Thursday, September 22, 2022 12:28 PM

Last time: $\mathbb{R} \cong \mathbb{R}_\ell$? No
$\mathbb{R} \cong \mathbb{R}_{fc}$? No
$\mathbb{R} \cong \mathbb{R}^2$? No
$\mathbb{R} \cong S^1$? No

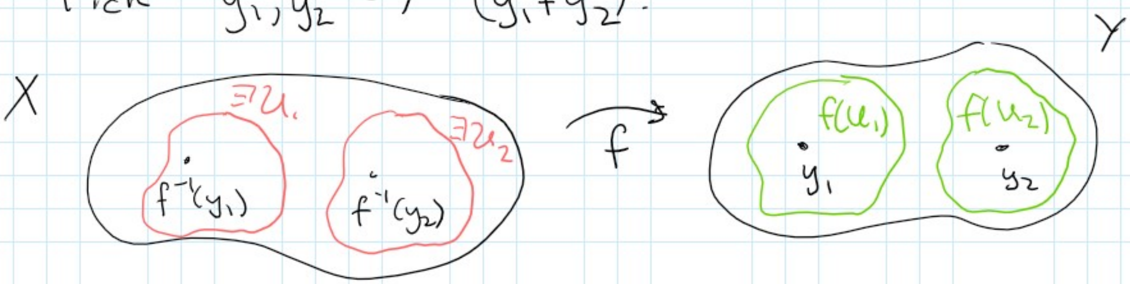
When we expect that the answer is "No" how to prove it?

Defn: A property P of spaces is a topological property if X has property P and $X \cong Y$ then Y has property P.

Example: The property of a space being Hausdorff (or not). is a topological property.

Proof: Suppose X is Hausdorff, and $X \cong Y$ by some homeomorphism $f: X \to Y$. WTS: Y is Hausdorff.

Pick $y_1, y_2 \in Y$ $(y_1 \neq y_2)$.



Note $f^{-1}(y_1) \neq f^{-1}(y_2)$ since f is a bijection.

X Hausdorff $\Rightarrow$ $\exists$ nbhds $\mathcal{U}_1$ of $f^{-1}(y_1)$, $\mathcal{U}_2$ of $f^{-1}(y_2)$ that satisfy $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$.

Then $f(\mathcal{U}_1)$, $f(\mathcal{U}_2)$ are open nbhds of y_1, y_2 and $f(\mathcal{U}_1) \cap f(\mathcal{U}_2) = \emptyset$ since $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$ and f is a bijection.

Note: if f is a homeomorphism $f: X \to Y$

and f is a bijection.

(Note: if f is a homeomorphism $f: X \to Y$ then f maps open sets in X to open sets in Y. Note $f^{-1}: Y \to X$ is continous and if $U \subset X$ is open then $(f^{-1})^{-1}(U) = f(U) \subset Y$ is open.) ◪

Cor: $\mathbb{R} \not\cong \mathbb{R}_{fc}$.

Pf: $\mathbb{R}$ is Hausdorff, but $\mathbb{R}_{fc}$ is not. ◪

Rules for constructing continuous functions

a) If $f: X \to Y$ has $f(x) = y_0$ for all $x \in X$ (f is constant) then f is continuous.

b) If $A \subset X$ w/ subspace topology then the inclusion $j: A \to X$ ($j(a) = a\ \forall a \in A$) is continuous.

c) If $f: X \to Y$, $g: Y \to Z$ are continuous then $g \circ f: X \to Z$ is continuous.

d) If $f: X \to Y$ is continuous and $A \subset X$ then $\underbrace{f|_A}_{\text{restriction}} : A \to Y$ is continuous.

e) If $f: X \to Y$ is cont. and $f(X) \subset Z \subset Y$ then $f': X \to Z$ (restricting the range) is continuous. Similarly, if $f(X) \subset Y \subset Z$ then $f'': X \to Z$ (expanding the range) is continuous.

f) Suppose $X = \bigcup U_i$ where each $U_i \subset X$ open

f) Suppose $X = \bigcup_{i \in I} \mathcal{U}_i$ where each $\mathcal{U}_i \subset X$ open.

Then $f : X \to Y$ is continuous if $f|_{\mathcal{U}_i} : \mathcal{U}_i \to Y$ is cont.

<u>Proof</u> of a) done last lecture.

c): Suppose $f: X \to Y$, $g : Y \to Z$ are continuous.

WTS $g \circ f : X \to Z$ is continuous.

Let $W \subset Z$ be open.

g cont $\Rightarrow$ $g^{-1}(W) \subset Y$ is open.

f cont $\Rightarrow$ $f^{-1}(g^{-1}(W)) \subset X$ is open.

$f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$.

Thus $g \circ f$ is continuous. //

The pasting lemma X top. space, $X = A \cup B$ where $A, B \subset X$ are closed sets.

Let $f: A \to Y$, $g : B \to Y$ be continuous maps s.t.

$$f|_{A \cap B} = g|_{A \cap B} \quad (*)$$

(i.e. for all $x \in A \cap B$, $f(x) = g(x)$). Then

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

$\quad \left\{ g(x) \quad x \in B \right.$

defines a continuous map $h : X \to Y$.

Pf. That h is a well-defined function $X \to Y$ follows from $(*)$.

Let's show that h is continuous.

Let $C \subset Y$ be closed.

Then
$$h^{-1}(C) = \{ x \in X \mid h(x) \in C \}$$
$$= \{ x \in A \mid f(x) \in C \} \cup \{ x \in B \mid g(x) \in C \}$$
$$= f^{-1}(C) \cup g^{-1}(C).$$

f cont $\Rightarrow f^{-1}(C) \subset A$ closed.

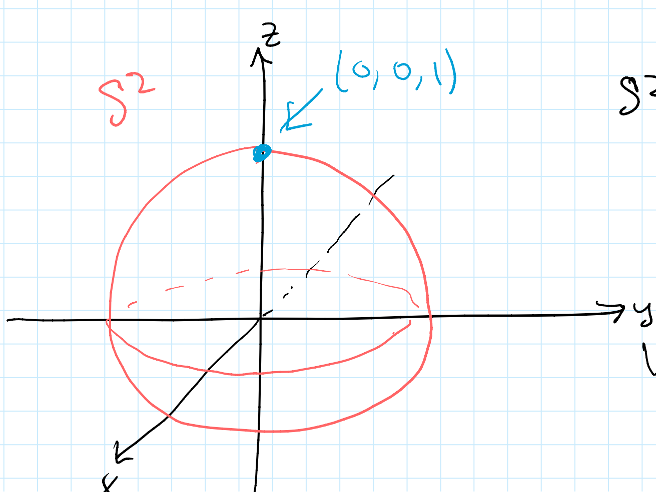
$\Rightarrow f^{-1}(C) = A \cap D$ where $D \subset X$ closed

$\Rightarrow f^{-1}(C)$ is closed in X b/c intersection of closed sets.

Similarly $g^{-1}(C)$ is closed

$\Rightarrow h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ closed in X. ∎

Let's see the pasting lemma in action.

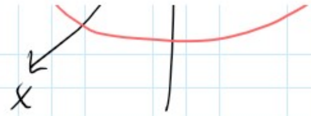


$S^2 = \{ (x, y, z) \mid x^2 + y^2 + z^2 = 1 \} \subset \mathbb{R}^3$ subspace topology.

Let $X = S^2 \setminus \{(0,0,1)\}$. $Y = \mathbb{R}^2$

We'll construct a continuous map homeomorphism

$f : X \to Y$.

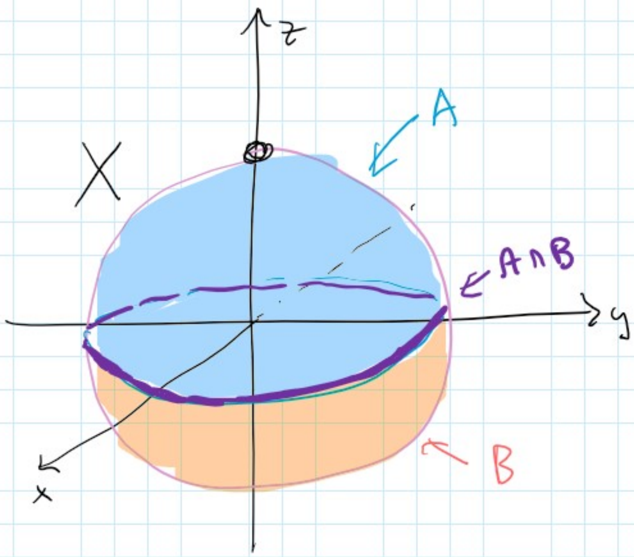


homeomorphism

$$f : X \to Y$$

$X = S^2 \setminus \{(0,0,1)\}$, $Y = \mathbb{R}^2$

Using the pasting lemma.



$$A = \{ (x,y,z) \in X \mid z \geq 0 \}$$

$$B = \{ (x,y,z) \in X \mid z \leq 0 \}$$

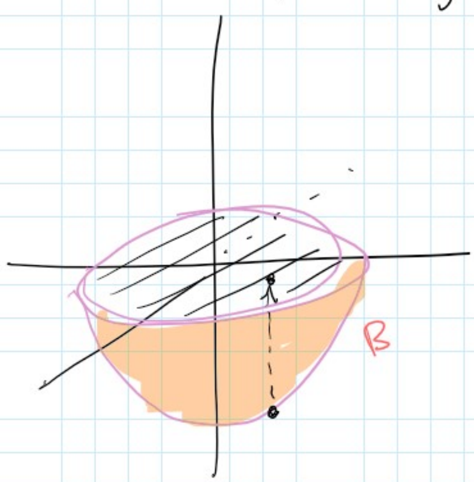
Note $A \cap B = S^1$ is the equator

Also $A \subset X$ is closed b/c $A = X \cap \left(\mathbb{R} \times \mathbb{R} \times [0,\infty) \right)$.

Similarly $B \subset X$ is closed.

Now define $f : A \to Y = \mathbb{R}^2$, $g : B \longrightarrow \mathbb{R}^2$.

First define $g : B \to \mathbb{R}^2$.



Let g project B to the unit disk in $\mathbb{R}^2$:

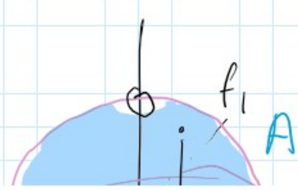
$$g(x,y,z) = (x,y)$$

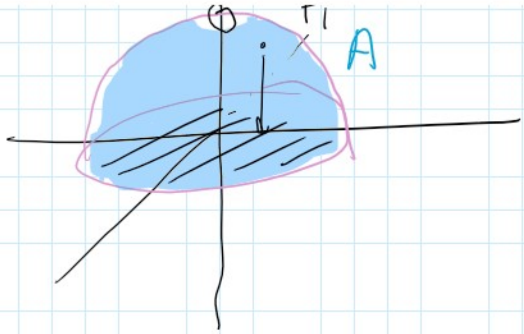
This is continuous.

Now define $f : A \longrightarrow \mathbb{R}^2$.

You can do the same as above:

$$f_1 : A \to \mathbb{R}^2$$



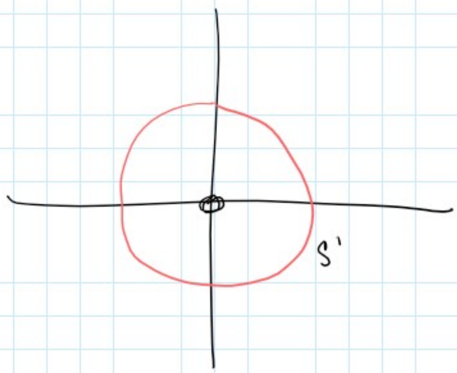


$f_1 : A \to \mathbb{R}^2$

$f_1(x, y, z) = (x, y)$

But f_1, g do not combine to give a bijection.

Compose w/ a map $f_2 : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\}$



$$f_2(x, y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$$

if $x^2+y^2 = 1$

$f_2(x, y) = (x, y)$

so f_2 fixes S^1

f_2 interchanges $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\} \setminus \{(0,0)\}$ with $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 > 1\}$.

Let $f = f_2 \circ f_1 : A \to \mathbb{R}^2$. $\quad x^2+y^2+z^2 = 1 \quad 1 - z^2 = x^2+y^2$

$$f(x,y,z) = f_2(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right) = \left(\frac{x}{1-z^2}, \frac{y}{1-z^2} \right).$$

Now have cont. functions $f : A \to \mathbb{R}^2$, $g : B \to \mathbb{R}^2$.

Note if $(x,y,z) \in A \cap B$ (so $z = 0$)

$$f(x,y,0) = (x,y) = g(x,y,0).$$

So $f|_{A \cap B} = g|_{A \cap B}$.

The pasting lemma applies to give a

The pasting lemma applies to give a continuous map $h: S^2 \setminus \{(0,0,1)\} \longrightarrow \mathbb{R}^2$.

In fact h is a bijection and its inverse is continuous. (exercise.)

$\Rightarrow$ h homeomorphism

$\Rightarrow$ $\boxed{S^2 \setminus \{(0,0,1)\} \cong \mathbb{R}^2.}$