A COMPETITION-DIFFUSION SYSTEM WITH A REFUGE

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(Communicated by Gail Wolkowicz)

ABSTRACT. In this paper, a model composed of two Lotka-Volterra patches is considered. The system consists of two competing species X, Y and only species Y can diffuse between patches. It is proved that the system has at most two positive equilibria and then that permanence implies global stability. Furthermore, to answer the question whether the refuge is effective to protect Y, the properties of positive equilibria and the dynamics of the system are studied when X is a much stronger competitor.

1. **Introduction.** In the study of ecology, we know that all species live in certain environments, where they spread out in space. Since the pioneering work of Skellam [13], spatial ecology has developed rapidly. Different diffusion modes correspond to different systems, that is, continuous diffusion corresponds to reaction-diffusion equations, while discrete diffusion corresponds to discrete diffusion equations. For the discrete systems with diffusion, many works considered the effect of diffusion on realizing system persistence and permanence. Among these, we mention Levin [11], Freedman and Waltman [5]. We point out especially that Takeuchi had many works in this field. Actually, his works include [19-24], Takeuchi and Lu [26], Freedman and Takeuchi [3, 4], Kuang and Takeuchi [10], and so forth. Also one can refer to his monograph [25] where part of his works are collected.

To save or protect certain species, a natural idea is to set up some refuges so that the protected species can enter or leave the refuges freely but its predators and competitors are kept out. Several biologically related questions then arise. Are such refuges effective to save certain species? Is there any better idea to protect the species? These questions are considered in [2] via a predator-prey reaction-diffusion model with two species and in [1] via a competitive reaction-diffusion model with two species.

In this paper, we will consider the following discrete diffusion model containing of two species with a refuge for one of them

$$\frac{dx_1}{dt} = r_1 x_1 (1 - x_1 - a_1 y_1),$$
$$\frac{dy_1}{dt} = y_1 (1 - y_1 - \mu x_1) + \delta(y_2 - y_1),$$

(1.1)

$$\frac{dy_2}{dt} = s_2 y_2 (1 - L_2^{-1} y_2) + \delta(y_1 - y_2),$$

where $x_1 \geq 0$ and $y_1 \geq 0$ are the densities of species X and Y in patch 1; $y_2 \geq 0$ is the density of species Y in patch 2; a_1 and μ represent the effects of species Y on species X (and vice versa); δ is the diffusion rate for species Y. All coefficients are positive.

This system describes an ecological model with two competing species, where Y can diffuse between patches while X is confined to one of the patches. It is easy to see that the system has exactly three boundary equilibria, i.e., $E_o := (0,0,0)$, $E_x := (1,0,0)$ and $E_y := (0, \tilde{y}_1, \tilde{y}_2)$. This model has been discussed by Takeuchi [22] and Takeuchi and Lu [26]. They gave some sufficient conditions for the permanence of the system, where permanence means there is a compact set S in the interior of the positive orthant such that the ω-limit set of any orbit with a positive initial value is contained in S. By constructing appropriate Liapunov functions they proved that (1.1) is globally stable if the competition in the two-species patch is weak enough. Their proof is somewhat technical. Still in [26], the authors pointed out "Unfortunately, we do not know in this case ($qr > 1$) if permanence implies global stability in general", where q and r describe the effects of competition in patch 1.

In this paper, we first prove that (1.1) has at most two positive equilibria. Then, using the method of monotone dynamical systems, we can prove that the system has a globally stable positive equilibrium if both E_x and E_y are linearly unstable. In other words, permanence implies global stability. This gives a positive answer to the question of [26]. To answer the biological questions whether the refuge is effective and whether there is some simpler idea to protect Y, regarding the effect of species X on species Y with μ as a parameter, we consider the properties of positive equilibria and the dynamics of the system for sufficiently large μ. We show that the stabilities of E_x and E_y are independent of sufficiently large μ and give the following conclusions.

Case 1. E_x is linearly stable and E_y is linearly unstable. In this case, E_x is globally stable.

Case 2. Both E_x and E_y are linearly stable. In this case, the attracting region of E_x increases in μ. Moreover, for any positive initial value v, there is some $\mu(v)$ such that v belongs to the attracting region of E_x for any $\mu > \mu(v)$.

Furthermore, we summarize an ecological conclusion, that is, the refuge is ineffective for Y in these two cases when species X is a much stronger competitor.

Case 3. E_x is linearly unstable and E_y is linearly stable. In this case, there are two positive equilibria, one linearly stable and one linearly unstable. The stable one, denoted by E_μ, converges to $(1, 0, \frac{s_2-\delta}{s_2} L_2)$ as $\mu \to \infty$. Moreover, the attracting region of E_μ increases in μ; For any positive initial value v, there is some $\mu(v)$ such that v belongs to the attracting region of E_μ for any $\mu > \mu(v)$.

Case 4. Both E_x and E_y are linearly unstable. In this case, it is shown that there is a globally stable positive equilibrium E_μ. Moreover, E_μ converges to $(1, 0, \frac{s_2-\delta}{s_2} L_2)$ as $\mu \to \infty$.

In addition, we conclude that in cases 3 and 4 the refuge is effective for Y, that is, Y always can survive in the second patch. But it is interesting that in these two cases, the density of Y in patch 2, $y_2 = \frac{s_2-\delta}{s_2} L_2$ at the stable coexistence, is less than the carrying capacity of Y in patch 2. This means that X affects the density of Y not only in patch 1 but also in patch 2. It is easy to see that to survive more species Y (and more species X), a simpler idea is to restrict not only the living

region of X but also the living region of Y. In other words, through restricting the living region of X in patch 1 and the living region of Y in patch 2, both the density of X in patch 1 and the density of Y in patch 2 can attain their carrying capacities.

2. **Model description and preliminaries.** We begin this section with some notations and definitions which will be used throughout this paper. Define the set $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for } 1 \geq i \geq n\}$ and $\text{Int}\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i > 0 \text{ for } 1 \geq i \geq n\}$. Let K be an orthant of $\mathbb{R}^n$. For any two points $x, y \in \mathbb{R}^n_+$, we write $x \leq_K y$ whenever $y - x \in K$, $x <_K y$ whenever $x \leq_K y$ and $x \neq y$, and $x \ll_K y$ when $y - x \in \text{Int}K$. If $x, y \in \mathbb{R}^n_+$ and $x \leq_K (\ll_K)y$, define $[x, y]_K = \{z \in \mathbb{R}^n_+ : x \leq_K z \leq_K y\}((x, y)_K = \{z \in \mathbb{R}^n_+ : x \ll_K z \ll_K y\})$.

Let $s(A) = \max\{\Re\lambda : \lambda \in \sigma(A)\}$, where A is a square matrix and $\sigma(A)$ is the set of eigenvalues of A. We denote the Jacobian matrix of a system of ordinary differential equations evaluated at an equilibrium P by $J(P)$. A semiflow ψ is said to be type-K monotone provided

$$\psi_t(x) \leq_K \psi_t(y) \text{ whenever } x \leq_K y \text{ and } t \geq 0.$$

The semiflow ψ is said to be strongly type-K monotone if ψ is type-K monotone and $\psi_t(x) \ll_K \psi_t(y)$ whenever $x <_K y$ and $t > 0$. An $n \times n$ A is called a cooperative matrix provided all off-diagonal entries of A are nonnegative, and a type-K cooperative matrix provided A has the form

$$\begin{pmatrix} A_1 & -A_2 \\ -A_3 & A_4 \end{pmatrix}$$

in which A_1 is a $k \times k$ cooperative matrix, A_2 is a $k \times (n-k)$ nonnegative matrix, A_3 is an $(n-k) \times k$ nonnegative matrix, A_4 is an $(n-k) \times (n-k)$ cooperative matrix. A is said to be competitive (type-K competitive) provided $-A$ is cooperative (type-K cooperative). A system of differential equations $\dot{x} = f(x)$ on $\mathbb{R}^n_+$ is called a type-K monotone (competitive) system if the Jacobian $Df(x)$ of f is type-K cooperative (competitive) at any $x \in \mathbb{R}^n_+$. Smith[17] showed that the flow generated by a type-K monotone system is type-K monotone. Furthermore, if $Df(x)$ is irreducible in some open set $\Omega \subset \mathbb{R}^n_+$, then the flow is strongly type-K monotone on Ω. Given any $z \in \mathbb{R}^n_+$, an orbit $\psi_t(z)$ is said to be K-monotonic nondecreasing (or nonincreasing), if $\psi_{t_1}(z) \geq_K \psi_{t_2}(z)$ whenever $t_1 > (\text{or} <)t_2$. For simplicity, we call $\psi_t(z)$ increasing (decreasing) to an equilibrium, when $\psi_t(z)$ K-monotonic nondecreasingly (nonincreasingly) converges to it as $t \to \infty$. Contextually, no confusion should result. In this paper, the important cones used are $\mathbb{R}^3_+$ and $K = \{(x_1, y_1, y_2) \in \mathbb{R}^3 : x_1 \geq 0, y_1 \leq 0, y_2 \leq 0\}$. We reserve the symbol K for this latter one. If the cone is $\mathbb{R}^n_+$, we drop the K and write " $\leq$ ", " $<$ ", " $\ll$ ".

In the spatial ecology, the following system:

$$\frac{dX_1}{d\tau} = \xi_1 X_1(1 - \frac{X_1}{M_1} - A_1\frac{Y_1}{M_1}),$$
$$\frac{dY_1}{d\tau} = \eta_1 Y_1(1 - \frac{Y_1}{N_1} - B_1\frac{X_1}{N_1}) + d(Y_2 - Y_1), \quad (2.1)$$
$$\frac{dY_2}{d\tau} = \eta_2 Y_2(1 - \frac{Y_2}{N_2}) + d(Y_1 - Y_2),$$

describes a model of competition-diffusion with a refuge for one species, where $X_1 \geq 0, Y_1 \geq 0$ are the population densities of competitors X and Y in patch 1; $Y_2 \geq 0$ is the population density of species Y in patch 2; A_1, B_1 describe the effects of competition in patch 1; $N_i, \ i = 1, 2$, (or M_1) are the carrying capacities for species Y (or X) in patch i (or 1); $\eta_i, i = 1, 2$, (or ξ_1) are the per capita growth rates for species Y (or X) in patch i (or 1); d is the per capita diffusion rate between two patches for species Y. Since species X is confined in patch 1 and only species Y can diffuse between the patches, patch 2 can be regarded as a refuge for species Y. Suppose all coefficients are positive.

We nondimensionalize equation (2.1) in order to describe the system in terms of a minimal set of parameters [12]. The following transformations:

$$x_1 = \frac{X_1}{M_1}, \ y_1 = \frac{Y_1}{N_1}, \ y_2 = \frac{Y_2}{N_1}, \ t = \eta_1 \tau, \ r_1 = \frac{\xi_1}{\eta_1},$$

$$s_2 = \frac{\eta_2}{\eta_1}, \ \delta = \frac{d}{\eta_1}, \ L_2 = \frac{N_2}{N_1}, \ a_1 = A_1 \frac{N_1}{M_1}, \ \mu = B_1 \frac{M_1}{N_1},$$

yield the nondimensional system

$$\frac{dx_1}{dt} = r_1 x_1 (1 - x_1 - a_1 y_1),$$

$$\frac{dy_1}{dt} = y_1 (1 - y_1 - \mu x_1) + \delta(y_2 - y_1), \qquad (2.2)$$

$$\frac{dy_2}{dt} = s_2 y_2 (1 - L_2^{-1} y_2) + \delta(y_1 - y_2).$$

The quantities $x_1 \geq 0$ and $y_1 \geq 0$ represent the densities of species X and Y in patch 1 scaled by their respective carrying capacities; $y_2 \geq 0$ represents the density of species Y in patch 2 scaled by the carrying capacity of Y in patch 1; a_1 and μ represent the per capita effect of species Y on species X (and vice versa) scaled by the ratio of respective carrying capacities; δ is the species specific diffusion rate scaled by the growth rate of species Y in patch 1; r_1 is the ratio of per capita growth rates of the two species in patch 1; a quantity L_2 represents the ratio of carrying capacities in the two patches for species Y; s_2 represents the ratio of per capita growth rates of species Y in the two patches; and t is a time metric that is a composite of τ and η_1. Here all coefficients are positive.

In order to obtain conditions for permanence, it is necessary to discuss the following subsystem:

$$\dot{y}_1 = y_1 (1 - y_1) + \delta(y_2 - y_1),$$

$$\dot{y}_2 = s_2 y_2 (1 - L_2^{-1} y_2) + \delta(y_1 - y_2). \qquad (2.3)$$

The following lemma has been proved many times: see for example [5] and [17].

Lemma 2.1. *For system* (2.3)*, there is a unique non-zero equilibrium, denoted by* $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)$. $\tilde{y} \gg 0$ *and it is globally stable with respect to* $\mathbb{R}_+^2 \backslash \{(0, 0)\}$.

Remark 2.1. If $\tilde{J}$ denotes the Jacobian matrix of (2.3) at $y = \tilde{y}$, then $s(\tilde{J}) < 0$. It is based on the equality

$$\tilde{J} = \begin{pmatrix} -\delta + 1 - 2\tilde{y}_1 & \delta \\ \delta & -\delta + s_2 - 2s_2 L_2^{-1} \tilde{y}_2 \end{pmatrix} = \begin{pmatrix} -\tilde{y}_1 - \delta \frac{\tilde{y}_2}{\tilde{y}_1} & \delta \\ \delta & -s_2 L_2^{-1} \tilde{y}_2 - \delta \frac{\tilde{y}_1}{\tilde{y}_2} \end{pmatrix}.$$

Obviously, $\tilde{J}$ is a negative definite quadratic form. Hence $s(\tilde{J}) < 0$.

It is easy to see that (2.2) has exactly three boundary equilibria, i.e., $(0,0,0)$, $(1,0,0)$ and $(0,\tilde{y}_1,\tilde{y}_2)$, denoted by E_o, E_x and E_y, respectively. Hereafter we consider only the generic case where the Jacobian matrices evaluated at E_o, E_x and E_y are hyperbolic; that is, no eigenvalue of the matrices has its real part equals to zero. By the Butler-McGehee lemma in [6], Takeuchi[22] gave the following lemma.

Lemma 2.2. *For system* (2.2), *we have the following properties:*
  (i) E_o *is linearly unstable and* $W^s(E_o) \cap (\mathbb{R}^3_+ \backslash \{E_o\}) = \emptyset$;
  (ii) *If* E_x *is linearly unstable, then* $W^s(E_x) \cap \mathbb{R}^3_+ = \{(x_1,y_1,y_2) \in \mathbb{R}^3_+ : x_1 > 0, y_1 = 0, y_2 = 0\}$;
  (iii) *If* E_y *is linearly unstable, then* $W^s(E_y) \cap \mathbb{R}^3_+ = \{(x_1,y_1,y_2) \in \mathbb{R}^3_+ : x_1 = 0, y_1 \geq 0, y_2 \geq 0, y_1 + y_2 > 0\}$.
    Here $W^s(P)$ *denote the strong stable manifold of an equilibrium* P.

The Jacobian matrix of the right-hand side of (2.2) is:

$$\begin{pmatrix} r_1 - 2r_1x_1 - r_1a_1y_1 & -r_1a_1x_1 & 0 \\ -\mu y_1 & 1 - 2y_1 - \mu x_1 - \delta & \delta \\ 0 & \delta & s_2 - 2s_2L_2^{-1}y_2 - \delta \end{pmatrix}.$$

It is apparent that (2.2) is cooperative with respect to K where

$$K = \{(x_1,y_1,y_2) \in \mathbb{R}^3 : x_1 \geq 0, y_1 \leq 0, y_2 \leq 0\}.$$

Furthermore, the Jacobian matrix is irreducible in Int$\mathbb{R}^3_+$. Let F denote the vector field described by (2.2), ψ_t the corresponding flow. Therefore, the flow ψ_t of (2.2) is type-K monotone in $\mathbb{R}^3_+$ and strongly type-K monotone in Int$\mathbb{R}^3_+$.

Lemma 2.1 and monotonicity imply that (2.2) is dissipative. In fact, if $z = (x,y) \in \mathbb{R}^3_+$ and $x = x_1 > 0, y = (y_1,y_2) > 0$, then $(0,y) \leq_K z \leq_K (x,0)$ and therefore,

$$\psi_t((0,y)) \leq_K \psi_t(z) \leq_K \psi_t((x,0)), t > 0.$$

Since $\psi_t((0,y)) \to E_y$ and $\psi_t((x,0)) \to E_x$ as $t \to \infty$, it follows that all positive orbits are attracted to the set $[E_y, E_x]_K$. If $z = (x,y)$ satisfies $x, y > 0$, then $\psi_t(z) \gg 0$ for $t > 0$. Define E and E^+ to be the sets of all nonnegative equilibria and all positive equilibria for ψ_t, respectively. Obviously, $[E_y, E_x]_K$ contains E and $E_* \in (E_y, E_x)_K$ for any $E_* \in E^+$. We can find the following results in [26], [15] and [18], respectively.

Lemma 2.3. *If* E_x *and* E_y *are both linearly unstable, then system* (2.2) *is permanent, or more precisely, there exist positive equilibria* E_* *and* E_{**} *with* $E_* \leq_K E_{**}$ *such that the* ω-*limit set of any solution initiating in* $\{z = (x,y) : x, y > 0\}$ *is contained in* $[E_*, E_{**}]_K$. *Furthermore, if* $E_* = E_{**}$, *then* E_* *is globally stable.*

Lemma 2.4. *Let* $\dot{z} = G(z)$ *be type-K monotone system on* $\mathbb{R}^n_+$ *and suppose* $G(z) \geq_K (\leq_K) 0$ *for some* $z \in \mathbb{R}^n_+$. *If* $\Psi_t(z)$ *is defined for all* $t \geq 0$, *then* $\Psi_t(z)$ *is K-monotonic nondecreasing(nonincreasing) for* $t > 0$, *where* Ψ_t *is the flow corresponding to* $\dot{z} = G(z)$. *If, in addition,* $O(z)$ *has compact closure in* $\mathbb{R}^n_+$, *then* $\omega(z)$ *is precisely one equilibrium, where* $O(z) = \{\Psi_t(z) : t \geq 0\}$.

Lemma 2.5. *Let* $\dot{z} = G_i(z), i = 1, 2$ *be type-K monotone systems on* $\mathbb{R}^n_+$ *and satisfy*

$$G_1(z) \geq_K G_2(z), \ \forall z \in \mathbb{R}^n_+.$$

Let Ψ_t^i be the flow corresponding to $\dot{z} = G_i(z), i = 1, 2$. If $z_1, z_2 \in \mathbb{R}_+^n$ and $z_1 \geq_K z_2$, $t > 0$ and $\Psi_t^i(z_i), i = 1, 2$ are defined, then $\Psi_t^1(z_1) \geq_K \Psi_t^2(z_2)$.

Note that the Jacobian of (2.2) is a tridiagonal matrix. By the theory of Smillie[14], we have the following result.

Theorem 2.1. *For system (2.2), $\omega(z)$ is a singleton for any $z \in \mathbb{R}_+^3$.*

Corollary 2.1. *Suppose (2.2) has no positive equilibrium. Then either E_x or E_y is globally stable.*

Note that system (2.2) is also type-K competitive when we consider the cone $K_1 = \{(x_1, y_1, y_2) \in \mathbb{R}^3 : x_1 \geq 0, y_1 \geq 0, y_2 \leq 0\}$. The dynamics of both the type-K monotone systems and the type-K competitive systems is 1-codimensional. Thus we prove the dynamics of 2-species competition-diffusion system with a refuge is 1-dimensional (see [9]) and give a different proof of Theorem 2.1.

3. **Main results.** Let $E_{\mu_0} = (x_1(\mu_0), y_1(\mu_0), y_2(\mu_0))$ be a positive equilibrium of system (2.2) when it has at least one at $\mu = \mu_0$. Let F^{μ_0} denote the vector field of F at $\mu = \mu_0$, $\psi_t^{\mu_0}$ the corresponding flow. If the field is at μ, we drop the μ and write F and ψ_t. First, let us calculate the Jacobian matrices at the three boundary equilibria.

$$J(E_o) = \begin{pmatrix} r_1 & 0 \\ 0 & J_o(E_o^2) \end{pmatrix}, \text{ where } J_o(E_o^2) = \begin{pmatrix} 1 - \delta & \delta \\ \delta & s_2 - \delta \end{pmatrix}.$$

$$J(E_x) = \begin{pmatrix} -r_1 & *_1 \\ 0 & J_x(E_x^2) \end{pmatrix}, \text{ where } *_1 = (-r_1 a_1, 0), \ J_x(E_x^2) = \begin{pmatrix} 1 - \mu - \delta & \delta \\ \delta & s_2 - \delta \end{pmatrix}.$$

$$J(E_y) = \begin{pmatrix} r_1 - r_1 a_1 \tilde{y}_1 & 0 \\ *_2 & J_y(E_y^2) \end{pmatrix}, \text{ where } *_2 = \begin{pmatrix} -\mu \tilde{y}_1 \\ 0 \end{pmatrix},$$

$$J_y(E_y^2) = \begin{pmatrix} 1 - 2\tilde{y}_1 - \delta & \delta \\ \delta & s_2 - 2s_2 L_2^{-1} \tilde{y}_2 - \delta \end{pmatrix}.$$

Remark 3.1. Since $J_o(E_o^2)$ is a symmetric matrix, its eigenvalues are both real numbers. It is easy to check that $s(J_o(E_o^2)) > 0$. Therefore E_o has at least two positive eigenvalues and it is linearly unstable. According to Remark 2.1, we know that $s(J_y(E_y^2)) < 0$.

Remark 3.2. Note that $J(E_\mu)$ is a tridiagonal, quasi-symmetric matrix (i.e., all nonzero off-diagonal entries $a_{i,i+1}$ and $a_{i+1,i}$ have the same sign). Hence all eigenvalues of $J(E_\mu)$ are real (see [27]).

Remark 3.3. E^+ is totally strongly ordered with respect to $\ll_K$, that is, $E^{(1)} \ll_K E^{(2)}$ or $E^{(1)} \gg_K E^{(2)}$ for any pair of points $E^{(i)} = (x_1^{(i)}, y_1^{(i)}, y_2^{(i)}) \in E^+, i = 1, 2$ with $E^{(1)} \neq E^{(2)}$. Without loss of generality, we suppose that $x_1^{(1)} > x_1^{(2)}$. Since $1 - x_1^{(i)} = a_1 y_1^{(i)}, i = 1, 2, \ y_1^{(1)} < y_1^{(2)}$. Assume to the contrary, $y_2^{(1)} > y_2^{(2)}$. Then $s_2 - s_2 L_2^{-1} y_2 - \delta < 0$ for any $y_2 \geq y_2^{(2)}$. Define $g(y_2) = s_2 y_2 (1 - L_2^{-1} y_2) - \delta y_2$. Therefore,

$$g(y_2^{(2)}) - g(y_2^{(1)}) = \delta y_1^{(1)} - \delta y_1^{(2)} < 0,$$

which contradicts to the fact that $g'(y_2) = s_2 - 2s_2 L_2^{-1} y_2 - \delta < 0$ for any $y_2 \geq y_2^{(2)}$.

Proposition 3.1. *Let $\mu^* = 1 + \frac{s_2 \delta}{\delta - s_2}$ when $s_2 < \delta$. We have*

(i) *E_x is linearly unstable if $\mu > 0$ and $s_2 \geq \delta$, or $\mu < \mu^*$ and $s_2 < \delta$; moreover, E_x is linearly stable if $s_2 < \delta$ and $\mu > \mu^*$.*

(ii) E_y *is linearly unstable (linearly stable) if* $1 - a_1\tilde{y}_1 > 0$ (< 0).

Proof. Since (ii) is obvious, we only prove (i). Let $\varrho = 1 - \mu$. E_x is linearly unstable if and only if $s(J_x(E_x^2)) > 0$. If $\varrho \geq 0$, then clearly $s(J_x(E_x^2)) > 0$;
If $\varrho < 0$ and $s_2 - \delta \geq 0$, then $\det J_x(E_x^2) < 0$, therefore $s(J_x(E_x^2)) > 0$;
If $\varrho < 0$ and $s_2 - \delta < 0$, then $\operatorname{tr} J_x(E_x^2) < 0$. Hence, $s(J_x(E_x^2)) > 0$ implies $\det J_x(E_x^2) = \rho s_2 - \rho\delta - s_2\delta < 0$. Thus $\varrho > \frac{s_2\delta}{s_2 - \delta}$. Substituting $\varrho = 1 - \mu$ into it yields the result. $\qquad\square$

Theorem 3.1. *For any* $\mu > 0$, *(2.2) has at most two positive equilibria.*

To prove this theorem, we first prove the following three weaker propositions.

Proposition 3.2. *For any* $\mu > 0$, *system (2.2) has at most three positive equilibria.*

Proof. Let (x_1, y_1, y_2) be a positive equilibrium of (2.2). Then it satisfies

$$
\begin{aligned}
1 - x_1 - a_1 y_1 &= 0, \\
y_1(1 - y_1 - \mu x_1) + \delta(y_2 - y_1) &= 0, \\
s_2 y_2(1 - L_2^{-1} y_2) + \delta(y_1 - y_2) &= 0.
\end{aligned}
\tag{3.1}
$$

Solve x_1, y_2 in terms of y_1, i.e., $x_1 = 1 - a_1 y_1$, $y_2 = (\delta y_1 - y_1(1 - y_1 - \mu(1 - a_1 y_1)))/\delta$, and substitute these into the third equation. After some algebraic manipulations it can be reduced to

$$
0 = \delta(y_1 - y_2) + s_2 y_2(1 - y_2 L_2^{-1}) = \delta^{-1}(-A y_1^4 + 2B y_1^3 + C y_1^2 + D y_1),
\tag{3.2}
$$

where

$$
A = \frac{(-1 + \mu a_1)^2 s_2}{\delta L_2}, \quad B = \frac{(-1 + \mu a_1)(\delta - 1 + \mu)s_2}{\delta L_2},
$$

$$
C = \frac{\delta(-1 + \mu a_1)(\delta - s_2)L_2 - (\delta - 1 + \mu)^2 s_2}{\delta L_2}, \quad D = -(-1 + \mu)(\delta - s_2) + \delta s_2.
$$

For the sake of contradiction, assume (2.2) has more than three positive equilibria. Then all coefficients of (3.2) must equal zero. From the coefficients of y_1^4 and y_1^2, we have $\mu = 1/a_1 = 1 - \delta > 0$. However, the constant term equals $\frac{(s_2 - \delta)(1 - \delta - 1) + \delta s_2}{\delta} = \delta \neq 0$, giving a contradiction. $\qquad\square$

Proposition 3.3. *For any* $\mu > \mu^{**}$, *system (2.2) has at most two positive equilibria, where* $\mu^{**} = 1/a_1$.

Proof. Deduce by contradiction, we assume that (2.2) has three positive equilibria. From (3.1), we have

$$
\begin{aligned}
(1 - \mu)y_1 - (1 - \mu a_1)y_1^2 + \delta(y_2 - y_1) &= 0, \\
s_2 y_2 - s_2 L_2^{-1} y_2^2 + \delta(y_1 - y_2) &= 0.
\end{aligned}
$$

Direct calculations yield

$$
\begin{aligned}
(\mu a_1 - 1)y_1 &= -(1 - \mu) - \delta(u^{-1} - 1) > 0, \\
s_2 L_2^{-1} y_2 &= s_2 + \delta(u - 1) > 0,
\end{aligned}
\tag{3.3}
$$

where $u = y_1/y_2$ and u satisfies

$$
(\mu a_1 - 1)\delta u^3 - (\mu a_1 - 1)(\delta - s_2)u^2 + s_2 L_2^{-1}(1 - \delta - \mu)u - (-s_2 L_2^{-1}\delta) = 0. \tag{3.4}
$$

Any positive equilibrium of (2.2) corresponds a positive root of (3.4). If $u_i, i = 1, 2, 3$ denote the roots of (3.4), then

$$u_1 u_2 u_3 = \frac{-s_2 L_2^{-1} \delta}{(\mu a_1 - 1)\delta}.$$

From the third equation of (3.1), if (x_1, y_1, y_2) is a positive equilibrium of (2.2), then $(1 - a_1 k y_1, k y_1, k y_2), k > 0$ is a positive equilibrium of (2.2) if and only if $k = 1$. Therefore different positive equilibria of (2.2) correspond to different positive roots of (3.4). Hence (3.4) must have three different positive roots, this contradicts to $u_1 u_2 u_3 < 0$. Thus the proposition is proved. □

Proposition 3.4. *Let $\mu \leq \mu^{**} = 1/a_1$. If system (2.2) has a positive equilibrium E_μ, then it is linearly stable.*

Proof. Let E_μ be a positive equilibrium of (2.2). The Jacobian matrix of the vector field at E_μ is given by $J(E_\mu)$, that is,

$$\begin{pmatrix} r_1 - 2r_1 x_1(\mu) - r_1 a_1 y_1(\mu) & -r_1 a_1 x_1(\mu) & 0 \\ -\mu y_1(\mu) & 1 - 2y_1(\mu) - \mu x_1(\mu) - \delta & \delta \\ 0 & \delta & s_2 - 2s_2 L_2^{-1} y_2(\mu) - \delta \end{pmatrix}$$

$$= \begin{pmatrix} -r_1 x_1(\mu) & -r_1 a_1 x_1(\mu) & 0 \\ -\mu y_1(\mu) & -y_1(\mu) - \delta \frac{y_2(\mu)}{y_1(\mu)} & \delta \\ 0 & \delta & -s_2 L_2^{-1} y_2(\mu) - \delta \frac{y_1(\mu)}{y_2(\mu)} \end{pmatrix}.$$

By Theorem 2.7 in [16], for a type-K matrix $J(E_\mu)$, $s(J(E_\mu)) < 0$ if and only if the following three inequalities are satisfied, that is,

$$(-1) \left| -r_1 x_1(\mu) \right| > 0,$$

$$(-1)^2 \begin{vmatrix} -r_1 x_1(\mu) & r_1 a_1 x_1(\mu) \\ \mu y_1(\mu) & -y_1(\mu) - \delta \frac{y_2(\mu)}{y_1(\mu)} \end{vmatrix} > 0, \text{ i.e., } \mu - \frac{1}{a_1} < \frac{\delta y_2(\mu)}{a_1 y_1^2(\mu)},$$

$$(-1)^3 \begin{vmatrix} -r_1 x_1(\mu) & r_1 a_1 x_1(\mu) & 0 \\ \mu y_1(\mu) & -y_1(\mu) - \delta \frac{y_2(\mu)}{y_1(\mu)} & \delta \\ 0 & \delta & -s_2 L_2^{-1} y_2(\mu) - \delta \frac{y_1(\mu)}{y_2(\mu)} \end{vmatrix} > 0,$$

i.e., $\mu - \frac{1}{a_1} < \frac{\delta y_2(\mu)}{a_1 y_1^2(\mu)} - \frac{\delta^2 y_2(\mu)}{a_1 y_1(\mu)(s_2 L_2^{-1} y_2^2(\mu) + \delta y_1(\mu))} := K(\mu).$

Obviously, we have $K(\mu) > 0$. It follows that E_μ is linearly stable if and only if $\mu < 1/a_1 + K(\mu)$. In particular, E_μ is linearly stable provided $\mu \leq 1/a_1$. □

By Theorem 2.1, together with Theorem 3.7 in [15], (2.2) has no more than one positive equilibrium when $\mu \leq \mu^{**}$. Combining with Proposition 3.3, we complete the proof of Theorem 3.1.

Remark 3.4. Proposition 3.4 provides us a criterion to determine whether a positive equilibrium is linearly stable or not, i.e., E_μ is linearly stable if and only if $\frac{1}{\mu}(1/a_1 + K(\mu)) > 1$, which will be frequently used throughout this paper.

From Proposition 3.4, we can obtain the following three corollaries about the global dynamics of the system for small μ which were given in [22].

Corollary 3.1. *Suppose $1 - a_1\tilde{y}_1 < 0$. Then for sufficiently small $\mu > 0$, especially, for $\mu \leq \mu^{**} = 1/a_1$, E_y is globally stable with respect to $\{(x_1, y_1, y_2) \in \mathbb{R}^3_+ : y_1 + y_2 > 0\}$.*

Remark 3.5. It is worth noting that, E_x is linearly unstable under the condition $\mu \leq \mu^{**}$ and $1 - a_1\tilde{y}_1 < 0$. In fact, by Proposition 3.1, if $s_2 \geq \delta$, then E_x is linearly unstable; else if $s_2 < \delta$, then to show that E_x is linearly unstable it suffices to prove $\mu^{**} < \mu^*$, or, equivalently, $\frac{1}{a_1} < 1 + \frac{s_2\delta}{\delta - s_2}$. Since $\tilde{y}_1 > \frac{1}{a_1}$, it suffices to prove $\tilde{y}_1 < 1 + \frac{s_2\delta}{\delta - s_2}$.

$$\tilde{y}_1 - 1 = \frac{\delta(\tilde{y}_2 - \tilde{y}_1)}{\tilde{y}_1} < \frac{s_2\delta}{\delta - s_2} \text{ implies } \delta(\tilde{y}_2 - \tilde{y}_1) < s_2\tilde{y}_2$$

Since $\delta(\tilde{y}_2 - \tilde{y}_1) - s_2\tilde{y}_2 = -s_2 L_2^{-1}\tilde{y}_2^2 < 0$, we prove the linear instability of E_x.

Corollary 3.2. *Suppose $s_2 < \delta$ and $\mu^* < \mu \leq \mu^{**}$. Then E_x is globally stable with respect to $\{(x_1, y_1, y_2) \in \mathbb{R}^3_+ : x_1 > 0, y_1 \geq 0, y_2 \geq 0\}$.*

Remark 3.6. Similar to Remark 3.5, we can prove the linear instability of E_y under the condition $s_2 < \delta$ and $\mu^* < \mu \leq \mu^{**}$.

Corollary 3.3. *Let $1 - a_1\tilde{y}_1 > 0$. Then system (2.2) has a unique positive equilibrium for sufficiently small $\mu > 0$. In particular, permanence implies global stability if $\mu < \min\{\mu^*, \mu^{**}\}$.*

More general than Corollary 3.3, we have the following theorem:

Theorem 3.2. *Suppose E_x and E_y are both linearly unstable. Then system (2.2) has a unique positive equilibrium for any $\mu > 0$. In other words, permanence implies global stability.*

Proof. Suppose, on the contrary, that there exists some μ_0 such that (2.2) has two positive equilibria $E^1_{\mu_0}$ and $E^2_{\mu_0}$ satisfying $E^1_{\mu_0} \ll_K E^2_{\mu_0}$. Clearly, we have $\mu_0 > 1/a_1$. We claim that $E^1_{\mu_0}$ and $E^2_{\mu_0}$ are both non-hyperbolic equilibria.

We argue by contradiction to prove the claim. By Theorem 3.1, one of $E^1_{\mu_0}$ and $E^2_{\mu_0}$ must be non-hyperbolic. If the other is hyperbolic, then it must be linearly stable. Without loss of generality, suppose $E^1_{\mu_0}$ is linearly stable and $E^2_{\mu_0}$ is non-hyperbolic, the other case can be proved similarly. Since $E^1_{\mu_0}$ is linearly stable, we can find $\epsilon > 0$ which is small so that for $\mu = \mu_0 + \epsilon$ system (2.2) has a linearly stable positive equilibrium, denoted by E^1_μ, satisfying $E^1_\mu \ll_K E^2_{\mu_0}$. Since E_x is linearly unstable and $F(E^2_{\mu_0}) >_K 0$, $\psi_t(E^2_{\mu_0})$ increases to a positive equilibrium, denoted by E^2_μ, satisfying $E^2_{\mu_0} \ll_K E^2_\mu$. Hence E^1_μ and E^2_μ are both stable in $[E^1_\mu, E^2_\mu]_K$. By the theory of connecting orbits in [7], there must exist a further positive equilibrium, a contradiction.

Since both $E^1_{\mu_0}$ and $E^2_{\mu_0}$ are non-hyperbolic, according to Remark 3.2, $E^i_{\mu_0}$ must satisfy (3.1) and $\det J(E^i_{\mu_0}) = 0$, $i = 1, 2$. Direct calculations yield

$$\delta^2 r_1 x_1 + (-r_1 x_1 + \delta r_1 x_1 + \mu r_1 x_1^2 + 2r_1 x_1 y_1 - \mu a_1 r_1 x_1 y_1)(-\delta + s_2 - 2s_2 y_2/L_2) = 0.$$

Substituting $x_1 = 1 - a_1 y_1$ and $y_2 = (\delta y_1 - y_1(1 - y_1 - \mu(1 - a_1 y_1)))/\delta$ into the above equation produces

$$-4Ay_1^3 + 6By_1^2 + 2Cy_1 + D = 0, \tag{3.5}$$

where A, B, C, D are defined as in Proposition 3.2. Multiplying (3.2) by $4\delta/y_1$ and then subtracting it from (3.5), there hold

$$2By_1^2 + 2Cy_1 + 3D = 0. \tag{3.6}$$

Therefore, (3.6) has two different positive roots. If $B = 0$, i.e., $\mu = 1 - \delta$, then $C = D = 0$. We have $s_2 = \delta$ from $C = 0$, then $D = \delta s_2 > 0$, a contradiction. Thus $B \neq 0$. Substituting $D = -(2By_1^2 + 2Cy_1)/3$ into (3.5) produces

$$-3Ay_1^2 + 4By_1 + C = 0. \tag{3.7}$$

Therefore, (3.7) also has two different positive roots. Hence, we have

$$\frac{2B}{-3A} = \frac{2C}{4B} = \frac{3D}{C}$$

and $C \neq 0, D \neq 0$. It follows that $C^2 = 6BD$. Since (3.6) has two different positive roots, $4C^2 - 24BD > 0$, a contradiction. The theorem is proved. $\square$

Note that the proof of Theorem 3.2 tells us that system (2.2) cannot have two non-hyperbolic positive equilibria simultaneously. To answer the biological questions that whether the refuge is effective and whether there is some better idea to protect species Y, in the rest of this section we focus on the properties of positive equilibria and the dynamics of (2.2) for sufficiently large μ.

Proposition 3.5. *If there exists a sequence $\{\mu_n\}$ with $\mu_n \to \infty$ as $n \to \infty$, corresponding to a convergent sequence of positive equilibria $\{E_{\mu_n}\}$, then precisely one of the three alternatives holds:*

(i) $E_{\mu_n} \to E_x$, *which holds only if $s_2 \leq \delta$. In this case, $y_2(\mu_n) \sim \mu_n y_1(\mu_n)\delta^{-1}$;*

(ii) $E_{\mu_n} \to (1, 0, \frac{s_2-\delta}{s_2}L_2)$, *denoted by E_a, which holds only if $s_2 \geq \delta$. In this case, $y_2(\mu_n) \sim \mu_n y_1(\mu_n)\delta^{-1}$, $y_2(\mu_n) > \frac{s_2-\delta}{s_2}L_2$;*

(iii) $E_{\mu_n} \to (0, \frac{1}{a_1}, \frac{(s_2-\delta)+\sqrt{(s_2-\delta)^2+4s_2L_2^{-1}a_1^{-1}\delta}}{2s_2L_2^{-1}})$, *denoted by E_b, which holds only if $1 - a_1\tilde{y}_1 \leq 0$.*

Proof. Let $E_{\mu_n} \to (x_1^*, y_1^*, y_2^*)$ as $n \to \infty$. Claim $x_1^* = 0$ or 1. If not, then $y_1^* = (1 - x_1^*)/a_1 > 0$. Passing to the limit $n \to \infty$ of the following equation

$$\mu_n x_1(\mu_n)y_1(\mu_n) = \delta(y_2(\mu_n) - y_1(\mu_n)) + y_1(\mu_n)(1 - y_1(\mu_n)),$$

we obtain that $\mu_n \to \frac{\delta(y_2^* - y_1^*) + y_1^*(1 - y_1^*)}{x_1^* y_1^*}$ as $n \to \infty$, which contradicts to $\mu_n \to \infty$.

If $x_1^* = 1$, then $y_1^* = 0$. Taking the limit $n \to \infty$ of the following equation

$$s_2 y_2(\mu_n)(1 - L_2^{-1}y_2(\mu_n)) + \delta(y_1(\mu_n) - y_2(\mu_n)) = 0, \tag{3.8}$$

we get $(s_2 - \delta - s_2L_2^{-1}y_2^*)y_2^* = 0$. If $y_2^* = 0$, then $E_{\mu_n} \to E_x$; else if $y_2^* > 0$, then $y_2^* = \frac{s_2-\delta}{s_2}L_2$, that is, $E_{\mu_n} \to E_a$.

If $x_1^* = 0$, then $y_1^* = 1/a_1$. Limiting (3.8) as $n \to \infty$ yields

$$s_2 L_2^{-1}y_2^{*2} - (s_2 - \delta)y_2^* - \delta/a_1 = 0.$$

This equation has two roots, one positive and one negative. We choose the positive one, i.e., E_b. Because of $x_1(\mu_n) = 1 - a_1 y_1(\mu_n) > 1 - a_1\tilde{y}_1$, we infer that $1 - a_1\tilde{y}_1 \leq 0$. $\square$

Remark 3.7. Actually from the discussion below we know that there exists a sequence $\{E_{\mu_n}\}$ with $E_{\mu_n} \to E_x$ as $\mu_n \to \infty$ if and only if $s_2 = \delta$ and $1 - a_1\tilde{y}_1 \neq 0$.

We will examine in detail the properties of positive equilibria and the dynamical behavior of (2.2). Our analysis will be carried out according to the following four cases:

1. $s_2 \geq \delta$, $1 - a_1\tilde{y}_1 > 0$, namely E_x and E_y are both linearly unstable.
2. $s_2 < \delta$, $1 - a_1\tilde{y}_1 > 0$, so E_y is linearly unstable, E_x is linearly stable if $\mu > \mu^*$ and linearly unstable if $\mu < \mu^*$.
3. $s_2 < \delta$, $1 - a_1\tilde{y}_1 < 0$, so E_y is linearly stable, E_x is linearly stable if $\mu > \mu^*$ and linearly unstable if $\mu < \mu^*$.
4. $s_2 \geq \delta$, $1 - a_1\tilde{y}_1 < 0$, namely E_x is linearly unstable, E_y is linearly stable.

Case 1: $s_2 \geq \delta$, $1 - a_1\tilde{y}_1 > 0$.

In this case, E_x and E_y are both linearly unstable. It follows from Theorem 3.2 that (2.2) has a unique positive equilibrium E_μ for any $\mu > 0$.

Theorem 3.3. *Let $s_2 > \delta$. Then*

(i) $E_\mu \to E_a$ *as $\mu \to \infty$, and $y_1(\mu) \sim \frac{s_2-\delta}{s_2}L_2\delta \cdot \frac{1}{\mu}$ as $\mu \to \infty$; moreover, $x_1(\mu) < 1$,*
$y_2(\mu) > \frac{s_2-\delta}{s_2}L_2$;

(ii) E_μ *is linearly stable for sufficiently large μ.*

Proof. (i) Since $s_2 > \delta$ and $1 - a_1\tilde{y}_1 > 0$, by Proposition 3.5 we directly get the result; (ii) Applying (i), then, we have

$$\lim_{\mu \to \infty} \frac{1}{\mu}\left(\frac{1}{a_1} + \frac{\delta s_2}{a_1} \cdot \frac{y_2^3(\mu)}{y_1^2(\mu)(s_2 y_2^2(\mu) + \delta L_2 y_1(\mu))}\right) = \lim_{\mu \to \infty} \frac{\delta s_2}{\mu a_1} \cdot \frac{y_2(\mu)}{s_2 y_1^2(\mu)}$$

$$= \lim_{\mu \to \infty} \frac{\delta}{\mu a_1} \frac{\frac{s_2-\delta}{s_2}L_2}{(\frac{s_2-\delta}{s_2}L_2\delta \cdot \frac{1}{\mu})^2} = \lim_{\mu \to \infty} \frac{s_2\mu}{a_1\delta L_2(s_2 - \delta)} = \infty.$$

According to Remark 3.4, E_μ is linearly stable for sufficiently large μ. $\qquad\square$

Remark 3.8. When $s_2 = \delta$, by Proposition 3.5 we can easily prove $E_\mu \to E_x$ and $y_2(\mu_n) \sim \mu_n y_1(\mu_n)\delta^{-1}$ as $\mu \to \infty$. Since

$$\lim_{n \to \infty} \frac{1}{\mu_n}\left(\frac{1}{a_1} + K(\mu_n)\right) = \lim_{n \to \infty} \frac{\delta s_2 y_2^3(\mu_n)}{\mu_n a_1 L_2 y_1^2(\mu_n)(s_2 y_2(\mu_n) + 2\delta y_1(\mu_n) - \delta y_2(\mu_n))}$$

$$= \lim_{n \to \infty} \frac{\delta s_2}{\mu_n a_1} \cdot \frac{y_2^3(\mu_n)}{2\delta L_2 y_1^3(\mu_n)} = \lim_{n \to \infty} \frac{s_2}{2a_1 L_2 \mu_n} \cdot \left(\frac{\mu_n}{\delta}\right)^3 = \infty,$$

it follows that E_μ is linearly stable for sufficiently large μ.

Case 2: $s_2 < \delta$, $1 - a_1\tilde{y}_1 > 0$.

In this case, E_y is linearly unstable, E_x is linearly stable if $\mu > \mu^*$ and linearly unstable if $\mu < \mu^*$.

Proposition 3.6. *E_x is globally stable as μ is sufficiently large. In particular, if $\mu^* < \mu^{**} = 1/a_1$, then E_x is globally stable for any $\mu > \mu^*$.*

Proof. Suppose by contradiction that there exists a sequence $\{\mu_n\}$ with $\mu_n \to \infty$ such that E_x is not globally stable. Then it corresponds to a sequence of positive equilibria $\{E_{\mu_n}\}$. By Proposition 3.5, we have $E_{\mu_n} \to E_x$ and $y_2(\mu_n) \sim$

$\mu_n y_1(\mu_n)\delta^{-1}$ as $n \to \infty$.

$$\lim_{n\to\infty} \frac{1}{\mu_n}\left(\frac{1}{a_1} + K(\mu_n)\right) = \lim_{n\to\infty} \frac{\delta s_2 y_2^3(\mu_n)}{\mu_n a_1 L_2 y_1^2(\mu_n)(s_2 y_2(\mu_n) + 2\delta y_1(\mu_n) - \delta y_2(\mu_n))}$$

$$\geq \lim_{n\to\infty} \frac{\delta s_2}{\mu_n a_1} \cdot \frac{y_2^3(\mu_n)}{2\delta L_2 y_1^3(\mu_n)} = \lim_{n\to\infty} \frac{s_2}{2a_1 L_2 \mu_n} \cdot \left(\frac{\mu_n}{\delta}\right)^3 = \infty.$$

Hence E_{μ_n} is linearly stable when n is sufficiently large. This contradicts to the linear stability of E_x when $\mu > \mu^*$. If $\mu^* < \mu^{**}$, suppose by contradiction that there exists $\mu_1 > \mu^{**}$ such that (2.2) has a positive equilibrium E_{μ_1} at $\mu = \mu_1$. So $F^{\mu_1}(E_{\mu_1}) = 0$. Since E_y is linearly unstable and $F^{\mu^{**}}(E_{\mu_1}) <_K 0$, $\psi_t^{\mu^{**}}(E_{\mu_1})$ decreases to a positive equilibrium, which contradicts to Corollary 3.2. $\square$

Corollary 3.4. *Suppose $\mu > \mu^*$. Then*

(i) *there exists $\mu' \geq \mu^*$ such that (2.2) has a positive equilibrium for any $\mu \in (\mu^*, \mu')$, and it has no positive equilibrium and E_x is globally stable for $\mu > \mu'$;*

(ii) *furthermore, if $\mu' > \mu^*$, then system (2.2) has exactly two positive equilibria for any $\mu \in (\mu^*, \mu')$.*

Proof. We only prove (ii) since the proof of (i) is obvious. First, we claim that (2.2) has a positive equilibrium $E_{\mu'}$ at $\mu = \mu'$. Indeed, by definition, we can find $\mu_n \in (\mu^*, \mu']$, corresponding to a positive equilibrium E_{μ_n}, such that $\mu_n \to \mu'$. Hence by passing to a subsequence we may assume that $E_{\mu_n} \to E_{\mu'}$. It is then easily seen that $E_{\mu'}$ is a nonnegative equilibrium of (2.2) at $\mu = \mu'$. We show that it is a positive equilibrium. Otherwise we have either $E_{\mu'} = E_o$, or E_x, or E_y. We can easily exclude the possibility that $E_{\mu'}$ can equal E_o or E_y according to the first equation of (3.1). Deducing by contradiction, we assume that $E_{\mu_n} \to E_x$. Dividing the second equation of (3.1) by y_1 and taking the limit $n \to \infty$ yields

$$\lim_{n\to\infty} y_2(\mu_n)/y_1(\mu_n) = 1 + (\mu' - 1)/\delta > 1 + (\mu^* - 1)/\delta = \delta/(\delta - s_2).$$

Similarly, according to the third equation of (3.1) we have

$$\lim_{n\to\infty} y_1(\mu_n)/y_2(\mu_n) = (\delta - s_2)/\delta,$$

a contradiction. Therefore $E_{\mu'}$ is a positive equilibrium of (2.2) at $\mu = \mu'$. Since for any $\mu \in (\mu^*, \mu')$, we have $F(E_{\mu'}) <_K 0$. There exists a positive equilibrium E_μ^1 for system (2.2) which is contained in $(E_y, E_{\mu'})_K$. Since E_x and E_μ^1 are both stable in $[E_\mu^1, E_x]_K$, there must exist a further positive equilibrium $E_\mu^2 \in (E_\mu^1, E_x)_K$. $\square$

Case 3: $s_2 < \delta$, $1 - a_1\tilde{y}_1 < 0$.

In this case, E_x is linearly stable for $\mu > \mu^*$ and is linearly unstable for $\mu < \mu^*$, E_y is linearly stable.

Proposition 3.7. *If $\mu > \mu^*$, then system (2.2) has a unique positive equilibrium E_μ. Moreover, E_μ is linearly unstable for sufficiently large μ and converges to E_b as $\mu \to \infty$. In addition, $\mu^* \geq \mu^{**}$.*

Proof. Since E_x and E_y are both linearly stable, we conclude that there exist a positive equilibrium for any $\mu > \mu^*$. Similar to Theorem 3.2, we can prove the uniqueness of the positive equilibrium. Arguing by contradiction, assume that there exists a sequence of positive equilibria $\{E_{\mu_n}\}$ with $E_{\mu_n} \to E_x$, where $\mu_n \to \infty$. We prove that the attracting region of E_x is nondecreasing in μ. In other words, for any $\mu_2 > \mu_1 > \mu^*$, if E_x attracts $z \in \mathbb{R}_+^3$ at $\mu = \mu_1$, so does $\mu = \mu_2$. Since

$F^{\mu_2}(z) \geq_K F^{\mu_1}(z)$ for any $z \in \mathbb{R}_+^3$, by Lemma 2.5, we have $\psi_t^{\mu_2}(z) \geq_K \psi_t^{\mu_1}(z)$ for any $t > 0$. Hence for sufficiently large n, E_{μ_n} are attracted by E_x, a contradiction. By Proposition 3.5, we conclude that $E_\mu \to E_b$ as $\mu \to \infty$. A simple calculation shows

$$\det J(E_\mu) = - r_1 x_1(\mu)[(y_1(\mu) + \delta \frac{y_2(\mu)}{y_1(\mu)})(s_2 L_2^{-1} y_2(\mu) + \delta \frac{y_1(\mu)}{y_2(\mu)})$$
$$- \delta^2 - a_1 \mu y_1(\mu)(s_2 L_2^{-1} y_2(\mu) + \delta \frac{y_1(\mu)}{y_2(\mu)})].$$

Thus $\det J(E_\mu) > 0$ as $\mu \to \infty$. Hence E_μ is linearly unstable for sufficiently large μ. $\qquad\square$

Let $\mathcal{A}_\mu$ be the attracting region of E_x, that is, $\mathcal{A}_\mu$ consists of all $z \in \mathbb{R}_+^3$ such that $\omega(z) = \{E_x\}$. In Proposition 3.7, we prove $\mathcal{A}_\mu$ is nondecreasing in μ. In fact, we have a much stronger result.

Proposition 3.8. *If $\mu > \mu^*$, then for any $(b_1, c_1, c_2) \in \mathbb{R}_+^3$ with $b_1 > 0$ and $c_1 + c_2 > 0$, there is some $\tilde\mu$ such that $(b_1, c_1, c_2) \in \mathcal{A}_\mu$ whenever $\mu > \tilde\mu$.*

Before proving this proposition, we need to analyze the following system:

$$\dot{y}_1 = y_1 - y_1^2 - \mu y_1 + \delta(y_2 - y_1),$$
$$\dot{y}_2 = s_2 y_2 - s_2 L_2^{-1} y_2^2 + \delta(y_1 - y_2). \tag{3.9}$$

Let $\kappa(\mu_0) = (\kappa_1(\mu_0), \kappa_2(\mu_0))$ be a positive equilibrium of system (3.9) when it has at least one at $\mu = \mu_0$. Let f^{μ_0} denote the vector field described by (3.9) at $\mu = \mu_0$, $\phi_t^{\mu_0}$ the corresponding flow. If the field is at μ, then we drop the μ and write f and ϕ_t.

Lemma 3.1. *(i) If $0 < s_2 < \delta$, then system (3.9) has no positive equilibrium and $(0,0)$ attracts all solutions for $\mu > \mu^*$, and has a unique positive equilibrium for $0 \leq \mu < \mu^*$ which attracts all non-trivial solutions and $\mu_1 > \mu_2 \geq 0$ implies $\kappa(\mu_2) \gg \kappa(\mu_1) \gg 0$; (ii) If $s_2 \geq \delta$, then (3.9) has a unique positive equilibrium for any $\mu \geq 0$ which attracts all non-trivial solutions and $\mu_1 > \mu_2 \geq 0$ implies $\kappa(\mu_2) \gg \kappa(\mu_1) \gg \kappa = (0, \frac{s_2 - \delta}{s_2} L_2)$. Furthermore, $\kappa(\mu) \to \kappa$ as $\mu \to \infty$.*

Proof. The Jacobian matrix of f at $(0,0)$ is

$$\begin{pmatrix} 1 - \mu - \delta & \delta \\ \delta & s_2 - \delta \end{pmatrix}.$$

By Proposition 3.1, we know that $(0,0)$ is linearly unstable if $\mu > 0$ and $s_2 \geq \delta$, or $\mu < \mu^*$ and $s_2 < \delta$, where $\mu^* = 1 + \frac{s_2 \delta}{\delta - s_2}$. Suppose for some μ there exists a positive equilibrium $\kappa(\mu)$, the Jacobian matrix of f at $\kappa(\mu)$ is

$$\begin{pmatrix} -\kappa_1(\mu) - \delta \frac{\kappa_2(\mu)}{\kappa_1(\mu)} & \delta \\ \delta & -s_2 L_2^{-1} \kappa_2(\mu) - \delta \frac{\kappa_1(\mu)}{\kappa_2(\mu)} \end{pmatrix}.$$

Hence $\kappa(\mu)$ is linearly stable when it exists. It is easy to show that (3.9) has finite positive equilibria. Therefore, (3.9) has no positive equilibrium if $(0,0)$ is linearly stable.

Obviously, (3.9) is cooperative and irreducible in $\text{Int}\mathbb{R}_+^2$. Therefore, ϕ_t is strongly monotone in $\text{Int}\mathbb{R}_+^2$. Then for all large positive k, $f(k1) \ll 0$, where $k1 = (k, k)$. Consequently $\phi_t(k1)$ decreasingly converges to an equilibrium for such k as $t \to \infty$.

Therefore for $\mu > 0$ such that $(0,0)$ is linearly unstable, there is a unique positive equilibrium $\kappa(\mu)$.

Let us prove the monotonicity of $\kappa(\mu)$ in μ. For any $\mu_1 > \mu_2 \geq 0$ such that $(0,0)$ is linearly unstable at $\mu = \mu_1$, we have $f^{\mu_1}(\kappa(\mu_2)) < f^{\mu_2}(\kappa(\mu_2)) = 0$. Hence, $\kappa(\mu_1) < \kappa(\mu_2)$. So strong monotonicity implies $\kappa(\mu_1) \ll \kappa(\mu_2)$. Since every trajectory is bounded and the system has a unique positive equilibrium, the proof of global stability is completed by Corollary 2.8 in [8]. We omit the proof of the last part which is similar to the arguments in Proposition 3.5. □

Proof of Proposition 3.8. Let $(x_1(t), y_1(t), y_2(t))$ be the unique solution of system (2.2) initiating in (b_1, c_1, c_2). Choose $\mu_1 > \mu^*$ and let $\alpha > \max\{1, L_2, c_1, c_2\}$. By Lemma 3.1, we know that the unique solution $y^0(t)$ of the following problem

$$\begin{cases} \dot{y}_1 = y_1 - y_1^2 - \mu_1 y_1 + \delta(y_2 - y_1), \\ \dot{y}_2 = s_2 y_2 - s_2 L_2^{-1} y_2^2 + \delta(y_1 - y_2), t > 1, \\ (y_1(1), y_2(1)) = \alpha 1, \text{ where } \alpha 1 = (\alpha, \alpha), \end{cases}$$

converges to $(0,0)$ as $t \to \infty$. Since $f^{\mu_1}(\alpha 1) \ll 0$, $y^0(t) \leq \alpha 1$ for all $t > 1$.

Define $\tilde{y}^0(t) = (\tilde{y}_1^0(t), \tilde{y}_2^0(t))$ such that it is continuous on $(0, \infty)$ and $\tilde{y}^0 = \alpha 1$ for $t \in [0, 1]$, $\tilde{y}^0 \geq y^0$ for $t \in [1, 2]$, $\tilde{y}^0 = y^0$ for $t > 2$. Let x^0 denote the unique solution of

$$\dot{x}_1 = r_1 x_1(1 - x_1 - a_1 \tilde{y}_1^0(t)), \ x_1(0) = b_1.$$

Since $\tilde{y}_1^0(t) \to 0$ as $t \to \infty$, it is easily shown that $x^0(t) > 0$ for $t > 0$ and $x^0(t) \to 1$ as $t \to \infty$. Let x^1 be the unique solution of the following auxiliary problem

$$\dot{x}_1 = r_1 x_1(1 - x_1 - a_1 \alpha), \ x_1(0) = b_1.$$

Obviously, we have $x^1(t) > 0$ for all $t > 0$. For any $\mu > \mu_1/x^1(1)$, we can find $t_\mu > 1$ such that $x^1(t) > \mu_1/\mu$ for $t \in [1, t_\mu]$. Moreover, we can choose $t_\mu \to \infty$ as $\mu \to \infty$.

By the definition of α, we know that $y(t) \ll \alpha 1$ for all $t > 0$ and hence $x_1(t) > x^1(t)$ for all $t > 0$. We claim that $y(t) \ll \tilde{y}^0(t)$ for $t \in [0, t_\mu]$. Indeed, since $y(t) \ll \alpha 1$ for all $t \geq 0$, we obviously have $y(t) \ll \tilde{y}^0(t)$ for $t \in [0, 1]$. For $t \in [1, t_\mu]$, from $x_1 > x^1 > \mu_1/\mu$ we deduce

$$\dot{y}_1 < y_1 - y_1^2 - \mu_1 y_1 + \delta(y_2 - y_1),$$
$$\dot{y}_2 = s_2 y_2 - s_2 L_2^{-1} y_2^2 + \delta(y_1 - y_2).$$

Since $y(1) \ll \alpha 1$, by Lemma 2.5 we have $y(t) \ll y^0(t) \leq \tilde{y}^0(t)$ for $t \in [1, t_\mu]$.

Let us fix $T > 2$ such that $x^0(t) > 1/2$ for $t \geq T$. Then choose $\tilde{\mu}$ large so that $t_\mu > T$ and $1/2 > \mu_1/\mu$ for $\mu \geq \tilde{\mu}$. We claim that whenever $\mu \geq \tilde{\mu}$, $y(t) \ll \tilde{y}^0(t)$ for all $t > 0$. Otherwise, for some fixed $\mu \geq \tilde{\mu}$, we can find $t^* > t_\mu$ such that

$$y(t) \ll \tilde{y}^0(t), \text{ for } t \in [0, t^*], \text{ and } y_1(t^*) = \tilde{y}_1^0(t^*) \text{ or } y_2(t^*) = \tilde{y}_2^0(t^*).$$

We show next that this is impossible. First, the above inequality implies $x_1(t) \geq x^0(t)$ for $t \in [0, t^*]$. Since $x^0(t) > 1/2$, for $t \in [T, t^*]$, by continuity, we can find $\gamma > 0$ small so that $x_1(t) > 1/2$ for $t \in [T, t^* + \gamma]$. It follows that, for $t \in [T, t^* + \gamma]$,

$$\dot{y}_1 < y_1 - y_1^2 - \mu_1 y_1 + \delta(y_2 - y_1),$$
$$\dot{y}_2 = s_2 y_2 - s_2 L_2^{-1} y_2^2 + \delta(y_1 - y_2).$$

Therefore, $y(T) \ll \tilde{y}^0(T) = y^0(T)$ implies that $y(t) \ll y^0(t)$ for $t \in [T, t^* + \gamma]$, which is a contradiction to the definition of t^*. Hence we have $0 \ll y(t) \ll \tilde{y}^0(t)$ for all

$t > 0$. Since $\tilde{y}^0(t) = y^0(t)$ for $t > 2$ and $y^0(t) \to 0$ as $t \to \infty$, we easily deduce $y(t) \to 0$ as $t \to \infty$. Since $y_1(t) < \tilde{y}_1^0(t)$ for all $t > 0$, we have $x^0(t) < x_1(t) \to 1$ as $t \to \infty$. $\square$

Case 4: $s_2 \geq \delta$, $1 - a_1\tilde{y}_1 < 0$.

In this case, we know E_x is linearly unstable and E_y is linearly stable. We first have the following theorem about the positive equilibria of (2.2).

Theorem 3.4. *(i) There exists some $\mu_* > 0$ such that (2.2) has no positive equilibrium for $\mu < \mu_*$, has at least one positive equilibria for $\mu = \mu_*$ and has exactly two positive equilibria for $\mu > \mu_*$; (ii) Given any $\rho \in (0,1)$, there exists $\hat{\mu}_\rho > 0$ such that for $\mu > \hat{\mu}_\rho$, (2.2) has a unique positive equilibrium E_μ with the property $x_1(\mu) \geq \rho$. Moreover, E_μ is linearly stable for sufficiently large μ and converges to E_a, while the other cluster of positive equilibria is linearly unstable for sufficiently large μ and converges to E_b as $\mu \to \infty$.*

Proof. (i) For any $\rho \in (0,1)$, consider the equations

$$y_1 - y_1^2 - \mu\rho y_1 + \delta(y_2 - y_1) = 0,$$
$$s_2 y_2 - s_2 L_2^{-1} y_2^2 + \delta(y_1 - y_2) = 0.$$

By Lemma 3.1, this system has a positive solution $\kappa(\rho\mu) = (\kappa_1(\rho\mu), \kappa_2(\rho\mu))$ and $\kappa(\rho\mu) \to \kappa$ as $\mu \to \infty$. Therefore, there exists some $\mu' > 0$ such that $\rho - \rho^2 - a_1\rho\kappa_1(\rho\mu) > 0$ for any $\mu \geq \mu'$. Now we fix $\mu \geq \mu'$. Then $F(E_\kappa) >_K 0$ and $E_\kappa \in (E_y, E_x)_K$, where $E_\kappa = (\rho, \kappa(\rho\mu'))$. Since E_x is linearly unstable, we conclude that $\psi_t(E_\kappa)$ increases to a positive equilibrium E_μ satisfying $E_\mu \in (E_\kappa, E_x)_K$.

Define $\mu_* = \inf\{\mu > 0 : (2.2)$ has a positive equilibrium$\}$. Then from Corollary 3.1, $\mu^{**} \leq \mu_* < \infty$, and there is $\mu_n \geq \mu_*$ with $\mu_n \to \mu_*$ as $n \to \infty$ such that for each μ_n system (2.2) has a positive equilibrium E_{μ_n}. We can choose a subsequence $\{E_{\mu_n}\}$ (still denoted by $\{E_{\mu_n}\}$) such that $\{E_{\mu_n}\} \to E_{\mu_*} = (x_1(\mu_*), y_1(\mu_*), y_2(\mu_*))$. Moreover, E_{μ_*} is a nonnegative equilibrium of (2.2) with $\mu = \mu_*$. We can easily prove that E_{μ_*} can be none of the three boundary equilibria E_o, E_x and E_y. Evidently, E_{μ_*} is not E_o or E_y. Similar to Corollary 3.4, suppose to the contrary, if $E_{\mu_*} = E_x$, then we have

$$\lim_{n\to\infty} y_2(\mu_n)/y_1(\mu_n) = 1 + (\mu_* - 1)/\delta < \infty \quad \text{and} \quad \lim_{n\to\infty} y_1(\mu_n)/y_2(\mu_n) = (\delta - s_2)/\delta \leq 0,$$

a contradiction. Hence E_{μ_*} is a positive equilibrium with $\mu = \mu_*$. Obviously, we have $F(E_{\mu_*}) >_K 0$ for any $\mu > \mu_*$. Therefore, similar to Corollary 3.4, there exist two positive equilibria $E_\mu^1 \in (E_{\mu_*}, E_x)_K$ and $E_\mu^2 \in (E_y, E_\mu^1)_K$ for any $\mu > \mu_*$.

(ii) We have shown in (i) that for any $\rho \in (0,1)$, we can find a positive equilibrium E_μ with $x_1(\mu) > \rho$ for sufficiently large μ. Since $f^{\mu x_1(\mu)}(y_1(\mu), y_2(\mu)) = 0$, $\kappa(\mu) \ll (y_1(\mu), y_2(\mu)) \ll \kappa(\mu\rho)$. By Lemma 3.1, $(y_1(\mu), y_2(\mu)) \to \kappa$ as $\mu \to \infty$. Therefore, E_μ converges to E_a as $\mu \to \infty$. Similar to Theorem 3.3, we can prove that E_μ is linearly stable for sufficiently large μ . Therefore, we prove the uniqueness and the other cluster of positive equilibria, denoted by $E'_\mu = (x'_1(\mu), y'_1(\mu), y'_2(\mu))$, satisfies $x'_1(\mu) \leq \rho$ for sufficiently large μ. Hence, E'_μ converges to E_b. Similar to Proposition 3.7, E'_μ is linearly unstable for sufficiently large μ. This complete the proof of theorem. $\square$

In the above discussion, we have proved that for any $\rho \in (0,1)$, there is $\hat{\mu}_\rho > 0$ such that for any $\mu > \hat{\mu}_\rho$, E_μ^1 is the unique positive equilibrium of (2.2) satisfying $x_1(\mu) \geq \rho$, and it is linearly stable. Let $\mathcal{B}_\mu$ be the attracting region of E_μ^1, we have

Corollary 3.5. *For any $\mu_1 > \mu_2 > \hat{\mu}_\rho$, $\mathcal{B}_{\mu_1} \supset \mathcal{B}_{\mu_2}$.*

Proof. Suppose $\mu_1 > \mu_2$. Then we have $F^{\mu_1}(E_{\mu_2}^1) >_K 0$. Therefore, $\psi_t^{\mu_1}(E_{\mu_2}^1)$ increases to $E_{\mu_1}^1$ as $t \to \infty$. Since $E_{\mu_1}^1$ is linearly stable, $\mathcal{B}_{\mu_1}$ contains a neighborhood U of $E_{\mu_2}^1$. Particularly, there is some $(h_1, g_1, g_2) \in \mathcal{B}_{\mu_1}$ with $0 < h_1 < x_1(\mu_2)$ and $(g_1, g_2) \gg (y_1(\mu_2), y_2(\mu_2))$. If $(b_1, c_1, c_2) \in \mathcal{B}_{\mu_2}$, then there exists some t_0 such that $(x_1^2(t_0), y_1^2(t_0), y_2^2(t_0)) \gg_K (h_1, g_1, g_2)$, where (x_1^2, y_1^2, y_2^2) is the solution of (2.2) initiating in (b_1, c_1, c_2) and $\mu = \mu_2$. Moreover, $(x_1^1(t_0), y_1^1(t_0), y_2^1(t_0)) \geq_K (x_1^2(t_0), y_1^2(t_0), y_2^2(t_0)) \gg_K (h_1, g_1, g_2)$, where (x_1^1, y_1^1, y_2^1) is the solution of (2.2) with initial value (b_1, c_1, c_2) and $\mu = \mu_1$. This implies that $(x_1^1(t_0), y_1^1(t_0), y_2^1(t_0)) \in \mathcal{B}_{\mu_1}$. It follows that $(b_1, c_1, c_2) \in \mathcal{B}_{\mu_1}$. Thus the corollary is proved. $\square$

Furthermore, we have the following result:

Proposition 3.9. *For any $(b_1, c_1, c_2) \in \mathbb{R}_+^3$ with $b_1 > 0$ and $c_1 + c_2 > 0$, there is some $\tilde{\mu}$ such that $(b_1, c_1, c_2) \in \mathcal{B}_\mu$ whenever $\mu > \tilde{\mu}$.*

To prove this proposition, we need some preparations. Recall that $\kappa(\mu)$ is the unique positive equilibrium of (3.9) and $\phi_t(c)$ denotes the unique solution of (3.9) initiating in $y(0) = c$, where $c = (c_1, c_2) > 0$.

Lemma 3.2. *Given any positive constants μ, θ satisfying $\mu > \theta$, there is a constant $T_\mu = T_\mu(\theta)$ such that $\phi_t(c) \leq \kappa(\theta)$ for any $t \geq T_\mu$. Moreover, T_μ is nonincreasing in μ.*

Proof. For any $\mu > \theta$, by Lemma 3.1, $\kappa(\mu) \ll \kappa(\theta)$. Still by Lemma 3.1, $\phi_t(c) \to \kappa(\mu)$ as $t \to \infty$. Hence there is some T_μ such that $\phi_t(c) \leq \kappa(\theta)$ for any $t > T_\mu$. On the other hand, $f^{\mu_1}(u) \leq f^{\mu_2}(u)$ for any $\mu_1 > \mu_2$ and $u \in \mathbb{R}_+^2$. Hence $\phi_t^{\mu_1}(c) \leq \phi_t^{\mu_2}(c)$. This implies that T_μ can be chosen to be nonincreasing in μ. $\square$

Since the equilibrium $\kappa(\mu)$ of (3.9) converges to κ as $\mu \to \infty$, given any $\epsilon > 0$, there is some $C > 0$ such that for any $\mu \geq C$, $\kappa_1(\mu) \leq \epsilon$. Fixing $\epsilon \in (0, 1/a_1)$, we have the following lemma.

Lemma 3.3. *Let $(x_1(t), y_1(t), y_2(t))$ be the solution of (2.2) with $x_1(0) = C/\mu$, $y(0) = \kappa(C)$. Then there is some μ_1^* such that for any $\mu > \mu_1^*$, $\lim\limits_{t \to \infty} (x_1(t), y_1(t), y_2(t)) = E_\mu^1$.*

Proof. Due to our choice of ϵ, it is easy to see that for sufficiently large $\mu > C$, $1 - \frac{C}{\mu} - a_1\epsilon > 0$ and therefore $F(C/\mu, \kappa(C)) >_K 0$. Hence $\psi_t(C/\mu, \kappa(C))$ increases to a positive equilibrium $E_\mu = (x_1(\mu), y_1(\mu), y_2(\mu))$. Moreover, $y_1(t) \leq y_1(0) = \kappa_1(C) \leq \epsilon$. It follows that $x_1(\mu) \geq 1 - a_1\epsilon > 0$. By Theorem 3.4, we know that $E_\mu = E_\mu^1$ provided that μ is large enough. This completes the proof. $\square$

Proof of Proposition 3.9. By Lemma 3.3, we only need to prove that for any $(b_1, c_1, c_2) \in \mathbb{R}_+^3$ satisfying $b_1 > 0$ and $c_1 + c_2 > 0$, there is a constant $\tilde{\mu}$ such that for any $\mu > \tilde{\mu}$, there is some positive number M such that $y(M) \leq \kappa(C)$ and $x_1(M) \geq C/\mu$, where (x_1, y_1, y_2) is the solution of (2.2) with initial value (b_1, c_1, c_2).

Choose a constant α such that $\alpha > \max\{1, L_2, c_1, c_2\}$. Obviously, $y(t) \ll \alpha \mathbf{1}$ for all $t > 0$. It follows that

$$\dot{x}_1 = r_1 x_1(1 - x_1 - a_1 y_1) > r_1 x_1(1 - x_1 - a_1\alpha).$$

Hence $x_1 > x^1$ for all $t > 0$, where x^1 denotes the unique solution of

$$\dot{x}_1 = r_1 x_1 (1 - x_1 - a_1 \alpha), \ x_1(0) = b_1.$$

Obviously, we have $x^1(t) > 0$ for $t > 0$. Therefore, for each $\mu > (x^1(1))^{-2}$, there is a positive number $\tau_\mu > 1$ such that $x^1(t) \geq 1/\sqrt{\mu}$ for $t \in [1, \tau_\mu]$, and we may choose τ_μ such that $\tau_\mu \to \infty$ as $\mu \to \infty$.

Let μ_1^* satisfy $\sqrt{\mu_1^*} > C$. Then by Lemma 3.2 there is some $M > 1$ such that for any $\mu > \mu_1^*$, $\phi_{M-1}^{\sqrt{\mu}}(\alpha 1) \leq \kappa(C)$. We then choose $\tilde{\mu}$ sufficiently large such that $\tau_\mu > M$ and $1/\sqrt{\mu} > C/\mu$ whenever $\mu \geq \tilde{\mu}$. Then for any $\mu \geq \tilde{\mu}$ and $t \in [1, M]$, $x_1(t) > x^1(t) \geq 1/\sqrt{\mu}$ and $y(t)$ satisfies

$$\dot{y}_1 - \delta(y_2 - y_1) = y_1 - y_1^2 - \mu x_1 y_1 < y_1 - y_1^2 - \sqrt{\mu} y_1,$$
$$\dot{y}_2 - \delta(y_1 - y_2) = s_2 y_2 - s_2 L_2^{-1} y_2^2.$$

It follows that $y(t) \leq \phi_{t-1}^{\sqrt{\mu}}(\alpha 1)$ for $t \in [1, M]$. In particular, $y(M) \leq \phi_{M-1}^{\sqrt{\mu}}(\alpha 1) \leq \kappa(C)$. Moreover, we have $x_1(M) \geq 1/\sqrt{\mu} > C/\mu$. The proof is complete. $\square$

In [1], similar conclusions to Proposition 3.8 and Proposition 3.9 were obtained for a reaction-diffusion model. From Theorem 3.3, Remark 3.8 and Proposition 3.6, 3.8, 3.9, we summarize a global result:

Theorem 3.5. *Suppose both E_x and E_y are hyperbolic. Then for any initial value $(b_1, c_1, c_2) \in \mathbb{R}_+^3$ with $b_1 > 0$ and $c_1 + c_2 > 0$, $S_\mu \to (1, 0, \max\{0, \frac{s_2 - \delta}{s_2} L_2\})$ as $\mu \to \infty$, where $\{S_\mu\} = \omega((b_1, c_1, c_2))$.*

The above result shows that when competitor X is much stronger than Y, species X tends to reach its carrying capacity, while species Y in patch 1 tends to extinction and in the refuge-patch 2, tends to a value less than its carrying capacity in patch 2. Hence the preceding analysis tells us that when the effect of species X on species Y is strong enough, it is advisable to protect more species Y by restricting the diffusion of Y.

Remark 3.9. Let δ_1 and δ_2 be the diffusion rates of species Y from patch 1 to patch 2 (and vice versa). If $\delta_1 \neq \delta_2$, then all corresponding results for system (2.2) still hold by the following transformations:

$$\bar{y}_2 = \frac{\delta_2}{\delta_1} y_2, \ \bar{L}_2 = \frac{\delta_2}{\delta_1} L_2, \ k = \frac{\delta_2}{\delta_1}, \ \bar{s}_2 = \frac{\delta_1}{\delta_2} s_2,$$

which yield

$$\frac{dx_1}{dt} = r_1 x_1 (1 - x_1 - a_1 y_1),$$
$$\frac{dy_1}{dt} = y_1 (1 - y_1 - \mu x_1) + \delta_1 (\bar{y}_2 - y_1),$$
$$\frac{d\bar{y}_2}{dt} = k(\bar{s}_2 \bar{y}_2 (1 - \bar{L}_2^{-1} \bar{y}_2) + \delta_1 (y_1 - \bar{y}_2)).$$

Rewriting the differential equations above, we have

$$\frac{dx_1}{dt} = r_1 x_1 (1 - x_1 - a_1 y_1),$$
$$\frac{dy_1}{dt} = y_1 (1 - y_1 - \mu x_1) + \delta(y_2 - y_1),$$
$$\frac{dy_2}{dt} = k(s_2 y_2 (1 - L_2^{-1} y_2) + \delta(y_1 - y_2)).$$

4. **Discussion.** In this paper we consider a two-competitor system with a refuge for one species. Based on the specific properties of the system and the method of monotone dynamical systems, we have solved Takeuchi and Lu's problem. We have also shown that establishing a refuge for the weak competitor may be ineffective when the effect of species X on species Y is strong enough. Figure 1 gives us a clear illustration of the main results of this paper.
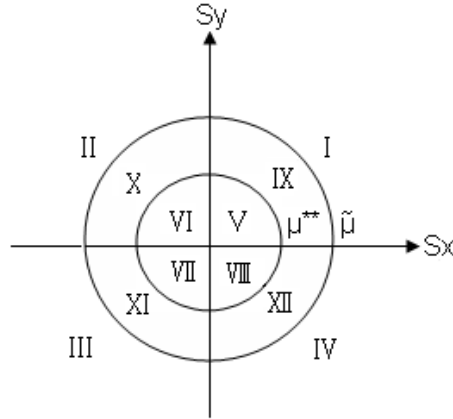

FIGURE 1. Denote $s_x = s(J(E_x))$ and $s_y = s(J(E_y))$. The small disc stands for $\mu \leq \mu^{**}$ and the area out of the big disc stands for $\mu > \tilde{\mu}$, i.e., for sufficiently large μ, and the annulus stands for medium μ. We divide $s_x - s_y$ plane into twelve parts. In part I, there exists a unique positive equilibrium which is linearly stable and converges to E_a as $\mu \to \infty$; in part II, E_x is globally stable; in part III, there exists a unique positive equilibrium which is linearly unstable and converges to E_b as $\mu \to \infty$, the attracting region of E_x can include any given interior point; in part IV, there exists exactly two positive equilibria, one is linearly stable and converges to E_a as $\mu \to \infty$, the other is linearly unstable and converges to E_b as $\mu \to \infty$, the attracting region of the linearly stable equilibrium can include any given interior point; in part V, there exists a unique positive equilibrium which is linearly stable; in part VI, E_x is globally stable or this part does not exist; in part VII, this part does not exist; in part VIII, E_y is globally stable; in part IX, there exists a unique positive equilibrium; in part XI, there exists a unique positive equilibrium.

In part X and XII, we have two questions: (1) In part X, whether the system can have a positive equilibrium, i.e., whether E_x must be globally stable. (2) In part XII, what the exact number of positive equilibria at $\mu = \mu_*$. By Theorem 3.2 and strong monotonicity of ψ_t in $\text{Int}\mathbb{R}^3_+$, obviously, we have the following.

Corollary 4.1. *Suppose E_y is linearly unstable. Then for any $\mu_1 > \mu_2 > 0$ such that E_x is linearly unstable at $\mu = \mu_1$, $E_{\mu_1} \gg_K E_{\mu_2} \gg_K (1-a_1\tilde{y}_1, \tilde{y}_1, \tilde{y}_2)$. Moreover, $s(J(E_\mu)) \leq 0$ for any $\mu > 0$ such that E_x is linearly unstable. In particular, $s(J(E_\mu)) < 0$ as μ is sufficiently small or large.*

In Corollary 4.1, we do not know whether the unique positive equilibrium is linearly stable in general. Suppose E_x and E_y are both linearly stable. We know the unique positive equilibrium E_μ satisfying $s(J(E_\mu)) > 0$ for sufficiently large μ. However, similarly, we do not know whether E_μ is linearly unstable in general.

We close with a modification of system (2.2) assuming that the diffusion rates of species Y between the two patches, denoted by $\delta_i(\mu)$ $(i = 1, 2)$, are dependent of the effects of species X on species Y. It is probably a more reasonable assumption, because many species have the ability to choose better habitats. It is natural to suppose that $\delta_i(\mu)$ $(i = 1, 2)$ are nonnegative and bounded for $\mu \in (0, \infty)$.
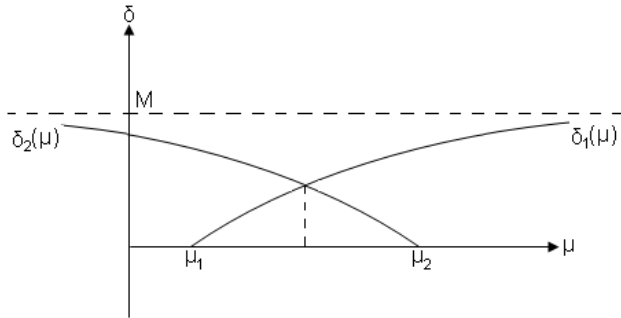

FIGURE 2. $0 < \delta_1(\mu) < M$ as $\mu > \mu_1$, $\delta_2(\mu) = 0$ as $\mu > \mu_2$, where $\mu_i (i = 1, 2)$ and M are all positive constants.

Here we consider a special case: $0 < \delta_1(\mu) < \infty$ and $\delta_2(\mu) \equiv 0$ for sufficiently large μ (see Figure 2 for example), which can be described as follows:

$$\frac{dx_1}{dt} = r_1 x_1(1 - x_1 - a_1 y_1),$$
$$\frac{dy_1}{dt} = y_1(1 - y_1 - \mu x_1) - \delta_1(\mu)y_1, \qquad (4.1)$$
$$\frac{dy_2}{dt} = s_2 y_2(1 - L_2^{-1} y_2) + \delta_1(\mu)y_1.$$

Similar to the proof of Proposition 3.8, for system (4.1) we have the following result.

Theorem 4.1. *For any initial value* $(b_1, c_1, c_2) \in \mathbb{R}_+^3$ *with* $b_1 > 0$ *and* $c_1 + c_2 > 0$, *then* $\omega((b_1, c_1, c_2)) = \{E_{x_1 y_2}\}$ *for sufficiently large* μ, *where* $E_{x_1 y_2} = (1, 0, L_2)$.

Comparing Theorem 3.5 with the above result, we can see that the latter is more biologically accepted.

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