Homework 4

1. Prove that for maps $f : \mathbb{R} \to \mathbb{R}$, the ε-δ definition of continuity implies the open set definition of continuity. (We showed the other direction in lecture.)

2. Let $f : A \to B$ and $g : C \to D$ be continuous. Define $f \times g : A \times C \to B \times D$ by $(f \times g)(a, c) = (f(a), g(c))$ for all $(a, c) \in A \times C$. Show that $f \times g$ is continuous.

3. Prove that each of the following is a topological property of spaces.
   (a) The property of having a countable topology.
   (b) The property of singleton sets $\{x\}$ being closed.
   (c) The property of having a countable subset which is dense.

4. Prove/disprove whether each of the following is a topological property of subspaces $X \subset \mathbb{R}$.
   (a) The property of X containing two points of distance d apart.
   (b) The property of containing infinitely many rational numbers.
   (c) The property of being bounded, in the sense that X is contained in some interval $[a, b]$.

   Consider the same problem but for subsets in $\mathbb{R}_{fc}$ (finite complement topology).

5. Given a map $F : X \times Y \to Z$, we say that it is *continuous in each variable separately* if for each $y_0 \in Y$, the map $h : X \to Z$ defined by $h(x) = F(x, y_0)$ is continuous, and for each $x_0 \in X$, the map $g : Y \to Z$ defined by $g(y) = F(x_0, y)$ is continuous. Show that if F is continuous, then it is continuous in each variable separately.

Extra problems

These problems need not be submitted. They are extra practice, for your benefit!

1. Show that $X \cong Y$ is an equivalence relation on the collection of topological spaces.

2. Let $A \subset X$. Suppose $f : A \to Y$ is a continuous map, and Y is Hausdorff. Show that if f can be extended to a map $g : \overline{A} \to Y$, then g is uniquely determined by f.