Exam 1

1. Let X, Y be topological spaces and $A \subset X$, $B \subset Y$. Prove that the closure of $A \times B$ in the product $X \times Y$ is equal to $\overline{A} \times \overline{B}$.

Let $(x, y) \in X \times Y$. Then $(x, y) \in \overline{A \times B}$ if and only if for every neighborhood U of (x, y) we have $U \cap (A \times B) \neq \varnothing$. It suffices in this statement to take $U = V \times W$ a basis open set of $X \times Y$ that contains (x, y). Then $U \cap (A \times B) \neq \varnothing$ becomes

$$(V \times W) \cap (A \times B) = (V \cap A) \times (W \cap B) \neq \varnothing,$$

In other words $V \cap A \neq \varnothing$ and $W \cap B \neq \varnothing$. Thus $(x, y) \in \overline{A \times B}$ if and only if for every open neighborhoods $V \subset X$ of x and $W \subset Y$ of y we have $V \cap A \neq \varnothing$ and $W \cap B \neq \varnothing$; thus $(x, y) \in \overline{A \times B}$ if and only if $x \in \overline{A}$ and $y \in \overline{B}$. This proves $\overline{A \times B} = \overline{A} \times \overline{B}$.

2. Let $\mathbb{R}$ be the real line with the standard topology and $\mathbb{R}_{fc}$ be the real line with the finite complement topology. Let L be a straight line in the plane, and describe the topology it inherits as a subspace of $\mathbb{R}_{fc} \times \mathbb{R}$. Do the same for $\mathbb{R}_{fc} \times \mathbb{R}_{fc}$.

$\mathbb{R}_{fc} \times \mathbb{R}$: If L is vertical, of the form $\{a\} \times \mathbb{R}$, then it inherits the standard topology. Similarly, if it is horizontal, it inherits the $\mathbb{R}_{fc}$ topology. Other cases: write $L = \{(x, y) \mid y = cx + d\}$ where $c, d \in \mathbb{R}$, $c > 0$. Note $U = L \cap (\mathbb{R} \times (a, b))$ is in bijection, via projection to the x-axis, with the standard open interval $((a - d)/c, (b - d)/c)$. As $\mathbb{R} \times (a, b)$ is open in $\mathbb{R}_{fc} \times \mathbb{R}$, $U \subset L$ is open. Thus standard open intervals are open in L. On the other hand, the topology on L is no finer than the standard topology, as $\mathbb{R}_{fc} \times \mathbb{R}$ is coarser than $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. Thus L has the standard topology. The case $c < 0$ is similar.

$\mathbb{R}_{fc} \times \mathbb{R}_{fc}$: Let $U \times V$ be a standard basis subset of $\mathbb{R}_{fc} \times \mathbb{R}_{fc}$. Then $\mathbb{R}^2 \backslash U \times V$ is a finite collection of horizontal and vertical lines $L_1, \ldots, L_n$. Then L intersects $\mathbb{R}^2 \backslash U \times V$ in either a finite number of points (at most one point in each $L \cap L_i$), or L is one of the lines L_i. In the first case $(U \times V) \cap L$ is a set in L with finite complement, and in the second case it is empty. Thus L has the finite complement topology.

3. Consider the sequence $a_n = -1/n$, $n \in \mathbb{Z}_+$. For each of the following topologies, determine whether this sequence converges, and if so, to which points.

(a) $\mathbb{R}_\ell$ (lower limit) (b) $\mathbb{R}_K$ (K-topology) (c) $\mathbb{R}_L$

The last topology $\mathbb{R}_L$ has basis the subsets $(-\infty, a)$ as a ranges over all real numbers.

(a) Let $A = \{-1/n \mid n \in \mathbb{Z}_+\}$. If $a \in \mathbb{R}$ and $a \geqslant 0$ consider the neighborhood of a given by $U = [a, a + 1)$. Then $A \cap U = \varnothing$ so the sequence cannot converge to a. If $a < 0$, choose $\epsilon \in \mathbb{R}$ such that $a + \epsilon < 0$. Then with $U = (a - \epsilon, a + \epsilon)$, if $-1/n \in A \cap U$ then $-1/n < a + \epsilon$, implying $n < |1/(a + \epsilon)|$. Thus $A \cap U$ is finite. So the sequence does not converge to a again, and in summary converges to no points.

(b) If $a \in \mathbb{R}$ is non-zero, a neighborhood $U = (a - \epsilon, a + \epsilon)$ with ϵ small enough will have $U \cap A$ finite (as in (a)), and so the sequence does not converge to non-zero a. On the other hand, every basis neighborhood of 0 is of the form $U = (a, b)$ or $U = (a, b) \backslash K$, and both types have

$U \cap A = \{-1/n \mid n > |a|\}$. Thus the sequence converges to 0.

(c) For any $a \in \mathbb{R}$, the neighborhood $U = \{a\}$ intersects A in at most one point, so the sequence does not converge to any point.

(d) Any neighborhood of $a \in \mathbb{R}$ is of the form $U = (-\infty, b)$ where $b > a$, or $U = \mathbb{R}$. Thus if $a \geqslant 0$ then $U \cap A$ contains A. If $a < 0$ choose $a < b < 0$, and then $U \cap A$ is finite. Thus the sequence converges to all non-negative real numbers.

4. Let (X, d) and (X', d') be a metric spaces, and $f : X \to X'$ is a map. Suppose there is some constant C such that $d'(f(x), f(y)) \leqslant Cd(x, y)$ for $x, y \in X$. Prove that f is continuous (where each of X and X' are given the metric topologies).

It suffices to show that for every $x \in X$, and every basis neighborhood $B_{d'}(f(x), \varepsilon)$ of $f(x) \in X'$, there exists a basis neighborhood $B_d(x, \delta)$ of x such that

$$f\left(B_d(x, \delta)\right) \subset B_{d'}(f(x), \varepsilon).$$

Let $\delta = \varepsilon/C$. Suppose $y \in B_d(x, \delta)$. Then, using the assumption in the problem, we have:

$$d'(f(x), f(y)) \leqslant Cd(x, y) < C\delta = C(\frac{\varepsilon}{C}) = \varepsilon.$$

Thus $f(y) \in B_{d'}(f(x), \varepsilon)$. So with this choice of δ, the claim is proved.

5. Let X be a Hausdorff topological space.

(a) Show that every singleton set in X is closed.

(b) Show that $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

(c) Can X arise as a subspace $X \subset Y$ of a non-Hausdorff space Y?

(a) Let $y \in \overline{\{x\}}$. Then every neighborhood V of y contains x. As X is Hausdorff, if $y \neq x$, there would exist open neighborhoods V of y and U of x such that $U \cap V = \varnothing$, and in particular $x \notin V$. So it must be that $x = y$. We have shown $\overline{\{x\}} = \{x\}$, and thus $\{x\}$ is closed.

(b) We show the complement is open. Let $(x, y) \in X \times X \backslash \Delta$. This means that $x \neq y$. As X is Hausdorff, there exist open neighborhoods U of x and V of y such that $U \cap V = \varnothing$. Moreover, $U \cap V = \varnothing$ implies that $U \times V \cap \Delta = \varnothing$. Thus $U \times V$ is an open neighborhood of (x, y) which is contained in $X \times X \backslash \Delta$. This proves that $X \times X \backslash \Delta$ is open.

6. (Optional) Let $\mathbb{R}^\omega$, the set of real-valued sequences, have the topology whose basis is given by subsets of the form $U_1 \times U_2 \times U_3 \times \cdots$ where each U_i is an open subset of $\mathbb{R}$.

1. Determine whether $\mathbb{R}^\omega$ is Hausdorff.
2. Determine whether the sequence $a_1 = (1, 0, \ldots)$, $a_2 = (0, 1/2, 0 \ldots)$, $a_3 = (0, 0, 1/3, 0, \ldots)$, ... converges, and if so, to what point(s).

(a) We show $\mathbb{R}^\omega$ is Hausdorff. Let $s = (s_1, s_2, \ldots)$ and $t = (t_1, t_2, \ldots)$ be two distinct elements. Thus there is some $i \in \mathbb{Z}_+$ such that $s_i \neq t_i$. Using that $\mathbb{R}$ is Hausdorff, choose neighborhoods $U_i, V_i \subset \mathbb{R}$ of s_i, t_i such that $U_i \cap V_i = \varnothing$. For $j \neq i$ set $U_j = V_j = \mathbb{R}$. Then $U = U_1 \times U_2 \times \cdots$ and $V = V_1 \times V_2 \times \cdots$ are neighborhoods in $\mathbb{R}^\omega$ of s and t respectively, and we have $U \cap V = \varnothing$. Thus $\mathbb{R}^\omega$ is Hausdorff as claimed.

(b) Suppose $\{a_n\}_{n=1}^\infty$ converges to $t = (t_1, t_2, \ldots) \in \mathbb{R}^\omega$. Choose a neighborhood of t of the form $U = U_1 \times U_2 \times \cdots$ where if $t_i \neq 0$ then U_i is an open interval disjoint from 0, and if $t_i = 0$ we set $U_i = (-1/i, 1/i)$. As $\{a_n\}_{n=1}^\infty$ converges to t, there is some $N \in \mathbb{Z}_+$ such that $\{a_n \mid n \geqslant N\} \subset U$. Since $a_N = (0, \ldots, 0, 1/N, 0, \ldots,) \in U$, we must have $t_i = 0$ for $i > N$ by our choice of U_i. For $i > N$, as $a_i \in U$, we must have $1/i \in U_i = (-1/i, 1/i)$, a contradiction. Thus the sequence converges to no points.