Homework 5

1. Recall that if $f : X \to Y$ is a continuous map, then $B \subset Y$ closed implies $f^{-1}(B) \subset X$ is closed. Use this to argue that the following are closed subsets of the topological spaces in which they live:

$$A = \{(x, y) \mid xy = 1\} \subset \mathbb{R}^2$$

$$S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\| = 1\} \subset \mathbb{R}^{n+1} \qquad n\text{-sphere}$$

$$B^n = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leqslant 1\} \subset \mathbb{R}^n \qquad \text{(closed) } n\text{-ball}$$

2. Let X be a set with an equivalence relation $\sim$ and denote by $X/\sim$ the set of equivalence classes. Define $q : X \to X/\sim$ by $q(x) = [x] = \{y \mid y \sim x\}$. The *quotient topology* on $X/\sim$ is defined by:

$$U \subset X/\sim \text{ is open} \iff q^{-1}(U) \subset X \text{ is open}$$

Prove that this definition actually defines a topology on $X/\sim$.

3. For $X = [0, 1]$ with equivalence relation $\sim$, where the only non-trivial relation[1] is $0 \sim 1$, we drew $X/\sim$ as a circle, having identified 0 and 1. In each case below, draw a picture of the quotient space $X/\sim$.

   (i) $X = \{(x, y) \mid 1 \leqslant x^2 + y^2 \leqslant 2\} \subset \mathbb{R}^2$. Non-trivial relations: $(a, b) \sim (c, d)$ if $|a^2 + b^2 - c^2 - d^2| = 1$ and $(a, b) = e(c, d)$ for some $e > 0$.

   (ii) $X = \{(x, y) \mid x \geqslant 0, y \geqslant 0\} \subset \mathbb{R}^2$. Non-trivial relations: $(a, 0) \sim (0, a)$ and $(0, a) \sim (a, 0)$ for $a > 0$.

   (iii) $X = \{(x, y) \mid xy = 0\} \subset \mathbb{R}^2$. Non-trivial relations: $(a, b), (c, d) \in X$ satisfy $(a, b) \sim (c, d)$ if $a, b, c, d \in \mathbb{Z}$, $a = d$, $b = c$.

[1] Here "non-trivial relation" means any relation $x \sim y$ where $x \neq y$.

Extra problems

These problems need not be submitted. They are extra practice, for your benefit!

1. Define an equivalence relation on the plane $X = \mathbb{R}^2$ as follows: $(a, b) \sim (c, d)$ if and only if $a+b^2 = c+d^2$. Consider the topological space $X/\sim$ (the quotient set with quotient topology).
   - (i) Describe the equivalence classes of $\sim$ (these are subsets of $X = \mathbb{R}^2$).
   - (ii) $X/\sim$ is homeomorphic to a familiar space. What is it? (Hint: to begin, argue that $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(a, b) = a + b^2$ induces a well-defined map $X/\sim \to \mathbb{R}$.)

2. In lecture we discussed the following result. Let $r : S^{n-1} \to \mathbb{R}_+$ be a continuous map. Define

$$X = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{0} \text{ or } \|\mathbf{x}\| \leqslant r(\mathbf{x}/\|\mathbf{x}\|)\} \subset \mathbb{R}^n$$

   Then X is homeomorphic to the closed unit ball in $\mathbb{R}^n$.
   - (i) Given continuous maps $r : S^{n-1} \to \mathbb{R}_+$ and $R : S^{n-1} \to \mathbb{R}_+$ with $r(\mathbf{x}) < R(\mathbf{x})$ for all $\mathbf{x}$, prove

$$Y = \{\mathbf{x} \in \mathbb{R}^n \mid r(\mathbf{x}/\|\mathbf{x}\|) \leqslant \|\mathbf{x}\| \leqslant R(\mathbf{x}/\|\mathbf{x}\|)\} \subset \mathbb{R}^n$$

     is homemorphic to $S^{n-1} \times [0, 1]$. Draw some interesting examples of Y in the case that $n = 2$.
   - (ii) We gave an explicit formula for the choice of r which makes X a standard square in the plane. Find also a formula for a continuous map $r : S^1 \to \mathbb{R}$ so that $X \subset \mathbb{R}^2$ is an equilateral triangle.

3. Show $[0, 1) \times [0, 1]$ is homeomorphic to $[0, 1) \times [0, 1)$.