Homework 1

1. Prove the following set theoretic identities:
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (b) $A \smallsetminus (B \cup C) = (A \smallsetminus B) \cap (A \smallsetminus C)$

2. Determine whether each subset of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is a Cartesian product of two subsets of $\mathbb{R}$:
   (a) $\{(a, b) \mid a \in \mathbb{Z}\}$
   (b) $\{(a, b) \mid a < b\}$
   (c) $\{(a, b) \mid b > 1\}$
   (d) $\{(a, b) \mid a \notin \mathbb{Q}, b \in \mathbb{Q}\}$
   (e) $\{(a, b) \mid a^2 + b^2 < 1\}$

3. Let $f : A \to B$ be a map of sets, and $A_0, A_1 \subset A$, $B_0, B_1 \subset B$. Prove **four** of the following statements (and either prove the rest on your own, or make sure you understand them):
   (a) $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$
   (b) $A_0 \subset f^{-1}(f(A_0))$
   (c) $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$
   (d) $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$
   (e) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$
   (f) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$
   (g) $f^{-1}(B_0 \smallsetminus B_1) = f^{-1}(B_0) \smallsetminus f^{-1}(B_1)$
   (h) $f(A_0 \smallsetminus A_1) \supset f(A_0) \smallsetminus f(A_1)$

   Furthermore, give examples where equality fails in 3f and 3h.

4. Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if f and g are injective (resp. surjective), then so too is $g \circ f$. If $g \circ f$ is injective (resp. surjective), what can be said about the injectivity of f and g (resp. surjectivity)?

5. Prove that $f : A \to B$ is bijective if and only if there exists a function $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

6. Determine whether each of the following sets is countable or not, with justification.
   (a) The set of all functions $\{0, 1\} \to \mathbb{Z}_+$.
   (b) The set of all functions $\mathbb{Z}_+ \to \mathbb{Z}_+$.
   (c) The set of all functions $\mathbb{Z}_+ \to \{0, 1\}$.
   (d) The set of irrational numbers.
   (e) The set of possible passwords created using a standard keyboard.

Extra problems

These problems need not be submitted. They are extra practice, for your benefit!

1. Let $f : A \to B$ be a function. Suppose $g : B \to A$ is a function satisfying $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. If g' is another such function satisfying these properties, show that $g = g'$.

2. Suppose there is an injection $A \to B$, and $A \neq \varnothing$. Prove there is a surjection $B \to A$.

3. Let A be a set, and $\mathcal{P}(A)$ the set of all subsets of A. If A is finite with n elements, argue that $\mathcal{P}(A)$ has exactly 2^n elements.

4. Fix a set X. For $A, B \in \mathcal{P}(X)$, define $A \cdot B = A \cap B$ and $A + B = (A \cup B) \setminus (A \cap B)$. Show

   $$A \cdot (B + C) = A \cdot B + A \cdot C$$

   for any $A, B, C \in \mathcal{P}(X)$. If we define $1 = X$ and $0 = \varnothing$, then we also have $A \cdot 0 = 0$ and $A \cdot 1 = A$. In fact, all of the usual arithmetic rules that one has for the integers now holds with this notation. There are some extra rules as well: $A \cdot A = A$ and $A + A = 0$.

5. An *algebraic* number x is a real (or complex) number which is the root of a polynomial $a_n x^n + \cdots + a_1 x + a_0$ with $a_i \in \mathbb{Z}$. For example, $\sqrt{2}$ is algebraic. Show the set of algebraic numbers is countable.