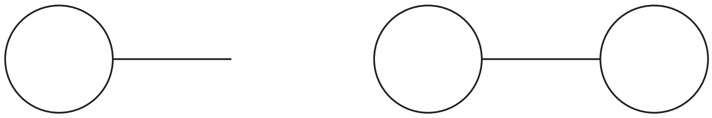
Homework 6

1. The n-sphere S^n is the union of a northern hemisphere and a southern hemisphere which intersect along an equatorial sphere of one less dimension. Recall also that S^n is homeomorphic to the quotient an n-disk D^n by collapsing its boundary. Use these two different descriptions of S^n to give two different proofs that it is a connected topological space.[1]

2. Let X be a topological space, and $C_X \subset X$ the subset of cut points, viewed as a topological space via the subspace topology. Show that the homeomorphism type of C_X is a topological property of X: that is, if $Y \cong X$ then $C_Y \cong C_X$. Consider the following two topological spaces (as subsets of $\mathbb{R}^2$):



   Determine C_X for each of these space. Make a conclusion about the relationship between the spaces.

3. Construct (draw) a subspace of $\mathbb{R}^2$ that has exactly 5 distinct cut points. Similarly, construct a subspace of $\mathbb{R}^2$ having exactly 5 distinct non-cut points.

4. Let $f : X \to X$ be continuous, where $X = [0, 1]$. Show that there is some $x \in [0, 1]$ such that $f(x) = x$. (Such an x is called a *fixed point* of the function f.) Is the same true for $X = (0, 1]$ or $X = (0, 1)$?

5. Using properties of connectedness, determine all continuous maps $f : \mathbb{R} \to \mathbb{R}_\ell$ where $\mathbb{R}_\ell$ has the lower limit topology.

6. Compute the connected components of each of the following spaces.
   (a) $\mathbb{Z}$ (with subspace topology inherited from $\mathbb{R}$)
   (b) $\mathbb{Q}$ (with subspace topology inherited from $\mathbb{R}$)
   (c) $\mathbb{R}_{fc}$ (finite complement topology)

[1] Use propositions we proved in lecture. (Also found in Munkres.)

Extra problems

These problems need not be submitted. They are extra practice, for your benefit!

1. Show that a space X is connected if and only if every continuous map $f : X \to \{0, 1\}$ is constant. (Here $\{0, 1\}$ is given the discrete topology.)

2. Show that the 2-dimensional torus is connected in two ways: first by using the quotient map that defines the torus; and second, by writing the torus as a union of subspaces all of which are homeomorphic to a disk and which all intersect in a common point. (For the latter method, draw a picture.)

3. Let $q : X \to X/\sim$ be the map in the definition of a quotient topology. Show that if $q^{-1}(\{[x]\})$ is connected for all $[x] \in X/\sim$ and $X/\sim$ is connected, then X is connected.