Homework 3

1. Show that if A is closed in Y and Y is closed in X, then A is closed in X.

2. Let A, B and A_i (where $i \in I$, for some index set I) be subsets of a topological space X. Prove:
   (a) If $A \subset B$ then $\overline{A} \subset \overline{B}$.
   (b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
   (c) $\overline{\bigcup A_i} \supset \bigcup \overline{A}_i$. Give an example where equality fails here.

3. Same setup as the previous problem. Determine which of the following equations hold; if an equality fails, determine if $\subset$ or $\supset$ holds.
   (a) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
   (b) $\overline{\bigcap A_i} = \bigcap \overline{A}_i$.
   (c) $\overline{A \backslash B} = \overline{A} \backslash \overline{B}$.

4. Show that X is Hausdorff if and only if the "diagonal" $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

5. Prove/disprove whether each of the following spaces is Hausdorff:
   (a) $\mathbb{R}_\ell$ (lower limit topology)
   (b) $\mathbb{R}_{fc}$ (finite complement topology)
   (c) $\mathbb{R}$ with topology generated by basis $(-\infty, a)$ where a runs over all reals
   (d) Trivial topology on any given set.
   (e) Every topology on a set with two elements.
   (f) $\mathbb{Z}$ with the topology whose basis consists of the sets $\{2k - 1, 2k\}$ where k runs over all integers.

Extra problems

These problems need not be submitted. They are extra practice, for your benefit!

1. Let X be a topological space, and let $Y \subset X$ be given the subspace topology. Show that $A \subset Y$ is closed in Y if and only if A is the intersection of Y with a closed set in X.

2. Show that the product of two Hausdorff spaces is Hausdorff.

3. For a subset A in a topological space X, define

$$\mathrm{Bd}(A) = \overline{A} \cap (\overline{X \backslash A})$$

We call $\mathrm{Bd}(A)$ the *boundary of* A.

(a) Show that $\mathrm{int}(A)$ and $\mathrm{Bd}(A)$ are disjoint, and $\overline{A} = \mathrm{int}(A) \cup \mathrm{Bd}(A)$.

(b) Show that $\mathrm{Bd}(A) = \varnothing$ if and only if A is both open and closed.

(c) Show that $U \subset X$ is open if and only if $\mathrm{Bd}(U) = \overline{U} \backslash U$.

(d) If U is open, is it true that $U = \mathrm{int}(\overline{U})$?