Homework 8

1. Let X be a topological space which is connected, normal, and has at least two distinct points. Prove that there is a continuous surjective function $f : X \to [0, 1]$.[1]

2. Prove that every compact Hausdorff space is regular.

3. Show that a closed subspace of a normal space is normal. (Surprisingly, we need the assumption of closedness here: there are normal spaces with non-normal subspaces!)

   Recall that for a (possibly infinite) product $\prod_{i\in I} X_i$ of topological spaces X_i, the *box topology* is defined using the basis whose open sets are products $\prod_{i\in I} U_i$ where each $U_i \subset X_i$ is open; and the *product topology* is defined using the basis whose open sets are products $\prod_{i\in I} U_i$ where each $U_i \subset X_i$ is open but $U_i \neq X_i$ for only finitely many $i \in I$.

4. Consider the set of infinite sequences in $\mathbb{R}$, viewed as a product:

$$\mathbb{R}^\omega = \prod_{i\in\mathbb{Z}_+} \mathbb{R} = \{\mathbf{x} = (x_1, x_2, \ldots) \mid x_i \in \mathbb{R}\}$$

   Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of sequences $\mathbf{x}$ such that $x_i \neq 0$ for only finitely many $i \in \mathbb{Z}_+$. What is the closure of $\mathbb{R}^\infty$ in the box and product topologies of $\mathbb{R}^\omega$? (Justify your answers.)

[1] You may use the Urysohn Lemma for this problem.

Extra problems

These problems need not be submitted. They are extra practice, for your benefit!

1. In the proof of the Urysohn Metrization Theorem, a key ingredient is that the space $\mathbb{R}^\omega$ with the product topology is metrizable. Show that the following defines a metric on $\mathbb{R}^\omega$:

$$d(\mathbf{x}, \mathbf{y}) = \sup_{i \in \mathbb{Z}_+} \left\{ \frac{1}{i} \min(|x_i - y_i|, 1) \right\}$$

   Furthermore, show that the metric topology induced by d agrees with the product topology on $\mathbb{R}^\omega$.

2. Prove the Urysohn Lemma in the special case of metric space (X, d) by directly defining the following function $f : X \to [0, 1]$, where $A, B \subset X$ are closed and disjoint subsets of X:

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}$$