Homework 2

1. Let X be a topological space, and $A \subset X$. Suppose that for each $x \in A$ there exists an open set U such that $x \in U$ and $U \subset A$. Show that A is open in X.

2. Let $\{\tau_i \mid i \in I\}$ be a collection of topologies on a set X. Show that the intersection of all these topologies, $\bigcap_{i \in I} \tau_i$, is a topology on X. What about the union?

3. Consider the following topologies $\tau_1, \ldots, \tau_5$ on $\mathbb{R}$ defined by the given bases:

$$\mathcal{B}_1 = \{(a,b) \mid a,b \in \mathbb{R},\ a < b\} \qquad (\tau_1 \text{ is the standard topology})$$
$$\mathcal{B}_2 = \{[a,b) \mid a,b \in \mathbb{R},\ a < b\} \qquad (\tau_2 \text{ is the lower-limit topology})$$
$$\mathcal{B}_3 = \{(a,b] \mid a,b \in \mathbb{R},\ a < b\} \qquad (\tau_3 \text{ is the upper-limit topology})$$
$$\mathcal{B}_4 = \{\mathbb{R} \backslash A \mid A \text{ a finite subset of } \mathbb{R}\} \qquad (\tau_4 \text{ is the finite complement topology})$$
$$\mathcal{B}_5 = \{(\infty, a) \mid a \in \mathbb{R}\}$$

Determine, for each of these topologies, the others that it contains.

4. Let X be an arbitrary set. Determine whether each of the following collections defines a topology on X. If so, prove it. If not, provide a counterexample (i..e for some set X, show that the axioms fail).
   (a) $\tau_{\text{count}} = \{A \subset X \mid X \backslash A \text{ is countable, or } A = \varnothing\}$.
   (b) $\tau_{\infty} = \{A \subset X \mid X \backslash A \text{ is infinite, or } A = \varnothing\}$.

5. Let X be a topological space. Suppose $A \subset Y \subset X$. Show that the topology that A inherits as a subspace of Y (where Y has the subspace topology that it inherits as a subspace of X) is the same as the topology that A inherits as a subspace of X.

6. Let $[-1, 1]$ have the subspace topology inherited from $\mathbb{R}$. Determine whether each of the following sets in $[-1, 1]$ is open, closed, or neither:

$$(1/2, 1), \qquad (1/2, 1], \qquad [1/2, 1), \qquad [1/2, 1], \qquad \{x \in (0,1) \mid 1/x \notin \mathbb{Z}_+\}$$

7. Suppose L is a straight line in the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Describe the topology that L inherits, viewing it as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. (Here we have written $\mathbb{R}_\ell$ for the reals with the lower-limit topology, and $\mathbb{R}$ for the reals with the standard topology.)

Extra problems

These problems need not be submitted. They are extra practice, for your benefit!

1. Let $X = \{a, b, c\}$. List the possible topologies on X. Determine the relationships (finer/coarser) between these topologies.

2. Define a topology on the set $\mathbb{R} \cup \{\infty\}$ (where ∞ is just some new element) as follows: we declare that a subset $U \subset \mathbb{R} \cup \{\infty\}$ is open if and only if either:

   (i) $U \subset \mathbb{R}$ and U is open as a subset of $\mathbb{R}$; or

   (ii) $\infty \in U$ and $U \backslash \{\infty\}$ is an open subset of $\mathbb{R}$ that contains $(-\infty, a)$ or (a, ∞) for some $a \in \mathbb{R}$.

   Show that this does in fact define a topology on $\mathbb{R} \cup \{\infty\}$.

3. Define a topology on $\mathbb{Z} \cup \{\infty\}$ as follows. A set $U \subset \mathbb{Z} \cup \{\infty\}$ is open if and only if either the complement of U is finite, or the complement includes ∞. Show that this indeed defines a topology.

4. Consider the topology on $\mathbb{Z} \cup \{\infty\}$ from Problem 3, and also the topology that it inherits as a subspace of $\mathbb{R} \cup \{\infty\}$ (with topology defined in Problem 2). How do these topologies compare?

5. Let p be a prime number. Consider the following subset of $\mathbb{Z}$, where $a \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geqslant 0}$:

   $$U_{a,k} = \{a + p^k b \mid b \in \mathbb{Z}\} = \text{"}a + p^k\mathbb{Z}\text{"}$$

   Show that $\mathcal{B} = \{U_{a,k} \mid a \in \mathbb{Z}, k \in \mathbb{Z}_{\geqslant 0}\}$ is a basis. The topology generated by this basis is called the *p-adic topology* on the integers, and is important in number theory.