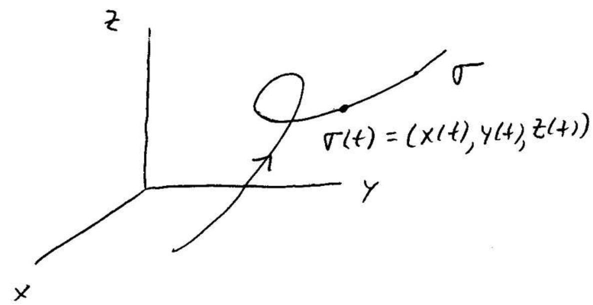
Chapter 2. Parameterized Curves in $\mathbb{R}^3$

Def. A smooth curve in $\mathbb{R}^3$ is a smooth map $\sigma : (a,b) \to \mathbb{R}^3$.



For each $t \in (a,b)$, $\sigma(t) \in \mathbb{R}^3$. As t increases from a to b, $\sigma(t)$ traces out a curve in $\mathbb{R}^3$. In terms of components,

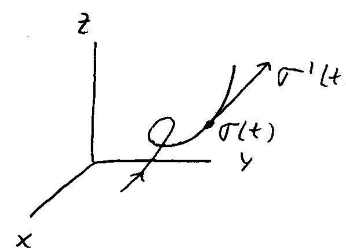
$$\sigma(t) = (x(t), y(t), z(t))\,, \tag{1}$$

or

$$\sigma : \begin{array}{l} x = x(t) \\ y = y(t) \\ z = z(t) \end{array} \quad a < t < b\,,$$

<u>velocity at time t:</u> $\dfrac{d\sigma}{dt}(t) = \sigma'(t) = (x'(t), y'(t), z'(t))$

<u>speed at time t:</u> $\left|\dfrac{d\sigma}{dt}(t)\right| = |\sigma'(t)|$



Ex. $\sigma : \mathbb{R} \to \mathbb{R}^3$, $\sigma(t) = (r\cos t, r\sin t, 0)$ - the standard parameterization of the unit circle,

$$\sigma : \begin{array}{l} x = r\cos t \\ y = r\sin t \\ z = 0 \end{array}$$

$$\begin{aligned} \sigma'(t) &= (-r\sin t, r\cos t, 0) \\ |\sigma'(t)| &= r \qquad \text{(constant speed)} \end{aligned}$$

Ex. $\sigma : \mathbb{R} \to \mathbb{R}^3$, $\sigma(t) = (r\cos t, r\sin t, ht)$, $r, h > 0$ constants (helix).

$$\begin{aligned} \sigma'(t) &= (-r\sin t, r\cos t, h) \\ |\sigma'(t)| &= \sqrt{r^2 + h^2} \quad \text{(constant)} \end{aligned}$$

Def A *regular* curve in $\mathbb{R}^3$ is a smooth curve $\sigma : (a, b) \to \mathbb{R}^3$ such that $\sigma'(t) \neq 0$ for all $t \in (a, b)$.

That is, a regular curve is a smooth curve with everywhere nonzero velocity.

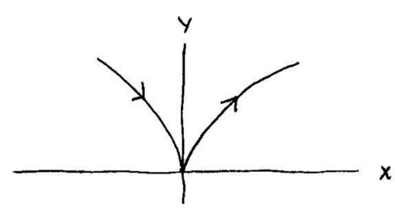
Ex. Examples above are regular.

Ex. $\sigma : \mathbb{R} \to \mathbb{R}^3$, $\sigma(t) = (t^3, t^2, 0)$. σ is smooth, but not regular:

$$\sigma'(t) = (3t^2, 2t, 0)\,, \qquad \sigma'(0) = (0, 0, 0)$$

Graph:

$$\sigma : \begin{matrix} x = t^3 \\ y = t^2 \\ z = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} y = t^2 = (x^{1/3})^2 \\ y = x^{2/3} \end{matrix}$$



There is a cusp, not because the curve isn't smooth, but because the velocity $= 0$ at the origin. A regular curve has a well-defined smoothly turning tangent, and hence its graph will appear smooth.

The Geometric Action of the Jacobian (exercise)

Given smooth map $F : U \subset \mathbb{R}^3 \to \mathbb{R}^3$, $p \in U$. Let X be any vector based at the point p. To X at p we associate a vector Y at $F(p)$ as follows.

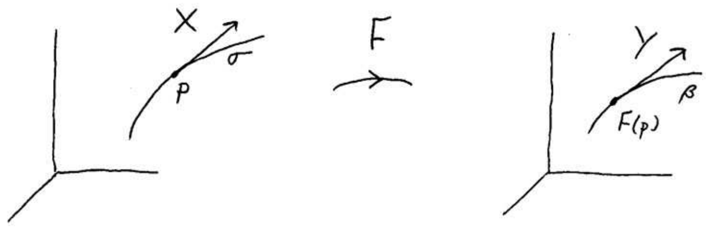
Let $\sigma : (-\epsilon, \epsilon) \to \mathbb{R}^3$ be any smooth curve such that,

$$\sigma(0) = p \quad \text{and} \quad \frac{d\sigma}{dt}(0) = X,$$

i.e. σ is a curve which passes through p at $t = 0$ with velocity X. (E.g. one can take $\sigma(t) = p + tX$.) Now, look at the image of σ under F, i.e. consider $\beta = F \circ \sigma$, $\beta : (-\epsilon, \epsilon) \to \mathbb{R}^3$, $\beta(t) = F \circ \sigma(t) = F(\sigma(t))$. We have, $\beta(0) = F(\sigma(0)) = F(p)$, i.e., β passes through $F(p)$ at $t = 0$. Finally, let

$$Y = \frac{d\beta}{dt}(0).$$

i.e. Y is the velocity vector of β at $t = 0$.



Exercise 2.1. Show that

$$Y = DF(p)X.$$

Note: In the above, X and Y are represented as column vectors, and the *rhs* of the equation involves matrix multiplication. Hint: Use the chain rule.

Thus, roughly speaking, the geometric effect of the Jacobian is to "send velocity vectors to velocity vectors". The same result holds for mappings $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ (i.e. it is not necessary to restrict to dimension three).

Reparameterizations

Given a regular curve $\sigma : (a, b) \to \mathbb{R}^3$. Traversing the same path at a different speed (and perhaps in the opposite direction) amounts to what is called a reparameterization.

Def. Let $\sigma : (a, b) \to \mathbb{R}^3$ be a regular curve. Let $h : (c, d) \subset \mathbb{R} \to (a, b) \subset \mathbb{R}$ be a diffeomorphism (i.e. h is 1-1, onto such that h and h^{-1} are smooth). Then $\tilde{\sigma} = \sigma \circ h : (c, d) \to \mathbb{R}^3$ is a regular curve, called a reparameterization of σ.

$$\tilde{\sigma}(u) = \sigma \circ h(u) = \sigma(h(u))$$

I.e., start with curve $\sigma = \sigma(t)$, make a change of parameter $t = h(u)$, obtain reparameterized curve $\tilde{\sigma} = \sigma(h(u))$; $t =$ original parameter, $u =$ new parameter.

Remarks.

1. σ and $\tilde{\sigma}$ describe the same path in space, just traversed at different speeds (and perhaps in opposite directions).

2. Compare velocities:

$$\begin{aligned} \tilde{\sigma} &= \sigma(h(u)) \quad \text{i.e.,} \\ \tilde{\sigma} &= \sigma(t), \text{ where } t = h(u)\,. \end{aligned}$$

By the chain rule,

$$\frac{d\tilde{\sigma}}{du} = \frac{d\sigma}{dt} \cdot \frac{dt}{du} = \frac{d\sigma}{dt} \cdot h'$$

$h' > 0$: orientation preserving reparameterization.
$h' < 0$: orientation reversing reparameterization.

Ex. $\sigma : (0, 2\pi) \to \mathbb{R}^3$, $\sigma(t) = (\cos t, \sin t, 0)$. Reparameterization function: $h : (0, \pi) \to (0, 2\pi)$,

$$h : t = h(u) = 2u\,, \qquad u \in (0, \pi)\,,$$

Reparameterized curve:

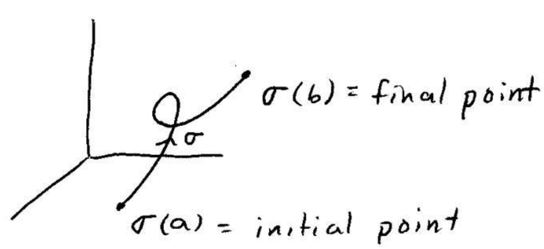
$$\begin{aligned} \tilde{\sigma}(u) &= \sigma(t) = \sigma(2u) \\ \tilde{\sigma}(u) &= (\cos 2u, \sin 2u, 0) \end{aligned}$$

$\tilde{\sigma}$ describes the same circle, but traversed twice as fast,

$$\text{speed of } \sigma = \left|\frac{d\sigma}{dt}\right| = 1\,, \qquad \text{speed of } \tilde{\sigma} = \left|\frac{d\tilde{\sigma}}{du}\right| = 2\,.$$

Remark Regular curves always admit a very important reparameterization: they can always be parameterized in terms of arc length.

<u>Length Formula:</u> Consider a smooth curve defined on a *closed* interval, $\sigma : [a, b] \to \mathbb{R}^3$.



σ is a smooth curve segment. Its *length* is defined by,

$$\text{length of } \sigma = \int_a^b |\sigma'(t)| dt.$$

I.e., to get the length, integrate speed *wrt* time.

Ex. $\sigma(t) = (r\cos t,\ r\sin t, 0) \quad 0 \le t \le 2\pi.$

$$\text{Length of } \sigma = \int_0^{2\pi} |\sigma'(t)| dt = \int_0^{2\pi} r dt = 2\pi r.$$

Fact. The length formula is independent of parameterization, i.e., if $\tilde{\sigma} : [c,d] \to \mathbb{R}^3$ is a reparameterization of $\sigma : [a,b] \to \mathbb{R}^3$ then length of $\tilde{\sigma}$ = length of σ.

Exercise 2.2 Prove this fact.

Arc Length Parameter:

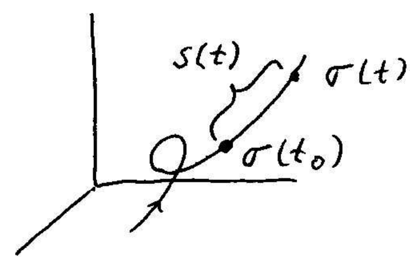
Along a regular curve $\sigma : (a,b) \to \mathbb{R}^3$ there is a distinguished parameter called *arc length* parameter. Fix $t_0 \in (a,b)$. Define the following function (*arc length function*).

$$s = s(t),\ \ t \in (a,b)\,, \qquad s(t) = \int_{t_0}^t |\sigma'(t)| dt\,.$$

Thus,

if $t > t_0$, $s(t) =$ length of σ from t_0 to t

if $t < t_0$, $s(t) = -$length of σ from t_0 to t.



$s = s(t)$ is smooth and by the Fundamental Theorem of calculus,

$$s'(t) = |\sigma'(t)| > 0 \ \text{ for all } t \in (a,b)$$

Hence $s = s(t)$ is strictly increasing, and so has a smooth inverse - can solve smoothly for t in terms of s, $t = t(s)$ (reparameterization function).
Then,

$$\tilde{\sigma}(s) = \sigma(t(s))$$

is the arc length reparameterization of σ.

Fact. A regular curve admits a reparameterization in terms of arc length.

Ex. Reparameterize the circle $\sigma(t) = (r\cos t, r\sin t, 0)$, $-\infty < t < \infty$, in terms of arc length parameter.

Obtain the arc length function $s = s(t)$,

$$\begin{aligned} s &= \int_0^t |\sigma'(t)|dt = \int_0^t r dt \\ s &= rt \quad \Rightarrow \quad t = \frac{s}{r} \quad \text{(reparam. function)} \end{aligned}$$

Hence,

$$\begin{aligned} \tilde{\sigma}(s) &= \sigma(t(s)) = \sigma\left(\frac{s}{r}\right) \\ \tilde{\sigma}(s) &= (r\cos\left(\frac{s}{r}\right),\ r\sin\left(\frac{s}{r}\right), 0). \end{aligned}$$

Remarks

1. Often one relaxes the notation and writes $\sigma(s)$ for $\tilde{\sigma}(s)$ (i.e. one drops the tilde).

2. Let $\sigma = \sigma(t)$, $t \in (a, b)$ be a *unit speed curve*, $|\sigma'(t)| = 1$ for all $t \in (a, b)$. Then,

$$\begin{aligned} s &= \int_{t_0}^t |\sigma'(t)|dt = \int_{t_0}^t 1 dt \\ s &= t - t_0\,. \end{aligned}$$

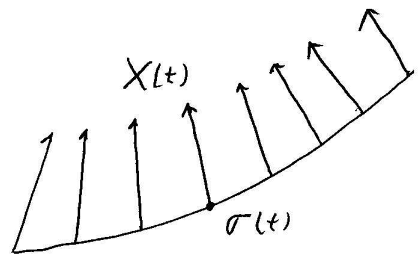
I.e. up to a trivial translation of parameter, $s = t$. Hence unit speed curves are already parameterized *wrt* arc length (as measured from some point). Conversely, if $\sigma = \sigma(s)$ is a regular curve parameterized *wrt* arc length s then σ is unit speed, i.e. $|\sigma'(s)| = 1$ for all s (why?). Hence the phrases "unit speed curve" and "curve parameterized *wrt* arc length" are used interchangably.

Exercise 2.3. Reparameterize the helix, $\sigma : \mathbb{R} \to \mathbb{R}^3$, $\sigma(t) = (r\cos t, r\sin t, ht)$ in terms of arc length.

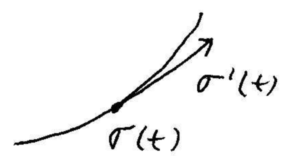
Vector fields along a curve.

We will frequently use the notion of a vector field along a curve σ.

Def. Given a smooth curve $\sigma : (a, b) \to \mathbb{R}^3$ a *vector field along* σ is a vector-valued map $X : (a, b) \to \mathbb{R}^3$ which assigns to each $t \in (a, b)$ a vector $X(t)$ at the point $\sigma(t)$.



Ex. Velocity vector field along $\sigma : (a,b) \to \mathbb{R}^3$.



$$\sigma' : (a,b) \to \mathbb{R}^3, \;\; t \to \sigma'(t)\,;$$

if $\sigma(t) = (x(t), y(t), z(t))$, $\sigma'(t) = (x'(t), y'(t), z'(t))$.

Ex. Unit tangent vector field along σ.

$$T(t) = \frac{\sigma'(t)}{|\sigma'(t)|}\,.$$

$|T(t)| = 1$ for all t. (Note σ must be regular for T to be defined).

Ex. Find unit tangent vector field along $\sigma(t) = (r\cos t, r\sin t, ht)$.

$$\begin{aligned} \sigma'(t) &= (-r\sin t, r\cos t, h) \\ |\sigma'(t)| &= \sqrt{r^2 + h^2} \\ T(t) &= \frac{1}{\sqrt{r^2+h^2}}(-r\sin t, r\cos t, h) \end{aligned}$$

Note. If $s \to \sigma(s)$ is parameterized *wrt* arc length then $|\sigma'(s)| = 1$ (unit speed) and so,

$$T(s) = \sigma'(s).$$

Differentiation. Analytically vector fields along a curve are just maps,

$$X : (a,b) \subset \mathbb{R} \to \mathbb{R}^3.$$

Can differentiate by expressing $X = X(t)$ in terms of components,

$$
\begin{aligned}
X(t) &= (X'(t), X^2(t), X^3(t)), \\
\frac{dX}{dt} &= \left(\frac{dX^1}{dt}, \frac{dX^2}{dt}, \frac{dX^3}{dt}\right).
\end{aligned}
$$

Ex. Consider the unit tangent field to the helix,

$$
\begin{aligned}
T(t) &= \frac{1}{\sqrt{r^2+h^2}}(-r\sin t, r\cos t, h) \\
T'(t) &= \frac{1}{\sqrt{r^2+h^2}}(-r\cos t, -r\sin t, 0).
\end{aligned}
$$

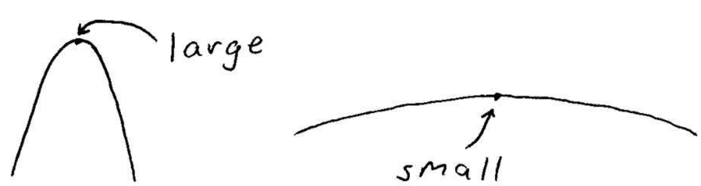
Exercise 2.4. Let $X = X(t)$ and $Y = Y(t)$ be two smooth vector fields along $\sigma : (a, b) \to \mathbb{R}^3$. Prove the following product rules,

$$
\begin{aligned}
&(1)\quad \frac{d}{dt}\langle X, Y\rangle = \langle \frac{dX}{dt}, Y\rangle + \langle X, \frac{dY}{dt}\rangle \\
&(2)\quad \frac{d}{dt}X \times Y = \frac{dX}{dt} \times Y + X \times \frac{dY}{dt}
\end{aligned}
$$

Hint: Express in terms of components.

<u>Curvature</u>

Curvature of a curve is a measure of how much a curve bends at a given point:



This is quantified by measuring the rate at which the unit tangent turns *wrt distance* along the curve. Given regular curve, $t \to \sigma(t)$, reparameterize in terms of arc length, $s \to \sigma(s)$, and consider the unit tangent vector field,

$$
T = T(s) \qquad (T(s) = \sigma'(s)).
$$

Now differentiate $T = T(s)$ *wrt* arc length,

$$
\frac{dT}{ds} = \text{curvature vector}
$$

The *direction* of $\frac{dT}{ds}$ tells us which way the curve is bending. Its magnitude tells us *how much* the curve is bending,

$$\left|\frac{dT}{ds}\right| = \text{curvature}$$

Def. Let $s \to \sigma(s)$ be a unit speed curve. The curvature $\kappa = \kappa(s)$ of σ is defined as follows,

$$\kappa(s) = |T'(s)| \quad (= |\sigma''(s)|)\,,$$

where $' = \frac{d}{ds}$.

Ex. Compute the curvature of a circle of radius r.

Standard parameterization: $\sigma(t) = (r\cos t, r\sin t, 0)$.
Arc length parameterization: $\sigma(s) = \left(r\cos\left(\frac{s}{r}\right), r\sin\left(\frac{s}{r}\right), 0\right)$.

$$\begin{aligned}
T(s) &= \sigma'(s) = \left(-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right), 0\right) \\
T'(s) &= \left(-\frac{1}{r}\cos\left(\frac{s}{r}\right), -\frac{1}{r}\sin\left(\frac{s}{r}\right), 0\right) \\
&= -\frac{1}{r}\left(\cos\left(\frac{s}{r}\right), \sin\left(\frac{s}{r}\right), 0\right) \\
\kappa(s) &= |T'(s)| = \frac{1}{r}
\end{aligned}$$

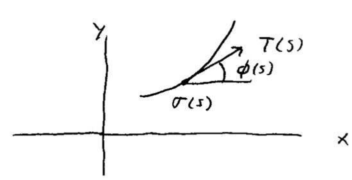
(Does this answer agree with intuition?)

Exercise 2.5. Let $s \to \sigma(s)$ be a unit speed *plane* curve,

$$\sigma(s) = (x(s), y(s), 0)\,.$$

For each s let,

$$\phi(s) = \text{angle between positive } x\text{-axis and } T(s).$$



Show: $\kappa(s) = |\phi'(s)|$ (i.e. $\kappa = \left|\frac{d\phi}{ds}\right|$).
Hint: Observe, $T(s) = \cos\phi(s)\mathbf{i} + \sin\phi(s)\mathbf{j}$ (why?).

Conceptually, the definition of curvature is the right one. But for computational purposes it's not so good. For one thing, it would be useful to have a formula for computing curvature which does not require that the curve be parameterized with respect to arc length. Using the chain rule, such a formula is easy to obtain.

Given a regular curve $t \to \sigma(t)$, it can be reparameterized *wrt* arc length $s \to \sigma(s)$. Let $T = T(s)$ be the unit tangent field to σ.

$$T = T(s), \quad s = s(t),$$

So by the chain rule,

$$\begin{aligned} \frac{dT}{dt} &= \frac{dT}{ds} \cdot \frac{ds}{dt} \\ &= \frac{dT}{ds} \left| \frac{d\sigma}{dt} \right| \\ \left| \frac{dT}{dt} \right| &= \left| \frac{d\sigma}{dt} \right| \underbrace{\left| \frac{dT}{ds} \right|}_{\kappa} \end{aligned}$$

and hence,

$$\kappa = \frac{\left| \frac{dT}{dt} \right|}{\left| \frac{d\sigma}{dt} \right|},$$

i.e.

$$\kappa(t) = \frac{|T'(t)|}{|\sigma'(t)|}, \quad ' = \frac{d}{dt}.$$

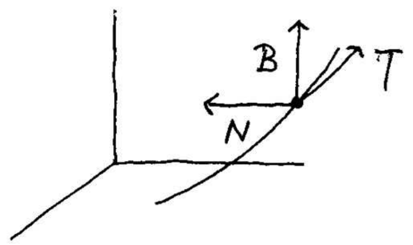
Exercise 2.6. Use the above formula to compute the curvature of the helix $\sigma(t) = (r \cos t, r \sin t, ht)$.

Frenet-Equations

Let $s \to \sigma(s)$, $s \in (a, b)$ be a regular unit speed curve such that $\kappa(s) \neq 0$ for all $s \in (a, b)$. (We will refer to such a curve as *strongly* regular). Along σ we are going to introduce the vector fields,

$T = T(s)$	- unit tangent vector field
$N = N(s)$	- principal normal vector field
$B = B(s)$	- binormal vector field

$\{T, N, B\}$ is called a Frenet frame.



At each point of σ $\{T, N, B\}$ forms an orthonormal basis, i.e. T, N, B are mutually perpendicular unit vectors.

To begin the construction of the Frenet frame, we have the unit tangent vector field,

$$T(s) = \sigma'(s), \qquad ' = \frac{d}{ds}$$

Consider the derivative $T' = T'(s)$.

Claim. $T' \perp T$ along σ.

Proof. It suffices to show $\langle T', T\rangle = 0$ for all $s \in (a, b)$. Along σ,

$$\langle T, T\rangle = |T|^2 = 1.$$

Differentiating both sides,

$$\begin{aligned}
\frac{d}{ds}\langle T, T\rangle = \frac{d}{ds}1 &= 0 \\
\langle \frac{dT}{ds}, T\rangle + \langle T, \frac{dT}{ds}\rangle &= 0 \\
2\langle \frac{dT}{ds}, T\rangle &= 0 \\
\langle T', T\rangle &= 0.
\end{aligned}$$

Def. Let $s \to \sigma(s)$ be a strongly regular unit speed curve. The *principal normal* vector field along σ is defined by

$$N(s) = \frac{T'(s)}{|T'(s)|} = \frac{T'(s)}{\kappa(s)} \qquad (\kappa(s) \neq 0)$$

The *binormal* vector field along σ is defined by

$$B(s) = T(s) \times N(s).$$

Note, the definition of $N = N(s)$ implies the equation

$$T' = \kappa N$$

Claim. For each s, $\{T(s), N(s), B(s)\}$ is an orthonormal basis for vectors in space based at $\sigma(s)$.

Mutually perpendicular:

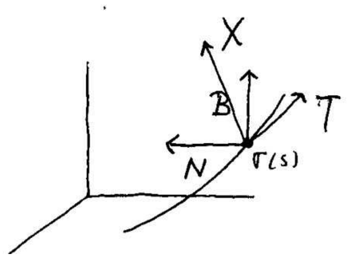
$$\langle T, N\rangle = \langle T, \frac{T'}{\kappa}\rangle = \frac{1}{\kappa}\langle T, T'\rangle = 0.$$

$$B = T \times N \Rightarrow \langle B, T\rangle = \langle B, N\rangle = 0.$$

Unit length: $|T| = 1$, and

$$\begin{aligned} |N| &= \left|\frac{T'}{|T'|}\right| = \frac{|T'|}{|T'|} = 1, \\ |B|^2 &= |T \times N|^2 \\ &= |T|^2|N|^2 - \langle T, N\rangle^2 = 1. \end{aligned}$$

Remark on o.n. bases.



X = vector at $\sigma(s)$.
X can be expressed as a linear combination of $T(s), N(s), B(s)$,

$$X = aT + bN + cB$$

The constants a, b, c are determined as follows,

$$\begin{aligned} \langle X, T\rangle &= \langle aT + bN + cB, T\rangle \\ &= a\langle T, T\rangle + b\langle N, T\rangle + c\langle B, T\rangle \\ &= a \end{aligned}$$

Hence, $a = \langle X, T\rangle$, and similarly, $b = \langle X, N\rangle$, $c = \langle X, B\rangle$. Hence X can be expressed as,

$$X = \langle X, T\rangle T + \langle X, N\rangle N + \langle X, B\rangle B.$$

Torsion: Torsion is a measure of "twisting". Curvature is associated with T'; torsion is associated with B':

$$\begin{aligned} B &= T \times N \\ B' &= T' \times N + T \times N' \\ &= \kappa N \times N + T \times N' \end{aligned}$$

Therefore $B' = T \times N'$ which implies $B' \perp T$, i.e.

$$\langle B', T \rangle = 0$$

Also, since $B = B(s)$ is a unit vector along σ, $\langle B, B \rangle = 1$ which implies by differentiation,

$$\langle B', B \rangle = 0$$

It follows that B' is a multiple of N,

$$\begin{aligned} B' &= \langle B', T \rangle T + \langle B', N \rangle N + \langle B', B \rangle B \\ B' &= \langle B', N \rangle N. \end{aligned}$$

Hence, we may write,

$$B' = -\tau N$$

where $\tau =$ *torsion* $:= -\langle B', N \rangle$.

Remarks

1. τ is a *function* of s, $\tau = \tau(s)$.
2. τ is *signed* i.e. can be positive or negative.
3. $|\tau(s)| = |B'(s)|$, i.e., $\tau = \pm |B'|$, and hence τ measuures how B wiggles.

Given a strongly regular unit speed curve σ, the collection of quantities T, N, B, κ, τ is sometimes referred to as the *Frenet apparatus*.

Ex. Compute T, N, B, κ, τ for the unit speed circle.

$$\begin{aligned} \sigma(s) &= \left(r\cos\left(\frac{s}{r}\right), r\sin\left(\frac{s}{r}\right), 0 \right) \\ T &= \sigma' = \left(-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right), 0 \right) \\ T' &= -\frac{1}{r}\left(\cos\left(\frac{s}{r}\right), \sin\left(\frac{s}{r}\right), 0 \right) \end{aligned}$$

$$
\begin{aligned}
\kappa &= |T'| = \frac{1}{r} \\
N &= \frac{T'}{k} = -\left(\cos\left(\frac{s}{r}\right), \sin\left(\frac{s}{r}\right), 0\right) \\
B &= T \times N \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -s & c & 0 \\ -c & -s & 0 \end{vmatrix} \\
&= \mathbf{k} = (0, 0, 1),
\end{aligned}
$$

(where $c = \cos\left(\frac{s}{r}\right)$ and $s = \sin\left(\frac{s}{r}\right)$). Finally, since $B' = 0$, $\tau = 0$, i.e. the torsion vanishes.

Conjecture. Let $s \to \sigma(s)$ be a strongly regular unit speed curve. Then, σ is a *plane curve* iff its torsion vanishes, $\tau \equiv 0$.

Exercise 2.7. Consider the helix,

$$\sigma(t) = (r\cos t, r\sin t, ht).$$

Show that, when parameterized *wrt* arc length, we obtain,

$$\sigma(s) = (r\cos\omega s, r\sin\omega s, h\omega s), \qquad (*)$$

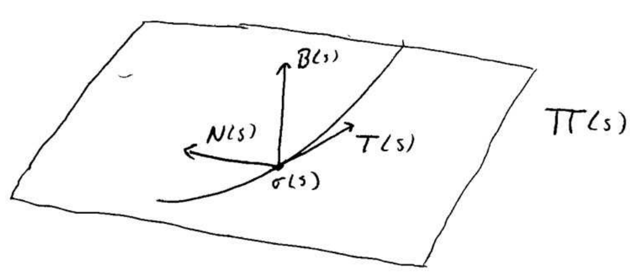
where $\omega = \dfrac{1}{\sqrt{r^2 + h^2}}$.

Ex. Compute T, N, B, κ, τ for the unit speed helix $(*)$.

$$
\begin{aligned}
T &= \sigma' = (-r\omega\sin\omega s, r\omega\cos\omega s, h\omega) \\
T' &= -\omega^2 r(\cos\omega s, \sin\omega s, 0) \\
\kappa &= |T'| = \omega^2 r = \frac{r}{r^2 + h^2} = \text{const.} \\
N &= \frac{T'}{\kappa} = (-\cos\omega s, -\sin\omega s, 0)
\end{aligned}
$$

$$
\begin{aligned}
B &= T \times N = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\omega \sin \omega s & r\omega \cos \omega s & h\omega \\ -\cos \omega s & -sin\omega s & 0 \end{vmatrix} \\
B &= (h\omega \sin \omega s, -h\omega \cos \omega s, r\omega) \\
B' &= (h\omega^2 \cos \omega s, h\omega^2 \sin \omega s, 0) \\
&= h\omega^2 (\cos \omega s, \sin \omega s, 0) \\
B' &= -h\omega^2 N \\
B' &= -\tau N \Rightarrow \tau = hw^2 = \frac{h}{r^2 + h^2}.
\end{aligned}
$$

Remarks.



$\Pi(s)$ = *osculating plane* of σ at $\sigma(s)$
= plane passing through $\sigma(s)$ spanned by $N(s)$ and $T(s)$
(or equivalently, perpendicular to $B(s)$).

(1) $s \to \Pi(s)$ is the family of osculating planes along σ. The Frenet equation $B' = -\tau N$ shows that the torsion τ measures how the osculating plane is twisting along σ.

(2) $\Pi(s_0)$ passes through $\sigma(s_0)$ and is spanned by $\sigma'(s_0)$ and $\sigma''(s_0)$. Hence, in a sense that can be made precise, $s \to \sigma(s)$ lies in $\Pi(s_0)$ " to second order in s". If $\tau(s_0) \neq 0$ then $\sigma'''(s_0)$ is *not* tangent to $\Pi(s_0)$. Hence the torsion τ gives a measure of the extent to which σ twists out of a given fixed osculating plane

Theorem. (Frenet Formulas) Let $s \to \sigma(s)$ be a strongly regular unit speed curve. Then the Frenet frame, T, N, B satisfies,

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N \end{aligned}$$

Proof. We have already established the first and third formulas. To establish the second, observe $B = T \times N \Rightarrow N = B \times T$. Hence,

$$\begin{aligned} N' &= (B \times T)' = B' \times T + B \times T' \\ &= -\tau N \times T + \kappa B \times N \\ &= -\tau(-B) + \kappa(-T) \\ &= -\kappa T + \tau B. \end{aligned}$$

We can express Frenet formulas as a matrix equation,

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}}_{A} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

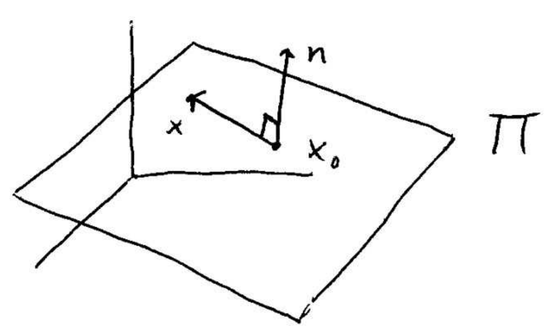
A is skew symmetric: $A^t = -A$. $A = [a_{ij}]$, then $a_{ji} = -a_{ij}$.

The Frenet equations can be used to derive various properties of space curves.

Proposition. Let $s \to \sigma(s)$, $s \in (a, b)$, be a strongly regular unit speed curve. Then, σ is a *plane curve* iff its torsion vanishes, $\tau \equiv 0$.

Proof. Recall, the plane Π which passes through the point $x_0 \in \mathbb{R}^3$ and is perpendicular to the unit vector n consists of all points $x \in \mathbb{R}^3$ which satisfy the equation,

$$\langle n, x - x_0 \rangle = 0$$



$\Rightarrow$: Assume $s \to \sigma(s)$ lies in the plane Π. Then, for all s,

$$\langle n, \sigma(s) - x_0 \rangle = 0$$

Since n is constant, differentiating twice gives,

$$\begin{aligned} \frac{d}{ds}\langle n, \sigma(s) - x_0 \rangle = \langle n, \sigma' \rangle = \langle n, T \rangle &= 0\,, \\ \frac{d}{ds}\langle n, T \rangle = \langle n, T' \rangle = \kappa \langle n, N \rangle &= 0\,, \end{aligned}$$

Since n is a unit vector perpendicular to T and N, $n = \pm B$, so $B = \pm n$. I.e., $B = B(s)$ is *constant* which implies $B' = 0$. Therefore $\tau \equiv 0$.

$\Leftarrow$: Now assume $\tau \equiv 0$. $B' = -\tau N \Rightarrow B' = 0$, i.e. $B(s)$ is constant,

$$B(s) = B = \text{ constant vector.}$$

We show $s \to \sigma(s)$ lies in the plane, $\langle B, x - \sigma(s_0) \rangle = 0$, passing through $\sigma(s_0)$, $s_0 \in (a, b)$, and perpendicular to B, i.e., will show,

$$\langle B, \sigma(s) - \sigma(s_0) \rangle = 0\,. \qquad (*)$$

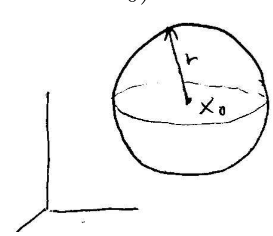
for all $s \in (a, b)$. Consider the function, $f(s) = \langle B, \sigma(s) - \sigma(s_0) \rangle$. Differentiating,

$$\begin{aligned} f'(s) &= \frac{d}{ds}\langle B, \sigma(s) - \sigma(s_0) \rangle \\ &= \langle B', \sigma(s) - \sigma(s_0) \rangle + \langle B, \sigma'(s) \rangle \\ &= 0 + \langle B, T \rangle = 0\,. \end{aligned}$$

Hence, $f(s) = c = \text{const}$. Since $f(s_0) = \langle B, \sigma(s_0) - \sigma(s_0) \rangle = 0.$, $c = 0$ and thus $f(s) \equiv 0$. Therefore $(*)$ holds, i.e., $s \to \sigma(s)$ lies in the plane $\langle B, x - \sigma(s_0) \rangle = 0$.

Sphere Curves. A sphere curve is a curve in $\mathbb{R}^3$ which lies on a sphere,

$$\begin{aligned} |x - x_0|^2 &= r^2\,, \qquad \text{(sphere of radius } r \text{ centered at } x_0) \\ \langle x - x_0, x - x_0 \rangle &= r^2 \end{aligned}$$



Thus, $s \to \sigma(s)$ is a sphere curve iff there exists $x_0 \in \mathbb{R}^3$, $r > 0$ such that

$$\langle \sigma(s) - x_0, \sigma(s) - x_0 \rangle = r^2\,, \quad \text{for all } s. \qquad (*)$$

If $s \to \sigma(s)$ lies on a sphere of radius r, it is reasonable to conjecture that σ has curvature $\kappa \geq \frac{1}{r}$ (why?). We prove this.

Proposition. Let $s \to \sigma(s)$, $s \in (a,b)$, be a unit speed curve which lies on a sphere of radius r. Then its curvature function $\kappa = \kappa(s)$ satisfies, $\kappa \geq \frac{1}{r}$.

Proof Differentiating (*) gives,

$$2\langle \sigma', \sigma - x_0 \rangle = 0$$

i.e.,

$$\langle T, \sigma - x_0 \rangle = 0.$$

Differentiating again gives:

$$\begin{aligned}
\langle T', \sigma - x_0 \rangle + \langle T, \sigma' \rangle &= 0 \\
\langle T', \sigma - x_0 \rangle + \langle T, T \rangle &= 0 \\
\langle T', \sigma - x_0 \rangle &= -1 \quad (\Rightarrow T' \neq 0) \\
\kappa \langle N, \sigma - x_0 \rangle &= -1
\end{aligned}$$

But,

$$\begin{aligned}
|\langle N, \sigma - x_0 \rangle| &= |N||\sigma - x_0||\cos\theta| \\
&= r|\cos\theta|,
\end{aligned}$$

and so,

$$\kappa = |\kappa| = \frac{1}{|\langle N, \sigma - x_0 \rangle|} = \frac{1}{r|\cos\theta|} \geq \frac{1}{r}$$

.

Exercise 2.8. Prove that any unit speed sphere curve $s \to \sigma(s)$ having constant curvature is a circle (or part of a circle). (Hints: Show that the torsion vanishes (why is this sufficient?). To show this differentiate $(*)$ a few times.

<u>Lancrets Theorem.</u>

Consider the unit speed circular helix $\sigma(s) = (r\cos\omega s, r\sin\omega s, h\omega s), \omega = 1/\sqrt{r^2 + h^2}$. This curve makes a constant angle *wrt* the z-axis: $T = \langle -r\omega\sin\omega s, r\cos\omega s, h\omega \rangle$,

$$\cos\theta = \frac{\langle T, \mathbf{k} \rangle}{|T||\mathbf{k}|} = h\omega = \text{const.}$$

Def. A unit speed curve $s \to \sigma(s)$ is called a generalized helix if its unit tangent T makes a constant angle with a fixed unit direction vector $\mathbf{u}$ ($\Leftrightarrow \langle T, \mathbf{u} \rangle = \cos\theta =$ const).

Theorem. (Lancret) *Let $s \to \sigma(s)$, $s \in (a, b)$ be a strongly regular unit speed curve such that $\tau(s) \neq 0$ for all $s \in (a, b)$. Then σ is a generalized helix iff $\kappa/\tau =$constant.*

Non-unit Speed Curves.

Given a regular curve $t \to \sigma(t)$, it can be reparameterized in terms of arc length $s \to \tilde{\sigma}(s)$, $\tilde{\sigma}(s) = \sigma(t(s))$, and the quantities T, N, B, κ, T can be computed. It is convenient to have formulas for these quantities which do not involve reparameterizing in terms of arc length.

Proposition. Let $t \to \sigma(t)$ be a strongly regular curve in $\mathbb{R}^3$. Then

(a) $T = \dfrac{\dot{\sigma}}{|\dot{\sigma}|}, \quad \cdot = \dfrac{d}{dt}$

(b) $B = \dfrac{\dot{\sigma} \times \ddot{\sigma}}{|\dot{\sigma} \times \ddot{\sigma}|}$

(c) $N = B \times T$

(d) $\kappa = \dfrac{|\dot{\sigma} \times \ddot{\sigma}|}{|\dot{\sigma}|^3}$

(e) $\tau = \dfrac{\langle \dot{\sigma} \times \ddot{\sigma}, \dddot{\sigma} \rangle}{|\dot{\sigma} \times \ddot{\sigma}|^2}$

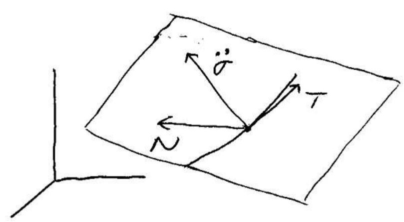
Proof. We derive some of these. See Theorem 1.4.5, p. 32 in Oprea for details. Interpreting physically, t=time, $\dot{\sigma}$=velocity, $\ddot{\sigma}$=acceleration. The unit tangent may be expressed as,

$$T = \frac{\dot{\sigma}}{|\dot{\sigma}|} = \frac{\dot{\sigma}}{v}$$

where $v = |\dot{\sigma}| =$ speed. Hence,

$$
\begin{aligned}
\dot{\sigma} &= vT \\
\ddot{\sigma} &= \frac{d}{dt}vt = \frac{dv}{dt}T + v\frac{dT}{dt} \\
&= \frac{dv}{dt}T + v\frac{dT}{ds}\cdot\frac{ds}{dt} \\
&= \frac{dv}{dt}T + v(\kappa N)v \\
\ddot{\sigma} &= \dot{v}T + v^2\kappa N
\end{aligned}
$$

Side Comment: This is the well-known expression for acceleration in terms of its tangential and normal components.



$$
\begin{aligned}
\dot{v} &= \text{tangential component of acceleration } (\dot{v} = \ddot{s}) \\
v^2\kappa &= \text{normal component of acceleration} \\
&= \text{centripetal acceleration (for a circle, } v^2\kappa = \frac{v^2}{r}\text{).}
\end{aligned}
$$

$\dot{\sigma}, \ddot{\sigma}$ lie in osculating plane; if $\tau \neq 0$, $\dddot{\sigma}$ does not.

Continuing the derivation,

$$
\begin{aligned}
\dot{\sigma}\times\ddot{\sigma} &= vT\times(\dot{v}T + v^2\kappa N) \\
&= v\dot{v}T\times T + v^3\kappa T\times N \\
\dot{\sigma}\times\ddot{\sigma} &= v^3\kappa B \\
|\dot{\sigma}\times\ddot{\sigma}| &= v^3\kappa|B| = v^3\kappa
\end{aligned}
$$

Hence,

$$
\kappa = \frac{|\dot{\sigma}\times\ddot{\sigma}|}{v^3} = \frac{|\dot{\sigma}\times\ddot{\sigma}|}{|\dot{\sigma}|^3}
$$

Also,

$$B = const \cdot \dot{\sigma} \times \ddot{\sigma} = \frac{\dot{\sigma} \times \ddot{\sigma}}{|\dot{\sigma} \times \ddot{\sigma}|}\,.$$

Exercise 2.9. Derive the expression for τ. Hint: Compute $\dddot{\sigma}$ and use Frenet formulas.

Exercise 2.10. Suppose σ is a regular curve in the x-y plane, $\sigma(t) = (x(t), y(t), 0)$, i.e.,

$$\sigma : \begin{array}{c} x = x(t) \\ y = y(t) \end{array}$$

(a) Show that the curvature of σ is given by,

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

(b) Use this formula to compute the curvature $\kappa = \kappa(t)$ of the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

Fundamental Theorem of Space Curves

This theorem says basically that any strongly regular unit speed curve is completely determined by its curvature and torsion (up to a Euclidean motion).

Theorem. *Let $\overline{\kappa} = \overline{\kappa}(s)$ and $\overline{\tau} = \overline{\tau}(s)$ be smooth functions on an interval (a, b) such that $\overline{\kappa}(s) > 0$ for all $s \in (a, b)$. Then there exists a strongly regular unit speed curve $s \to \sigma(s)$, $s \in (a, b)$ whose curvature and torsion functions are $\overline{\kappa}$ and $\overline{\tau}$, respectively. Moreover, σ is essentially unique, i.e. any other such curve $\tilde{\sigma}$ can be obtained from σ by a Euclidean motion (translation and/or rotation).*

Remarks

1. The FTSC shows that curvature and torsion are the *essential* quantities for describing space curves.

2. The FTSC also illustrates a very important issue in differential geometry. The problem of establishing the existence of some geometric object having certain geometric properties often reduces to a problem concerning the existence of a solution to some differential equation, or system of differential equations.

Proof: Fix $s_0 \in (a,b)$, and in space fix $P_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ and a positively oriented orthonormal frame of vectors at P_0, $\{T_0, N_0, B_0\}$.

We show that there exists a *unique* unit speed curve $\sigma : (a,b) \to \mathbb{R}^3$ having curvature $\overline{\kappa}$ and torsion $\overline{\tau}$ such that $\sigma(s_0) = P_0$ and σ has Frenet frame $\{T_0, N_0, B_0\}$ at $\sigma(s_0)$.

The proof is based on the Frenet formulas:

$$\begin{aligned} T' &= \kappa N \\ N' &= -\kappa T + \tau B \\ B' &= -\tau N \end{aligned}$$

or, in matrix form,

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

The idea is to mimmick these equations using the *given functions* $\overline{\kappa}$, $\overline{\tau}$. Consider the following *system* of O.D.E.'s in the (as yet unknown) vector-valued functions $e_1 = e_1(s), e_2 = e_2(s), e_3 = e_3(s)$,

$$\left.\begin{aligned} \frac{de_1}{ds} &= \overline{\kappa} e_2 \\ \frac{de_2}{ds} &= -\overline{\kappa} e_1 + \overline{\tau} e_3 \\ \frac{de_3}{ds} &= -\overline{\tau} e_2 \end{aligned}\right\} (*)$$

We express this system of ODE's in a notation convenient for the proof:

$$\frac{d}{ds}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \overline{\kappa} & 0 \\ -\overline{\kappa} & 0 & \overline{\tau} \\ 0 & -\overline{\tau} & 0 \end{bmatrix}}_{\Omega} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

Set,

$$\Omega = \begin{bmatrix} 0 & \overline{\kappa} & 0 \\ -\overline{\kappa} & 0 & \overline{\tau} \\ 0 & -\overline{\tau} & 0 \end{bmatrix} = \left[\Omega_i{}^j\right],$$

i.e. $\Omega_1{}^1 = 0,\ \Omega_1{}^2 = \overline{\kappa},\ \Omega_1{}^3 = 0$, etc. Note that Ω is skew symmetric, $\Omega^t = -\Omega \iff \Omega_j{}^i = -\Omega_i{}^j,\ 1 \le i, j \le 3$. Thus we may write,

$$\frac{d}{ds}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \left[\Omega_i{}^j\right]\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

or,

$$\frac{d}{ds}e_i = \sum_{j=1}^{3}\Omega_i{}^j e_j\,, \quad 1 \le i \le 3$$

$$\text{IC}: \quad \begin{aligned} e_1(s_0) &= T_0 \\ e_2(s_0) &= N_0 \\ e_3(s_0) &= B_0 \end{aligned}$$

Now, basic existence and unique result for systems of linear ODE's guarantees that this system has a unique solution:

$$s \to e_1(s),\ s \to e_2(s),\ s \to e_3(s),\ s \in (a, b)$$

We show that $e_1 = T$, $e_2 = N$, $e_3 = B$, $\overline{\kappa} = \kappa$ and $\overline{\tau} = \tau$ for some unit speed curve $s \to \sigma(s)$.

Claim $\{e_1(s), e_2(s), e_3(s)\}$ is an orthonormal frame for all $s \in (a, b)$, i.e.,

$$\langle e_i(s), e_j(s)\rangle = \delta_{ij} \ \ \forall\ s \in (a, b)$$

where δ_{ij} is the "Kronecker delta" symbol:

$$\delta_{ij} = \begin{cases} 0 & i \ne j \\ 1 & i = j. \end{cases}$$

Proof of the claim: We make use of the "Einstein summation convention":

$$\frac{d}{ds}e_i = \sum_{j=1}^{3}\Omega_i{}^j e_j = \Omega_i{}^j e_j$$

Let $g_{ij} = \langle e_i, e_j\rangle,\ g_{ij} = g_{ij}(s), 1 \le i, j \le 3$. Note,

$$\begin{aligned} g_{ij}(s_0) &= \langle e_i(s_0), e_j(s_0)\rangle \\ &= \delta_{ij} \end{aligned}$$

The g_{ij}'s satisfy a system of linear ODE's,

$$\begin{aligned}
\frac{d}{ds} g_{ij} &= \frac{d}{ds}\langle e_i, e_j\rangle \\
&= \langle e_i', e_j\rangle + \langle e_i, e_j'\rangle \\
&= \langle \Omega_i{}^k e_k, e_j\rangle + \langle e_i, \Omega_j{}^\ell e_\ell\rangle \\
&= \Omega_i{}^k \langle e_k, e_j\rangle + \Omega_j{}^\ell \langle e_i, e_\ell\rangle
\end{aligned}$$

Hence,

$$\frac{d}{ds} g_{ij} = \Omega_i{}^k g_{kj} + \Omega_j{}^\ell g_{i\ell}$$

$$\text{IC}: \; g_{ij}(s_0) = \delta_{ij}$$

Observe, $g_{ij} = \delta_{ij}$ is a solution to this system,

$$\begin{aligned}
LHS &= \frac{d}{ds}\delta_{ij} = \frac{d}{ds}\text{const} = 0. \\
RHS &= \Omega_i{}^k \delta_{kj} + \Omega_j{}^\ell \delta_{i\ell} \\
&= \Omega_i{}^j + \Omega_j{}^i \\
&= 0 \text{ (skew symmetry!)}.
\end{aligned}$$

But ODE theory guarantees a *unique* solution to this system. Therefore $g_{ij} = \delta_{ij}$ is *the* solution, and hence the claim follows.

How to define σ: Well, if $s \to \sigma(s)$ is a unit speed curve then

$$\sigma'(s) = T(s) \quad \Rightarrow \quad \sigma(s) = \sigma(s_0) + \int_{s_0}^{s} T(s)ds.$$

Hence, we *define* $s \to \sigma(s), s \in (a,b)$ by,

$$\sigma(s) = P_0 + \int_{s_0}^{s} e_1(s)ds$$

Claim σ is unit speed, $\kappa = \overline{\kappa}$, $\tau = \overline{\tau}$, $T = e_1$, $N = e_2$, $B = e_3$.

We have,

$$
\begin{aligned}
\sigma' &= \frac{d}{ds}(P_0 + \int_{s_0}^{s} e_1(s)ds) = e_1 \\
|\sigma'| &= |e_1| = 1\,, \quad \text{therefore } \sigma \text{ is unit speed,} \\
T &= \sigma' = e_1 \\
\kappa &= |T'| = |e_1'| = |\overline{\kappa} e_2| = \overline{\kappa} \\
N &= \frac{T'}{\kappa} = \frac{e_1'}{\overline{\kappa}} = \frac{\overline{\kappa} e_2}{\overline{\kappa}} = e_2 \\
B &= T \times N = e_1 \times e_2 = e_3 \\
B' &= e_3' = -\overline{\tau} e_2 = -\overline{\tau} N \quad \Rightarrow \\
\tau &= \overline{\tau}\,.
\end{aligned}
$$