Homework 7

1. Let $A, B \subset X$ be subspaces of a topological space X. Show that if A and B are compact then so is the union $A \cup B$.

2. Prove/disprove whether the following spaces are compact:
   (a) $\mathbb{R}_\ell$ (lower limit topology)
   (b) $\mathbb{R}_{fc}$ (finite complement topology)
   (c) the graph of $y = x^{100}$ in $\mathbb{R}^2$

3. Let $A, B \subset X$ where X is a topological space. Suppose X is Hausdorff. Show that if A and B are compact then so is the intersection $A \cap B$.[1]

   This result requires X to be Hausdorff. See the extra problems on the next page for a counterexample to the general case.

4. Prove/disprove whether $[0,1] \times [0,1]$ is homeomorphic to $[0,1] \times [0,1)$.

5. Let X and Y be topological spaces, with Y compact. Consider the projection map $\pi : X \times Y \to X$, $\pi(x, y) = x$. In this exercise you will show that π is a *closed map*, i.e. it sends every closed set in the domain to a closed set in the codomain.
   (a) Let $A \subset X \times Y$ be closed. Prove that $\pi(A) \subset X$ is closed if and only if $\forall x \in X$ such that $\{x\} \times Y$ is disjoint from A, $\exists$ a neighborhood $U \subset X$ of x such that $U \times Y$ is disjoint from A.
   (b) Let $x \in X$ be such that $\{x\} \times Y$ is disjoint from A. Argue the existence of a covering of $\{x\} \times Y$ by basis open sets in $X \times Y$, such that the cover is disjoint from A. Then use this cover and the compactness of Y to define U.[2] Conclude that π is a closed map.
   (c) Show by example that if Y is not compact, then π may fail to be a closed map. (Hint: consider the graph of the real function $1/x$.)

6. **Theorem:** Let $f : X \to Y$ be a map between spaces, where Y is compact Hausdorff. Then f is continuous if and only if the graph of f, defined by

   $$\Gamma_f = \{(x, f(x)) \mid x \in X\} \subset X \times Y,$$

   is closed in $X \times Y$.

   Prove the "$\Leftarrow$" direction. (Hint: if Γ_f is closed and $V \subset Y$ is a neighborhood of $f(x_0)$, then the intersection of Γ_f with $X \times (Y \backslash V)$ is closed. Apply the result of the previous exercise.)

[1] While a direct argument is possible, you should try to use propositions we proved in lecture to prove this.
[2] This argument is similar to a step in our proof that the product of compact spaces is compact.

Extra problems

These problems need not be submitted. They are extra practice, for your benefit!

1. Let X be the line with two origins. Recall that as a set, $X = \mathbb{R}\backslash\{0\} \cup \{o_1, o_2\}$. Basis open sets are (a,b) where $a < b < 0$ or $0 < a < b$, and also $(a,0) \cup \{o_i\} \cup (0,b)$ where $a < 0 < b$, $i \in \{1,2\}$.

   (a) There are two natural injections $f_1, f_2 : \mathbb{R} \to X$. Define them and show they are continuous.

   (b) Consider the image of $[0,1]$ under the two different maps f_1, f_2. Show that these give compact subsets of X with a non-compact intersection.

2. Prove the "$\Rightarrow$" direction of exercise 6 above.

3. Let X be a non-empty set, and $x_0 \in X$ a fixed point. Define

$$\tau = \{U \subset X \mid x_0 \notin U \text{ or } X\backslash A \text{ is finite }\}$$

   Show that τ is a topology on X. Verify that this topology is both Hausdorff and compact. This shows that every non-empty set has a topology which is compact and Hausdorff.

4. Parametrizing S^1 by the angle θ of its points, define a map $S^1 \to S^1 \times S^1$ by $\theta \mapsto (2\theta, 3\theta)$. Show that this map is a homeomorphism onto its image. Draw the image on $S^1 \times S^1$ viewed as (1) a square with sides identified in the appropriate manner; (2) the surface of a donut. Also: draw the graph of f!