rangle \\
&= \sum_{i,j=1}^{2} g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}.
\end{aligned}$$

Hence, for the *speed* in surface coordinates, we have,

$$\left|\frac{d\sigma}{dt}\right| = \sqrt{\sum_{i,j=1}^{2} g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}}.$$

For length, we then have,

$$\begin{aligned}
\text{Length of } \sigma &= \int_a^b \sqrt{\sum_{i,j} g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}}\; dt \\
&= \int_a^b \sqrt{g_{11}\left(\frac{du^1}{dt}\right)^2 + 2g_{12}\frac{du^1}{dt}\frac{du^2}{dt} + g_{22}\left(\frac{du^2}{dt}\right)^2}\; dt
\end{aligned}$$

Arc length. Let s denote arc length along σ, s can be computed in terms of t as follows. $s = s(t)$, $a \le t \le b$,

$$\begin{aligned}
s(t) &= \text{length of } \sigma \text{ from time } a \text{ to time } t \\
&= \int_a^t \sqrt{\sum_{i,j} g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}}\; dt\,.
\end{aligned}$$

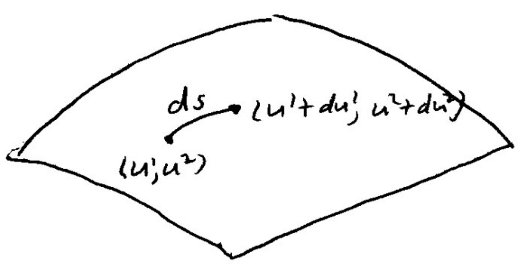
Arc length element:

$$\frac{ds}{dt} = \sqrt{\sum_{i,j} g_{ij} \frac{du^i}{dt}\frac{du^j}{dt}}$$

In terms of differentials,

$$\begin{aligned} ds &= \sqrt{\sum_{i,j} g_{ij} \frac{du^i}{dt}\frac{du^j}{dt}}\, dt \\ ds^2 &= \left(\sum_{i,j} g_{ij} \frac{du^i}{dt}\frac{du^j}{dt}\right) dt^2 \\ &= \sum_{i,j} g_{ij} \left(\frac{du^i}{dt} dt\right)\left(\frac{du^j}{dt} dt\right) \\ ds^2 &= \sum_{i,j=1}^{2} g_{ij}\, du^i du^j \ . \end{aligned}$$

Heuristics: element of arc length



$$ds^2 = \sum_{i,j} g_{ij}\, du^i du^j$$

Traditionally, one displays the metric (or, metric components g_{ij}) by writing out the arc length element.

Notations:

$$ds^2 = g_{11}(du^1)^2 + 2g_{12}du^1du^2 + g_{22}(du^2)^2$$

$$ds^2 = g_{uu}du^2 + 2g_{uv}du\,dv + g_{vv}dv^2$$

$$(u^1 = u, u^2 = v)$$

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \qquad \text{(Gauss).}$$

Remark. These expressions for arc length element of a surface M generalize the expression for the arc length element in the Euclidean u-v plane we encounter in calculus,

$$ds^2 = du^2 + dv^2$$

(i.e. $g_{uu} = 1, \ \ g_{uv} = 0, \ \ g_{vv} = 1$).

Ex. Write out the arc length element for the sphere S^2_r parameterized in terms of geographic coordinates,

$$\mathbf{x}(\theta, \phi) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r\cos\theta) \ .$$

We previously computed the g_{ij}'s,

$$[g_{ij}] = \begin{bmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{bmatrix}$$

i.e., $g_{\theta\theta} = r^2, \ \ g_{\theta\phi} = g_{\phi\theta} = 0, \ \ g_{\phi\phi} = r^2 \sin^2\theta$. So,

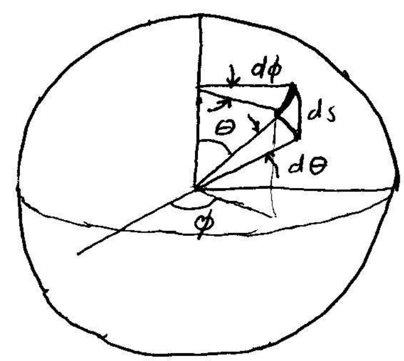
$$\begin{aligned} ds^2 &= g_{\theta\theta} d\theta^2 + 2 g_{\theta\phi} d\theta d\phi + g_{\phi\phi} d\phi^2 \\ ds^2 &= r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \ . \end{aligned}$$

But this expression is familiar from calculus as the arc length element which can be derived from heuristic geometric considerations.

$$ds^2 = d\ell_1^2 + d\ell_2^2.$$

$$d\ell_1 = r d\theta, \ \ d\ell_2 = r\sin\theta d\phi$$

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

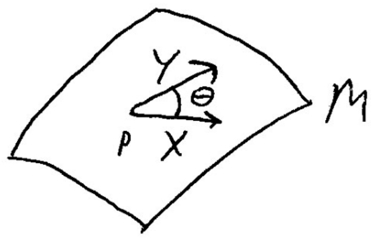


Exercise 4.2. Consider the parameterization of the x-y plane in terms of *polar coordinates*,

$$\mathbf{x}: \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = 0 \end{array} \ , 0 < r < \infty, \ \ 0 < \theta < 2\pi \ ,$$

i.e., $\mathbf{x}(r, \theta) = (r\cos\theta, r\sin\theta, 0), \ 0 < r < \infty, \ 0 < \theta < 2\pi$. Compute the g_{ij}'s with respect to these coordinates. Show that the arc length element in this case is: $ds^2 = dr^2 + r^2 d\theta^2$.

Angle Measurement.



$$X = \sum_i X^i \mathbf{x}_i,$$
$$Y = \sum_j Y^j \mathbf{x}_j,$$

$$\begin{aligned}\cos\theta &= \frac{\langle X, Y\rangle}{|X||Y|}\\ &= \frac{\sum g_{ij} X^i Y^j}{\sqrt{\sum g_{ij} X^i X^j}\ \sqrt{\sum g_{ij} Y^i Y^j}}\ .\end{aligned}$$

Ex. Determine the angle between the coordinate vectors $\mathbf{x}_1 = \dfrac{\partial \mathbf{x}}{\partial u^1}$ and $\mathbf{x}_2 = \dfrac{\partial \mathbf{x}}{\partial u^2}$ in terms of the g_{ij}'s.

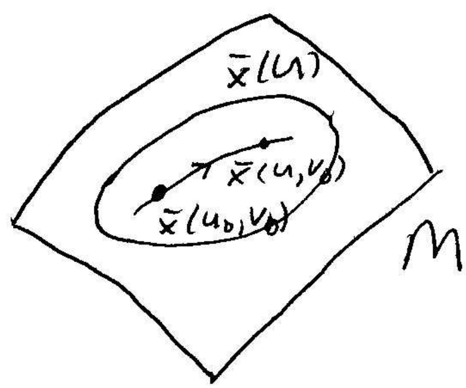
$$\cos\theta = \frac{\langle \mathbf{x}_1, \mathbf{x}_2\rangle}{|\mathbf{x}_1|\ \mathbf{x}_2|} = \frac{g_{12}}{\sqrt{g_{11}}\sqrt{g_{22}}}$$

($|\mathbf{x}_1| = \sqrt{\langle \mathbf{x}_1, \mathbf{x}_1\rangle} = \sqrt{g_{11}}$, etc.)

The Metric is intrinsic:

This discussion is somewhat heuristic. We claim that the g_{ij}'s are *intrinsic*, i.e. in principle they can be determined by measurements made in the surface.

Let $\mathbf{x} : U \to M$ be a proper patch in M; $\mathbf{x} = \mathbf{x}(u^1, u^2) = \mathbf{x}(u, v)$ (i.e., $u^1 = u$, $u^2 = v$. Consider the coordinate curve $u \xrightarrow{\sigma} \mathbf{x}(u, v_0)$ passing through $\mathbf{x}(u_0, v_0)$.



Let $s = s(u)$ be the arc length function along σ, i.e.,

$$\begin{aligned} s(u) &= \text{length of } \sigma \text{ from } u_0 \text{ to } u \\ &= \int_{u_0}^{u} \left| \frac{\partial \mathbf{x}}{\partial u} \right| du \\ &= \int_{u_0}^{u} \sqrt{g_{uu}} du \qquad \left(\left| \frac{\partial \mathbf{x}}{\partial u} \right| = \sqrt{g_{uu}} \right). \end{aligned}$$

By making length measurements in the surface the function $s = s(u)$ is known. Then by calculus, the derivative,

$$\frac{ds}{du} = \sqrt{g_{uu}} \,.$$

is known. Therefore $g_{11} = g_{uu}$, and similarly $g_{22} = g_{vv}$, can in principal be determined by measurements made in the surface.

The metric component g_{12} can then be determined by angle measurement,

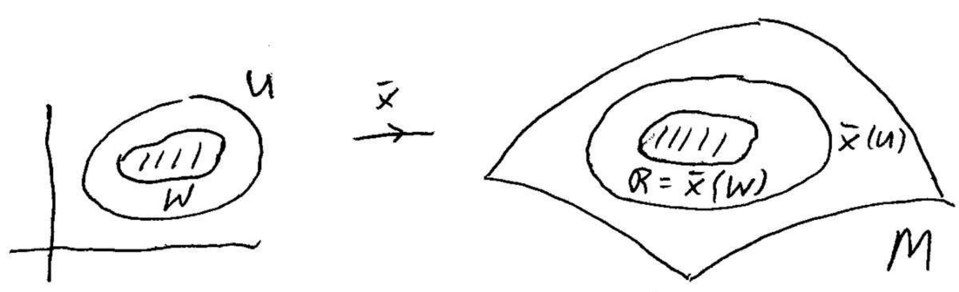
$$\begin{aligned} g_{12} &= \langle \mathbf{x}_1, \mathbf{x}_2 \rangle = |\mathbf{x}_1||\mathbf{x}_2| \cos\theta \\ &= \sqrt{g_{11}}\sqrt{g_{22}} \cdot \cos(\text{angle between } \mathbf{x}_1, \mathbf{x}_2) \,. \end{aligned}$$

Hence g_{12} is also measurable. Thus *all* metric components can be determined by measurements made in the surface, i.e.

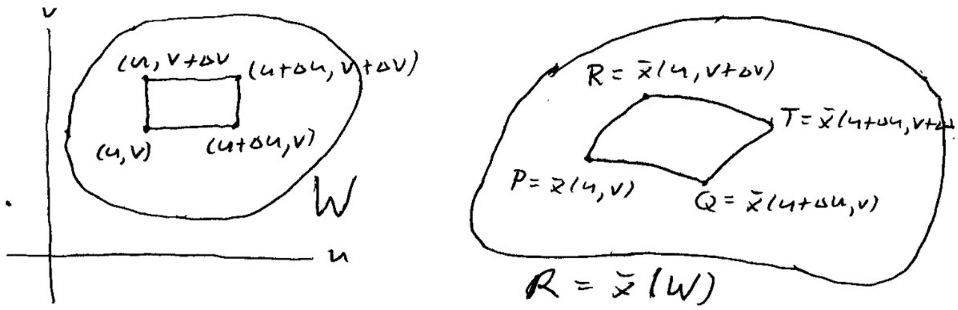
the metric components and all quantities determined from them are intrinsic.

Surface Area.

Let M be a surface, and let $\mathbf{x} : U \to M$ be a proper patch in M. Consider a bounded region $\mathcal{R}$ contained in $\mathbf{x}(U)$; we have $\mathcal{R} = \mathbf{x}(\mathcal{W})$ for some bounded region W in U:



We want to obtain (i.e. heuristically motivate) a formula for the area of $\mathcal{R} = \mathbf{x}(W)$. Restrict attention to $\mathcal{R} = \mathbf{x}(W)$; partition W into small rectangles:



Let $\Delta S =$ area of the small patch corresponding to the coordinate rectangle. Then,

$$\begin{aligned} \Delta S &\approx \text{area of the parallelogram spanned by } \vec{PQ} \text{ and } \vec{PR}, \\ \Delta S &\approx |\, \vec{PQ} \times \vec{PR} \,| \,. \end{aligned}$$

But,

$$\begin{aligned} \vec{PQ} &= \mathbf{x}(u+\Delta u, v) - \mathbf{x}(u,v) \approx \frac{\partial \mathbf{x}}{\partial u}\Delta u \,, \\ \vec{PR} &= \mathbf{x}(u, v+\Delta v) - \mathbf{x}(u,v) \approx \frac{\partial \mathbf{x}}{\partial v}\Delta v \,, \end{aligned}$$

and thus,

$$\begin{aligned} \Delta S &\approx \left| \frac{\partial \mathbf{x}}{\partial u}\Delta u \times \frac{\partial \mathbf{x}}{\partial v}\Delta v \right| \\ &\approx \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| \Delta u \Delta v \,. \end{aligned}$$

The smaller the increments Δu and Δv, the better the approximation.

$$\begin{aligned} dS &= \text{the area element of the surface corresponding to the} \\ & \quad\ \text{coordinate increments } du,\ dv \,, \\ dS &= \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| du\ dv \,. \end{aligned}$$

To obtain the total area of $\mathcal{R}$, we must sum up all these area elements - but the summing up process is integration:

$$\begin{aligned} \text{Area of } \mathcal{R} &= \iint dS \,, \\ \text{Area of } \mathcal{R} &= \iint_W \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| du\, dv \,, \end{aligned}$$

where $\mathcal{R} = \mathbf{x}(W)$.

This is a perfectly reasonable formula for computing surface area - but *not* for 2-dimensional creatures living in the surface. It involves the cross product which is an $\mathbb{R}^3$ concept. We now show how this area formula can be expressed in an *intrinsic* way (i.e. involving the g_{ij}'s).

Using generic notation, $u^1 = u,\ u^2 = v,\ \mathbf{x} = \mathbf{x}(u^1, u^2)$ we write,

$$\begin{aligned}\text{Area of } \mathcal{R} &= \iint_W \left| \frac{\partial \mathbf{x}}{\partial u^1} \times \frac{\partial \mathbf{x}}{\partial u^2} \right| du^1 du^2 \\ &= \iint_W |\mathbf{x}_1 \times \mathbf{x}_2| du^1 du^2\end{aligned}$$

Now introduce the notation,

$$g = \det[g_{ij}], \quad g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle \,.$$

Lemma. $g = |\mathbf{x}_1 \times \mathbf{x}_2|^2$

Proof. Recall the vector identity,

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \,.$$

Hence,

$$\begin{aligned}|\mathbf{x}_1 \times \mathbf{x}_2|^2 &= |\mathbf{x}_1|^2 |\mathbf{x}_2|^2 - \langle \mathbf{x}_1, \mathbf{x}_2 \rangle^2 \\ &= \langle \mathbf{x}_1, \mathbf{x}_1 \rangle \langle \mathbf{x}_2, \mathbf{x}_2 \rangle - \langle \mathbf{x}_1, \mathbf{x}_2 \rangle^2 \\ &= g_{11} g_{22} - g_{12}^2 = g,\end{aligned}$$

$$g = \det[g_{ij}] = \det \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = g_{11} g_{22} - g_{12}^2 \,,$$

where we have used $g_{21} = g_{12}$. Thus, the surface area formula may be expressed as,

$$\begin{aligned}\text{Area of } \mathcal{R} &= \iint_W \sqrt{g}\, du^1 du^2 \qquad (\mathcal{R} = \mathbf{x}(W)) \\ &= \iint_W dS \,,\end{aligned}$$

where,

$$dS = \sqrt{g}\, du^1 du^2.$$

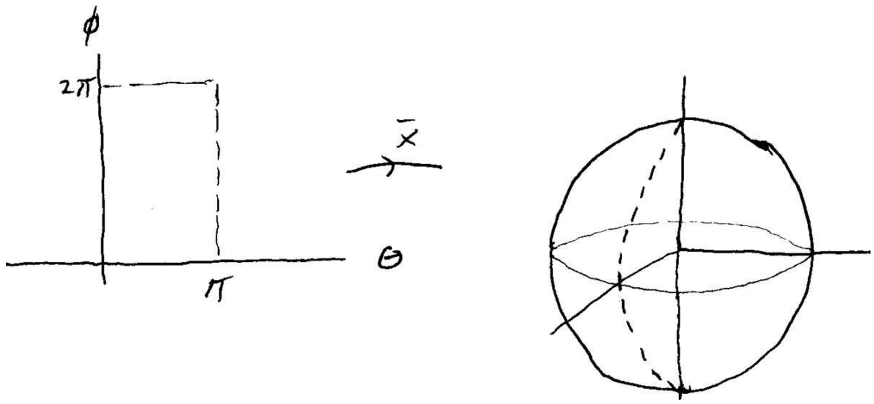
Ex. Compute the area of the sphere of radius r.

$$S_r^2 : x^2 + y^2 + z^2 = r^2 \,.$$

Parameterize with respect to geographical coordinates, $\mathbf{x} : U \to S_r^2$,

$$\mathbf{x}(\theta, \phi) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta) \,,$$

$U : 0 < \theta < \pi,\ 0 < \phi < 2\pi$.



We have,

$$\text{Area of } S_r^2 = \iint_U dS \,, \quad \text{where} \quad dS = \sqrt{g} d\theta d\phi \,.$$

Now,

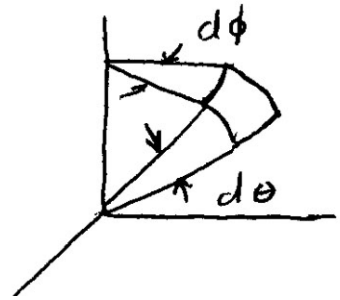
$$\begin{aligned} g &= \det[g_{ij}] = \det \begin{bmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{bmatrix} \\ &= \det \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{bmatrix} \\ g &= r^4 \sin^2\theta \end{aligned}$$

Thus,

$$dS = \sqrt{r^4 \sin^2\theta}\, d\theta d\phi = r^2 \sin\theta d\theta d\phi$$

Remark: This expression for the surface area element of a sphere is familiar from calculus or physics where it is usually derived by heuristic considerations:

$$\begin{aligned} dS &= d\ell_1 d\ell_2, \\ d\ell_1 &= r d\theta,\ d\ell_2 = r\sin\theta d\phi \\ dS &= (r d\theta)(r\sin\theta d\phi) \\ &= r^2 \sin\theta d\theta d\phi. \end{aligned}$$



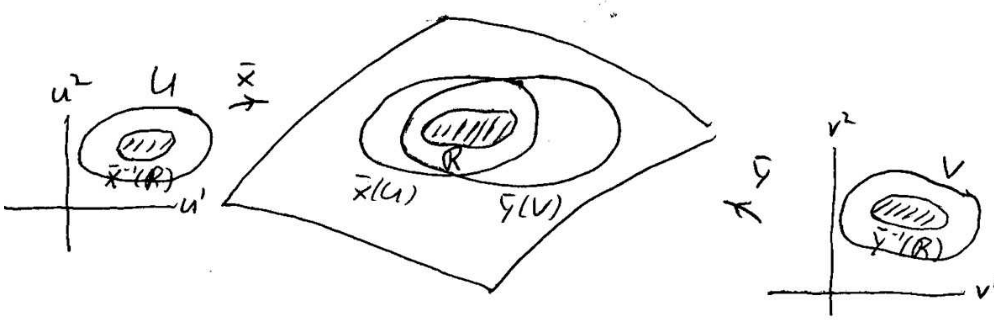
Continuing the computation of the surface area of S^2_r,

$$\begin{aligned} \text{Area of } S^2_r &= \iint_U r^2 \sin\theta d\theta d\phi = \iint_{\overline{U}} r^2 \sin\theta d\theta d\phi \\ &= \int_0^{2\pi}\int_0^{\pi} r^2\sin\theta d\theta d\phi = \int_0^{2\pi} r^2[-\cos\theta]|_0^{\pi}\ d\phi \\ &= \int_0^{2\pi} 2r^2 d\phi = 2r^2\phi|_0^{2\pi} = 4\pi r^2\,. \end{aligned}$$

The surface area formula involves a choice of coordinates, i.e. a choice of proper patch. It is important to recognize that the formula is independent of this choice.

Proposition. *The area formula is independent of the choice of coordinate patch.*

Let $\mathbf{x} : U \to M$, $\mathbf{y} : V \to M$ be proper patches, and suppose $\mathcal{R}$ is contained in $\mathbf{x}(U) \cap \mathbf{y}(V)$:



Set,

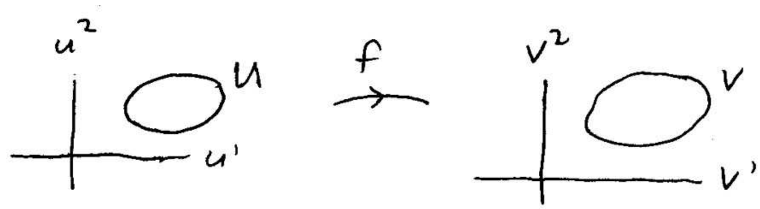
$$\begin{aligned} g_{ij} &= \langle \mathbf{x}_i, \mathbf{x}_j \rangle, \quad g = \det[g_{ij}]\,, \\ \tilde{g}_{ij} &= \langle \mathbf{y}_i, \mathbf{y}_j \rangle, \quad \tilde{g} = \det[\tilde{g}_{ij}]\,. \end{aligned}$$

Then the claim is that,

$$\iint_{\mathbf{x}^{-1}(\mathcal{R})} \sqrt{g}\,du^1du^2 = \iint_{\mathbf{y}^{-1}(\mathcal{R})} \sqrt{\tilde{g}}\,dv^1dv^2$$

Proof: The proof is an application of the change of variable formula for double integrals.

Let $f : U \subset I\!R^2 \to V \subset I\!R^2$ be a diffeomorphism, where U, V are bounded regions in $I\!R^2$.



$$f : \begin{array}{l} v^1 = f^1(u^1, u^2) \\ v^2 = f^2(u^1, u^2) \end{array}.$$

Then, the change of variable formula for double integrals is as follows,

$$\begin{aligned} \iint_V h(v^1, v^2)dv^1dv^2 &= \iint_U h \circ f(u^1, u^2)|\det Df|du^1du^2 \\ &= \iint_U h(f^1(u^1, u^2), f^1(u^1, u^2)) \left|\frac{\partial(v^1, v^2)}{\partial(u^1, u^2)}\right| du^1du^2 \end{aligned},$$

or, in briefer notation,

$$\iint_V h dv^1dv^2 = \iint_U h \left|\frac{\partial(v^1, v^2)}{\partial(u^1, u^2)}\right| du^1du^2 .$$

In the case at hand, $f = \mathbf{y}^{-1} \circ \mathbf{x} : \mathbf{x}^{-1}(\mathcal{R}) \to \mathbf{y}^{-1}(\mathcal{R})$, and $h = \sqrt{\tilde{g}}$. So, by the change of variable formula,

$$\iint_{y^{-1}(\mathcal{R})} \sqrt{\tilde{g}}dv^1dv^2 = \iint_{\mathbf{x}^{-1}(\mathcal{R})} \sqrt{\tilde{g}} \left|\frac{\partial(v^1, v^2)}{\partial(u^1, u^2)}\right| du^1du^2$$

Thus, to complete the proof, it suffices to establish the following lemma.

Lemma. $g = \det[g_{ij}]$, $\tilde{g} = \det[\tilde{g}_{ij}]$. *are related by,*

$$\sqrt{g} = \sqrt{\tilde{g}} \left|\frac{\partial(v^1, v^2)}{\partial(u^1, u^2)}\right| .$$

Proof of the lemma: By the Exercise 3.4 (but with role of $\mathbf{x}$ and $\mathbf{y}$ reversed),

$$\frac{\partial \mathbf{x}}{\partial u^1} \times \frac{\partial \mathbf{x}}{\partial u^2} = \frac{\partial(v^1, v^2)}{\partial(u^1, u^2)} \frac{\partial \mathbf{y}}{\partial v^1} \times \frac{\partial \mathbf{y}}{dv^2}$$

or,

$$\mathbf{x}_1 \times \mathbf{x}_2 = \frac{\partial(v^1, v^2)}{\partial(u^1, u^2)} \mathbf{y}_1 \times \mathbf{y}_2$$

Hence,

$$\begin{aligned} g &= \det[g_{ij}] = |\mathbf{x}_1 \times \mathbf{x}_2|^2 \\ &= \left(\frac{\partial(v^1, v^2)}{\partial(u^1, u^2)}\right)^2 |\mathbf{y}_1 \times \mathbf{y}_2|^2 \\ &= \left(\frac{\partial(v^1, v^2)}{\partial(u^1, u^2)}\right)^2 \tilde{g}. \end{aligned}$$

Taking square roots yields the result.

Exercise 4.3 Consider the torus of large radius R and small radius r described in Exercise 3.3. Use the *intrinsic* surface area formula and the parameterization given in Exercise 3.3 to compute the surface area of the torus. Answer: $4\pi^2 Rr$.

Exercise 4.4 Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ be a smooth function of two variables. Let M be the graph of $f|_w = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y), (x, y) \in W\}$, where W is a bounded subset of U. Derive the following standard formula from calculus for the surface area of M,

$$\text{Area of } M = \iint_W \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy\,,$$

by considering the Monge patch associated to f.

More Tensor Analysis

Let $\mathbf{x} : U \to M$, $\mathbf{y} : V \to M$ be overlapping patches in a surface M, $W := \mathbf{x}(U) \cap \mathbf{y}(V) \neq \emptyset$. Let $f = \mathbf{y}^{-1} \circ \mathbf{x} : \mathbf{x}^{-1}(W) \to \mathbf{y}^{-1}(W)$,

$$f : \begin{array}{l} v^1 = f^1(u^1, u^2) \\ v^2 = f^2(u^1, u^2) \end{array}$$

be the smooth overlap map, cf., p. 15 of Chapter 3. Introduce the metric components with respect to each patch,

$$g_{ij} = \langle \mathbf{x}_i, \mathbf{x}_j \rangle, \quad \tilde{g}_{ij} = \langle \mathbf{y}_i, \mathbf{y}_j \rangle \,.$$

How are these metric components related on the overlap?

Exercise 4.5 Show that,

$$g_{ij} = \sum_{a,b=1}^{2} \tilde{g}_{ab} \frac{\partial v^a}{\partial u^i} \frac{\partial v^b}{\partial u^j}, \quad i, j = 1, 2.$$

These equations can be expressed as a single matrix equation,

$$[g_{ij}] = \left[\frac{\partial v^a}{\partial u^i} \right]^t [\tilde{g}_{ab}] \left[\frac{\partial v^b}{\partial u^j} \right].$$

Taking determinants we obtain,

$$\begin{aligned} g &= \det[g_{ij}] = \det[*]^t[*][*] \\ &= \det[*]^t \det[*] \det[*] \\ &= \det \left[\frac{\partial v^a}{\partial u^i} \right] \det[\tilde{g}_{ij}] \det \left[\frac{\partial v^b}{\partial u^j} \right] \\ &= \tilde{g} (\det Df)^2 \\ g &= \tilde{g} \left[\frac{\partial (v^1, v^2)}{\partial (u^1, u^2)} \right]^2, \end{aligned}$$

our second derivation of this formula.

Remark: Interchanging the roles of $\mathbf{x}$ and $\mathbf{y}$ above we obtain,

$$\tilde{g}_{ab} = \sum_{i,j} g_{ij} \frac{\partial u^i}{\partial v^a} \frac{\partial u^j}{\partial v^b},$$

which involves the Jacobian of f^{-1}. Compare this "transformation law" for the metric components to the transformation law for vector components considered in Exercise 3.8. Vector fields are "contravariant" tensors. The metric $\langle \ , \ \rangle$ is a "covariant" tensor.