Thursday, September 29, 2022 4:31 PM

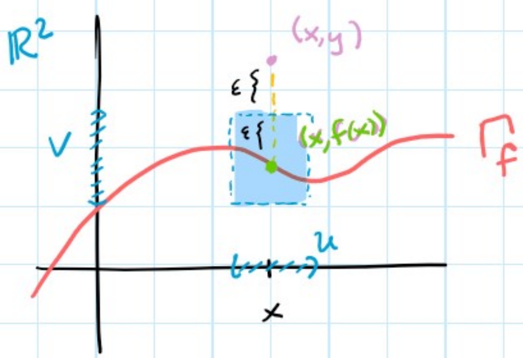
#6 (practice for EXAM 1)

$f: \mathbb{R} \to \mathbb{R}$ continuous

Show $\Gamma_f = \{ (x, f(x)) \mid x \in \mathbb{R} \} \subset \mathbb{R}^2$ is closed.

Proof #1: We show $\mathbb{R}^2 \setminus \Gamma_f$ is open.
Suffices to show that for each $(x,y) \in \mathbb{R}^2 - \Gamma_f$
$\exists$ open nbhd W of (x,y) s.t. $W \subset \mathbb{R}^2 \setminus \Gamma_f$.

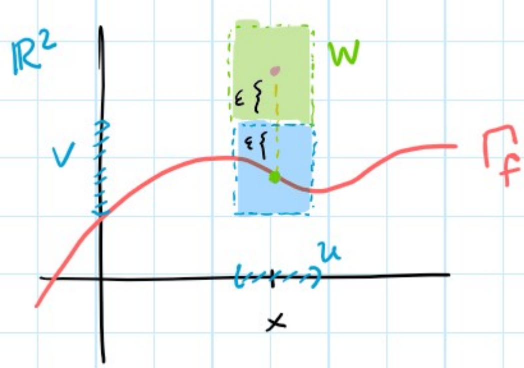
So let $(x,y) \in \mathbb{R}^2 \setminus \Gamma_f$. Then $y \neq f(x)$.



Let $\boxed{\varepsilon = |f(x) - y|/2}$

f cont. $\Rightarrow \exists \delta > 0$ s.t.

$$f(\underbrace{(x-\delta, x+\delta)}_{U}) \subset \underbrace{(f(x) - \varepsilon,\ f(x) + \varepsilon)}_{V}$$



Consider $W = (x-\delta, x+\delta) \times (y-\varepsilon, y+\varepsilon)$.
Then W is a nbhd of (x,y).
We must check that $W \subset \mathbb{R}^2 \setminus \Gamma_f$.

Let $(x', y') \in W$. Must show $y' \neq f(x')$.

We compute (triangle inequality 2x):

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$|y-y'|<\varepsilon$ since $(x',y')\in W$

$$|y'-f(x')| \geq |y-f(x')| - |y'-y| > |y-f(x')| - \varepsilon$$

$\varepsilon = \frac{|y-f(x)|}{2}$

$$\geq |y-f(x)| - |f(x)-f(x')| - \varepsilon$$

$$= 2\varepsilon - |f(x)-f(x')| - \varepsilon = \varepsilon - |f(x)-f(x')|$$

$|f(x)-f(x')|<\varepsilon$ since $|x-x'|<\delta$.

$$> \varepsilon - \varepsilon = 0$$

Finally, $|y'-f(x')| > 0 \Rightarrow y' \neq f(x')$, as desired. ∎

Proof #2:

Consider $F: \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x,y) = f(x) - y$.

Note $F = g \circ (\mathrm{id}_{\mathbb{R}} \times f)$ where $g: \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = x - y$.

It's a composition of continuous functions, hence continuous.

Now $F^{-1}(\{0\}) = \{(x,y) \mid F(x,y) = 0\}$

$$= \{(x,y) \mid f(x) - y = 0\}$$

$$= \{(x, f(x))\} = \Gamma_f.$$

F cont. & $\{0\} \subset \mathbb{R}$ closed $\Rightarrow F^{-1}(\{0\}) = \Gamma_f$ is closed. ∎