Practice Problems for Exam #2.

1. Let $A_1, A_2, \ldots$ be connected subspaces of X.

Suppose $A_i \cap A_{i+1} \neq \emptyset$ $\forall i \in \mathbb{Z}_+$.

Prove that $\bigcup_{i \in \mathbb{Z}_+} A_i$ is connected.

Pf. Suppose $\bigcup_{i \in \mathbb{Z}_+} A_i = \mathcal{U} \cup V$ where $\mathcal{U}, V$ are disjoint & open.

(note: $\mathcal{U}, V$ open in subspace top. of $\bigcup_{i \in \mathbb{Z}_+} A_i$)

$A_1 = (A_1 \cap \mathcal{U}) \cup (A_1 \cap V)$

A_1 connected & $(A_1 \cap \mathcal{U}), (A_1 \cap V)$ open, disjoint in A_1

$\Rightarrow$ one of them is empty.

WLOG, suppose $A_1 \cap V = \emptyset$. Then $A_1 \subset \mathcal{U}$.

$A_2 = (A_2 \cap \mathcal{U}) \cup (A_2 \cap V)$

A_2 connected $\Rightarrow$ one of $A_2 \cap \mathcal{U}$, $A_2 \cap V$ is empty

since $A_1 \cap A_2 \neq \emptyset$, and $A_1 \subset \mathcal{U}$, $A_2 \cap \mathcal{U} \neq \emptyset$.

$\Rightarrow A_2 \cap V = \emptyset$.

Inductively, $A_i = (A_i \cap \mathcal{U}) \cup (A_i \cap V)$

A_i connected $\Rightarrow$ one of $A_i \cap \mathcal{U}$, $A_i \cap V$ empty

$A_i \cap A_{i-1} \neq \phi \Rightarrow A_i \cap V = \phi$.

So we get $A_i \subset \mathcal{U} \ \forall i \in \mathbb{Z}_+ \ (\Leftrightarrow A_i \cap V = \phi \ \forall i \in \mathbb{Z}_+)$.

So $\bigcup_{i \in \mathbb{Z}_+} A_i \subset \mathcal{U}$, so in fact $V = \phi$, contradiction.

$\square$

#2: Show every subspace $A \subset \mathbb{R}_{fc}$ is compact.

<u>Pf.</u> Let $A \subset \bigcup_{i \in I} \mathcal{U}_i$ where each $\mathcal{U}_i \subset \mathbb{R}_{fc}$ is open.

Assume $A \neq \phi$. Pick $\mathcal{U}_{i_0}$ $(i_0 \in I)$ that $\neq \phi$.

Then $A \setminus \mathcal{U}_{i_0}$ is finite since $\mathcal{U}_{i_0} = \mathbb{R} \setminus$ finite set.

Thus $A \setminus \mathcal{U}_{i_0} = \{p_1, \ldots, p_n\}$.

For each p_j choose $\mathcal{U}_{i_j}$ $(i_j \in I)$ s.t. $p_{i_j} \in \mathcal{U}_{i_j}$.

Then $A \subset \mathcal{U}_{i_0} \cup \mathcal{U}_{i_1} \cup \cdots \cup \mathcal{U}_{i_n}$. $\square$

#3: $\left.\begin{array}{l} f: X \to Y \text{ continuous bijective} \\ X \text{ compact}, \ Y \text{ Hausdorff} \end{array}\right\} \Rightarrow f$ is a homeomorphism.

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Pf: Suffices to show $f^{-1}: Y \to X$ is continuous.

i.e. given any closed $A \subset X$, need to argue that $(f^{-1})^{-1}(A) \subset Y$ is closed.

Note $(f^{-1})^{-1}(A) = f(A)$. So suffices to show:

$$A \subset X \text{ closed} \Rightarrow f(A) \subset Y \text{ closed.}$$

We see that:

$$A \subset X \text{ closed, } X \text{ compact} \Rightarrow A \text{ compact}$$

$$f \text{ cont., } A \text{ compact} \Rightarrow f(A) \text{ compact}$$

$$\underset{\text{compact}}{f(A)} \subset \underset{\text{Hausdorff}}{Y} \Rightarrow f(A) \text{ closed in } Y.$$

$\square$

#4. Prove every nonconstant path $f: [0,1] \to \mathbb{R}^2$ contains (in its image) some point such that one of the coordinates is rational.

Pf. $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ $\quad \pi_1(x,y) = x$
$\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ $\quad \pi_2(x,y) = y$ $\Big\}$ these are continuous

We have $f = (f_1, f_2)$ where $f_1 = \pi_1 \circ f$, $f_2 = \pi_2 \circ f$.

$f_1 : [0,1] \to \mathbb{R}$
$f_2 : [0,1] \to \mathbb{R}$ $\Big\}$ these are comp. of cont. fns. so they are cont.

Since f is nonconstant, one of f_1 or f_2 is nonconstant, say f_1.

Then $\exists\, s,t \in [0,1]$ s.t. $f_1(s) \neq f_1(t)$.
In fact assume $f_1(s) < f_1(t)$.

Then choose $q \in \mathbb{Q}$ s.t. $f_1(s) < q < f_1(t)$

By the IVT applied to $f_1 : [0,1] \to \mathbb{R}$: $\exists\, r \in [0,1]$ s.t.
connected
$f_1(s) < q < f_1(t)$ $\quad$ $f_1(r) = q$.

Then $f(r) = (f_1(r), f_2(r))$ has 1st coordinate rational. $\square$

#5: X top. space, $\sim$ equivalence reln
Give examples illustrating:

X Hausdorff $\overset{(i)}{\cancel{\Rightarrow}}$ $\underset{(ii)}{\cancel{\Leftarrow}}$ $X/\sim$ Hausdorff

X Hausdorff $\;\not\Rightarrow\;$ $X/\sim$ Hausdorff

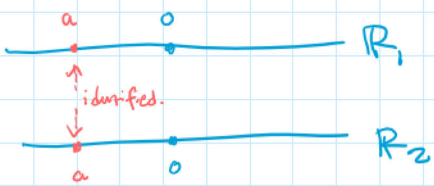
$\not\Leftarrow$ (ii)

(i) $\underline{\text{Ex. 1}}$ $X = \mathbb{R}$ $\quad\sim$ identifies all of $\mathbb{Q}$.

X Hausdorff. $\quad [\mathbb{Q}], \{r\} \in X/\sim \quad$ (r irrational)

cannot separate these two pts by disjoint open nbhds in $X/\sim$.

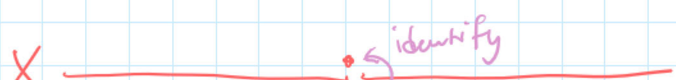
$\underline{\text{Ex. 2}}$ $X = \mathbb{R}_1 \sqcup \mathbb{R}_2$ is Hausdorff. $\quad\sim$ identifies $a \in \mathbb{R}_1 \setminus \{0\}$ with $a \in \mathbb{R}_2 \setminus \{0\}$.



$X/\sim$ is the line with two origins, which is not Hausdorff.

(ii) $\underline{\text{Ex. 1}}$: Line w/ two origins $= X$ (not Hausdorff)

$\sim$ identifies the two origins.



$\rightsquigarrow$ $X/\sim = \mathbb{R}$ which is Hausdorff.

$\underline{\text{Ex. 2}}$: X any space (in particular can be non-Hausdorff)

Define $\sim$ to identify all pts of X.

Then $X/\sim = \{X\}$ is a singleton set.

And 1-point space has a unique (Hausdorff) topology.