So, are the examples where $(X,<)$ is an ordered set, $A \subset X$, and the two topologies (i), (ii) are the same?

Defn. Let $(X,<)$ be an ordered set. A subset $A \subset X$ is convex if for all $a,b \in A$ with $a<b$ we have $(a,b) \subset A$.

In this definition, $(a,b) = \{x \in X \mid a<x<b\}$.

Proposition. Let $(X,<)$ be an ordered set and $A \subset X$ a convex subset. Then topologies (i), (ii) in this case are the same.

Metric Spaces

Defn. A metric space (X,d) is a set X together with a function $d: X \times X \to \mathbb{R}$, a metric, which satisfies, for all $x,y,z \in X$:

(1) $d(x,y) \geq 0$ and $d(x,y) = 0 \iff x=y$

(2) $d(x,y) = d(y,x)$

(3) $d(x,y) + d(y,z) \geq d(x,z)$ (triangle inequality)

Given a metric space (X,d) define for any $x \in X$, $\varepsilon > 0$:

$$B_d(x,\varepsilon) = \{y \in X \mid d(x,y) < \varepsilon\}$$

This is the ε-ball centered at $x \in X$. Now set

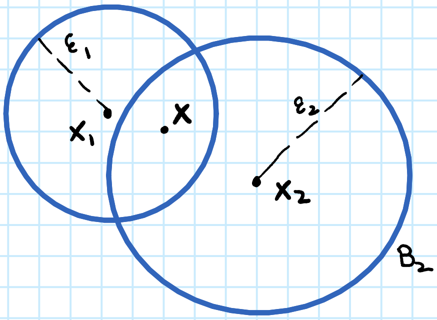
$$\mathcal{B} = \{B_d(x,\varepsilon) \mid x \in X,\ \varepsilon \in \mathbb{R}_+\}$$

We claim that $\mathcal{B}$ is a basis for a topology on X.
Let's verify the two conditions that a basis must satisfy.

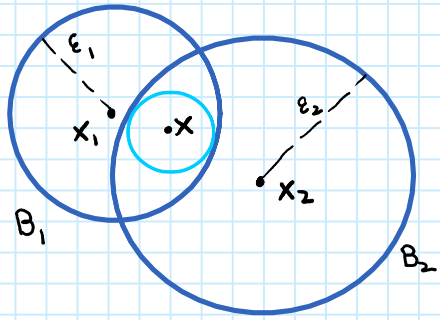
(1) For each $x \in X$ there's some $B \in \mathcal{B}$ s.t. $x \in B$.
We can take $B = B_d(x,\varepsilon)$ for any $\varepsilon > 0$. ✓

(2) For $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there's some $B_3 \in \mathcal{B}$ that satisfies $x \in B_3 \subset B_1 \cap B_2$.

As $B_1, B_2 \in \mathcal{B}$ we can write $B_1 = B_d(x_1, \varepsilon_1)$, $B_2 = B_d(x_2, \varepsilon_2)$ for some $x_1, x_2 \in X$ and $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_+$. Intuitively we have a picture as follows:



We would like to take a small ε-ball around x as our B_3, i.e. $B_3 = B_d(x, \varepsilon)$.



Of course $x \in B_3 = B_d(x,\varepsilon)$, regardless of our choice of $\varepsilon > 0$.
Can we choose ε so that $B_3 \subset B_1 \cap B_2$?

Note $z \in B_3 = B_d(x,\varepsilon) \Leftrightarrow d(x,z) < \varepsilon$ while $z \in B_i \Leftrightarrow d(z, x_i) < \varepsilon_i$ $(i = 1,2)$

So $B_3 \subset B_1 \cap B_2$ amounts to: $d(z,x) < \varepsilon \Rightarrow d(z,x_1) < \varepsilon_1, \; d(z,x_2) < \varepsilon_2$. $(*)$

Suppose $d(x,z) < \varepsilon$. By the triangle inequality, we have

$$d(z,x_1) \le d(z,x) + d(x,x_1) < \varepsilon + d(x,x_1)$$

and thus if $\varepsilon \le \varepsilon_1 - d(x,x_1)$, then $d(z,x_1) < \varepsilon_1$.
Similarly, if $\varepsilon \le \varepsilon_2 - d(x,x_2)$, then $d(z,x) < \varepsilon_2$.

Thus if $\varepsilon = \min\{\varepsilon_1 - d(x,x_1), \varepsilon_2 - d(x,x_2)\}$ then $(*)$ holds, i.e. $B_3 \subset B_1 \cap B_2$.
We see that condition (2) is satisfied. ✓

The topology on X defined by the basis $\mathcal{B}$ is called the <u>metric topology</u>.

<u>Note</u>: Given any ε-ball $B_d(x,\varepsilon)$ and $y \in B_d(x,\varepsilon)$, there exists some $\delta > 0$ s.t. $B_d(y,\delta) \subset B_d(x,\varepsilon)$. The argument is contained above.

From this note and the definition of a basis generating a topology, we have the following characterization of open sets:

$\mathcal{U} \subset X$ is open in metric topology $\Longleftrightarrow$ for all $x \in \mathcal{U}$ there is some $\varepsilon > 0$ s.t. $B_d(x,\varepsilon) \subset \mathcal{U}$.

Metric spaces continued

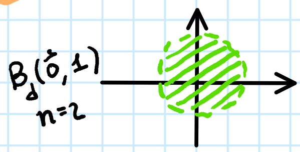
Recall that given a metric space (X, d), X inherits a metric topology with basis given by the ε-balls

$$B_d(x,\varepsilon) = \{ y \in X \mid d(x,y) < \varepsilon \}$$

as x varies over X and ε over $\mathbb{R}_+$.

Examples

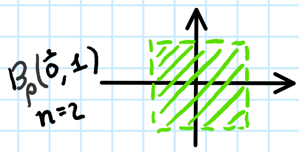
1. $\mathbb{R}^n$ has the "standard" Euclidean metric: for $\vec{x} = (x_1, \dots, x_n)$, $\vec{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$



$$d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\| = \left(\sum_{i=1}^{n}(x_i-y_i)^2\right)^{1/2}$$

also called d_{ℓ^2} "ℓ^2-metric"

2. $\mathbb{R}^n$ also has the "square metric" ρ:



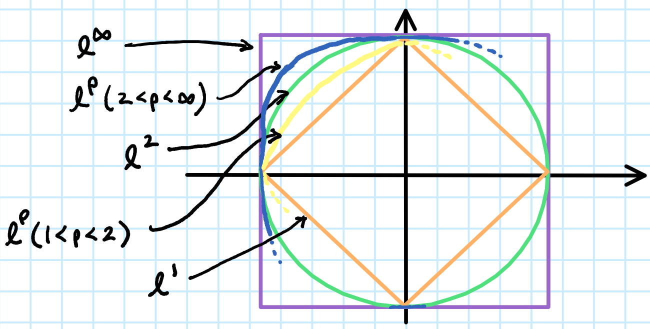
$$\rho(\vec{x},\vec{y}) = \max_{1 \le i \le n} \{ |x_i - y_i| \}$$

also called d_{ℓ^∞} "ℓ^∞-metric"

3. Generalizing the previous examples, $\mathbb{R}^n$ has the "ℓ^p-metric" for any $p \ge 1$:

$$d_{\ell^p}(\vec{x},\vec{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}$$

Here are the unit balls centered at $\vec{0}$ in the case $n=2$:



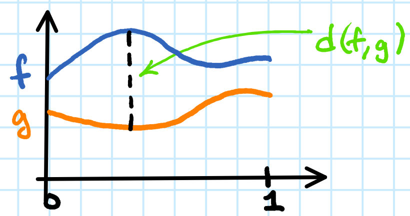
drawn is the boundary of each unit ball $B_{d_{\ell^p}}(\vec{0},1)$

4. X any set. Define $d: X \times X \to \mathbb{R}$ by $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$. Then d is a metric. Note $B_d(x,1) = \{x\}$. Thus the metric topology is discrete.

5. Consider $X = \{ f: [0,1] \to \mathbb{R} \mid f \text{ continuous} \}$. Define for $f, g \in X$:

$$d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$$

Then d makes X into a metric space.



Proposition X a set with two metrics d, d' and associated metric topologies $\tau_d, \tau_{d'}$. Then $\tau_d \subset \tau_{d'} \iff$ for all $x \in X$, $\varepsilon > 0$ there is a $\delta > 0$ s.t.

$$B_{d'}(x,\delta) \subset B_d(x,\varepsilon)$$

Proof follows easily from earlier results.

Sometimes different metrics on a set lead to the same topology.

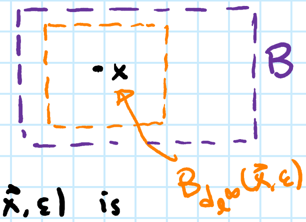
Proposition The metric topologies τ_{ℓ^p} on $\mathbb{R}^n$ for all ℓ^p-metrics d_{ℓ^p} $(1 \le p \le \infty)$ all agree with the standard (product) topology τ.

Proof: Let $\mathcal{B} = \{ (a_1,b_1) \times \cdots \times (a_n,b_n) \}$ be the basis for τ, and

$\mathcal{B}_{\ell^p} = \{ B_{d_{\ell^p}}(\vec{x},\varepsilon) \mid \vec{x} \in \mathbb{R}^n, \varepsilon > 0 \}$ the basis for τ_{ℓ^p}.

First we show $\tau_{\ell^\infty} = \tau$.

Let $B = (a_1,b_1) \times \cdots \times (a_n,b_n) \in \mathcal{B}$, $\vec{x} \in B$. Choose $\varepsilon > 0$ s.t. $a_i < x_i - \varepsilon < x + \varepsilon < b_i$ for $i = 1, \ldots, n$. Then we have



$$B_{d_{\ell^\infty}}(\vec{x},\varepsilon) = (x_1-\varepsilon, x_1+\varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon) \subset B$$

This shows $\tau \subset \tau_{\ell^\infty}$. On the other hand, each $B_{d_{\ell^\infty}}(\vec{x},\varepsilon)$ is in fact a member of $\mathcal{B}$ and thus $\mathcal{B}_{\ell^\infty} \subset \mathcal{B} \Rightarrow \tau_{\ell^\infty} \subset \tau$.

Thus $\tau_{\ell^\infty} = \tau$.

Now let $1 \le p < \infty$. Then we have for any $\vec{x}, \vec{y} \in \mathbb{R}^n$:

$$d_{\ell^\infty}(\vec{x},\vec{y}) = \max_{1 \le i \le n} \{ |x_i - y_i| \} = \max_{1 \le i \le n} \{ (|x_i - y_i|^p)^{1/p} \} \le \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{1/p}$$

$$= d_{\ell^p}(\vec{x},\vec{y}) \le \left(n \cdot \max_{1 \le i \le n} \{ |x_i - y_i|^p \} \right)^{1/p} = n^{1/p} \cdot \max_{1 \le i \le n} \{ (|x_i - y_i|^p)^{1/p} \}$$

$$= n^{1/p} \cdot d_{\ell^\infty}(\vec{x},\vec{y})$$

$$\Rightarrow d_{\ell^\infty}(\vec{x},\vec{y}) \le d_{\ell^p}(\vec{x},\vec{y}) \le n^{1/p} \cdot d_{\ell^\infty}(\vec{x},\vec{y}).$$

$B_{d_{\ell^p}}(\vec{x},\varepsilon) \subset B_{d_{\ell^\infty}}(\vec{x},\varepsilon)$ $\qquad$ $B_{d_{\ell^\infty}}(\vec{x}, \varepsilon / n^{1/p}) \subset B_{d_{\ell^p}}(\vec{x},\varepsilon)$

Then $\tau_{\ell^p} = \tau_{\ell^\infty}$ by the previous proposition. ∎