Tuesday, September 20, 2022 11:29 AM

Continuous Functions

We have realized one of our initial goals: to make precise what we mean by a "space" (topological space). Recall that we also aimed to make sense of when two spaces are "the same" / different in a qualitative way. Today we realize this latter goal.

Defn X, Y topological spaces. A function/map $f: X \to Y$ is continuous if $\forall$ open $V \subset Y$, $f^{-1}(V) \subset X$ is open.

Suppose $\mathcal{B}_Y$ is a basis for Y.

$$\underset{\text{open}}{V \subset Y} \iff V = \bigcup B_i \quad \text{for } \{B_i\} \subset \mathcal{B}_Y$$

$$\text{Then } f^{-1}(V) = f^{-1}\left(\bigcup B_i\right) = \bigcup f^{-1}(B_i) \quad (*)$$

If f is cont. then $f^{-1}(B)$ is open $\forall B \in \mathcal{B}_Y$ since each B is open in Y.

Conversely, suppose $f^{-1}(B)$ is open $\forall B \in \mathcal{B}_Y$. Then by $(*)$, for any open $V \subset Y$, $f^{-1}(V)$ is a union of open sets, so is open in X.

Thus we've shown:

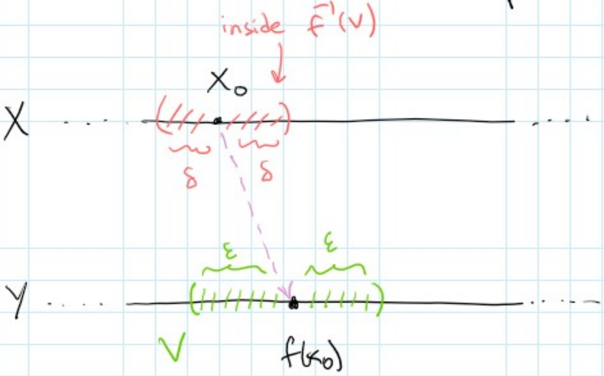
$$\boxed{f \text{ is continuous} \iff f^{-1}(B) \text{ is open } \forall B \in \mathcal{B}_Y}$$

The case $X = Y = \mathbb{R}$

Suppose $\underset{X}{f: \underset{\parallel}{\mathbb{R}}} \to \underset{\underset{Y}{\parallel}}{\mathbb{R}}$ is continuous.

$X \quad Y$

Let $x_0 \in X$. Consider the nbhd $V = (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$ of x_0 where $\varepsilon > 0$.



f cont $\Rightarrow$ $f^{-1}(V)$ open $\subset X$

note $x_0 \in f^{-1}(V)$.

$\exists$ nbhd $(x_0-\delta, x_0+\delta)$ of x_0 where $\delta > 0$

s.t. $(x_0-\delta, x_0+\delta) \subset f^{-1}(V)$.

$\Rightarrow$ $f((x_0-\delta, x_0+\delta)) \subset V = (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$

$\Rightarrow$ If $x \in (x_0-\delta, x_0+\delta)$ then $f(x) \in (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$

$\Rightarrow$ If $|x-x_0| < \delta$ then $|f(x)-f(x_0)| < \varepsilon$.

Defn (ε-δ defn of conti) A function $f: \mathbb{R} \to \mathbb{R}$ is continuous (in the ε-δ sense) if for all $x_0 \in \mathbb{R}$ and all $\varepsilon > 0$ $\exists \delta > 0$ such that for all $x \in \mathbb{R}$

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

We've shown: $f: \mathbb{R} \to \mathbb{R}$ continuous $\Rightarrow$ f is cont. in the ε-δ sense.

("$\Leftarrow$" is also true, exercise)

Examples

① Let $f: X \to Y$ be any function

Suppose X has the discrete topology

$\underset{\text{open}}{V \subset Y}$ is $f^{-1}(V) \subset X$ open? Yes.

So f is continuous.

open

So f is continuous.

(2) X with two topologies τ, τ'

$$f = id_X : X_\tau \longrightarrow X_{\tau'}$$

$$f \text{ cont} \iff \forall V \underset{\text{open}}{\subset} X_{\tau'}, \quad \underset{\substack{\parallel \\ V}}{f^{-1}(V)} \subset X_\tau \text{ is open.}$$

So: f cont $\iff$ τ finer than τ'.

ex. $id : \mathbb{R} \longrightarrow \mathbb{R}_\ell$ is not continuous

but $id : \mathbb{R}_\ell \longrightarrow \mathbb{R}$ is continuous.

(3) Constant map $f : X \to Y$ $f(x) = y_0 \in Y$ for all $x \in X$.

$$f^{-1}(V) = \begin{cases} \emptyset & \text{if } y_0 \notin V \\ X & \text{if } y_0 \in V \end{cases} \Rightarrow f \text{ continuous.}$$

Prop. X, Y top. spaces TFAE:

(1) f is continuous

(2) $\forall A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$.

(3) $\forall$ closed $B \subset Y$, $f^{-1}(B)$ is closed in X.

(4) $\forall x \in X$ and nbhds V of $f(x)$ $\exists$ nbhd U of x s.t. $f(U) \subset V$.

Pf. Let's prove (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3).

(1) $\Rightarrow$ (2): Suppose f is continuous. Let $A \subset X$.

Want to show $f(\bar{A}) \subset \overline{f(A)}$.

① ⇒ ②: Suppose f is continuous. Let $A \subset X$.

Want to show $f(\bar{A}) \subset \overline{f(A)}$.

Let $y \in f(\tilde{A})$. Then $y = f(x)$ where $x \in \bar{A}$.

Thus $\forall$ open nbhds $\mathcal{U} \subset X$ of x, $\mathcal{U} \cap A = \phi$.

Let $V \subset Y$ be a nbhd of $f(x)$.

Then $f^{-1}(V)$ is open (since f is continuous)

and it's a nbhd of x. So $f^{-1}(V) \cap A \neq \phi$.

$\Longrightarrow V \cap f(A) \neq \phi. \Longrightarrow y = f(x) \in \overline{f(A)}$.

$\Longrightarrow f(\bar{A}) \subset \overline{f(A)}$.

① ⇒ ③: f continuous. Let $B \subset Y$ be closed.

WTS: $f^{-1}(B)$ is closed.

$$Y \setminus B \text{ is open} \underset{f \text{ cont}}{\Longrightarrow} f^{-1}(Y \setminus B) \text{ is open in } X.$$

$$f^{-1}(Y \setminus B) = f^{-1}(Y) \setminus f^{-1}(B) = X \setminus f^{-1}(B)$$

$\Longrightarrow f^{-1}(B)$ is closed in X.

(proof of $1 \Rightarrow 2$, $1 \Rightarrow 3$)

Defn. X, Y top. spaces. A map $f: X \to Y$ is a **homeomorphism** if it is a continuous bijection and $f^{-1}: Y \to X$ is continuous.

Note: $\exists$ continuous bijections s.t. $f^{-1}: Y \to X$ is not continuous!

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ex. $id: \mathbb{R}_\ell \to \mathbb{R}$ is continuous, bijective

but the inverse, which is $id: \mathbb{R} \to \mathbb{R}_\ell$ is not cont.

Defn X and Y are **homeomorphic** if $\exists$ homeomorphism $f: X \to Y$.

Notation: $X \cong Y$ ← homeomorphic

Recall an **equivalence reln** $\sim$ on a set S:

(i) $\forall a \in S$, $a \sim a$

(ii) $\forall a, b \in S$ $a \sim b \Rightarrow b \sim a$

(iii) $\forall a, b, c \in S$ $a \sim b$ & $b \sim c \Rightarrow a \sim c$.

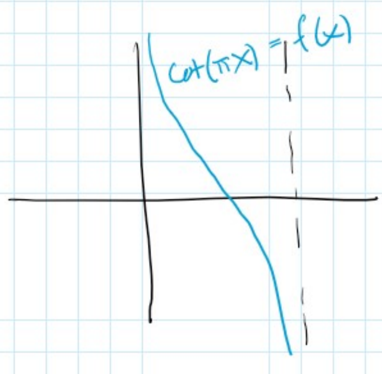
Exercise: $\underset{\text{homeomorphism}}{\cong}$ is an equivalence reln. on the collection of all topological spaces.

Examples

① $\mathbb{R} \cong (0,1)$

$f: (0,1) \to \mathbb{R}$

$f(x) = \cot(\pi x)$



② $(0,1) \cong (0,2)$ $f(x) = 2x$ $f: (0,1) \to (0,2)$

$(0,1) \cong (1,2)$ $f(x) = x+1$

In general, $(a,b) \cong (c,d)$.

In general, $(a, b) \cong (c, d)$.

(3) Any bijection b/w w/ discrete topologies $f: X \to Y$ is a homeomorphism.

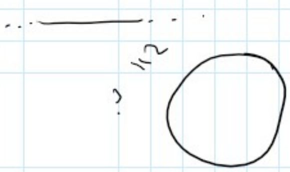
Questions

$\mathbb{R} \cong \mathbb{R}_\ell$? No.

$\mathbb{R} \cong \mathbb{R}_{fc}$? No.

$\mathbb{R} \cong \mathbb{R}^2$? No.

$\mathbb{R} \cong S^1$? No.
circle



$\mathbb{R}^2 \not\cong \mathbb{R}^3$