Practice problems for Exam 1

1. Let X be a topological space. Show that a set $A \subset X$ is open if and only if $x \in A$ has an open neighborhood $U(x)$ such that $U(x) \subset A$.

2. Let $\mathbb{R}$ be the real line with the standard topology and $\mathbb{R}_t$ be the real line with the trivial topology. Let L be a straight line in the plane, and describe the topology it inherits as a subspace of $\mathbb{R} \times \mathbb{R}_t$ (cover all cases).

3. Let (X, d) be a metric space and $a \in X$ some fixed point. Prove that the function $f : X \to \mathbb{R}$ given by $f(x) = d(x, a)$ is continuous.

4. Consider the sequence $a_n = 1/n$ (where $n \in \mathbb{Z}_+$) of points on the real line. With respect to which of the following topologies on $\mathbb{R}$ does this sequence converge? When it does, what does it converge to?

   (a) $\mathbb{R}_\ell$ (b) $\mathbb{R}_K$ (c) $\mathbb{R}_{fc}$ (finite complement topology)

5. Let X be a Hausdorff topological space and $A \subset X$ a subset. Prove that A with the subspace topology is Hausdorff.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $\Gamma_f = \{(x, f(x)) | x \in \mathbb{R}\} \subset \mathbb{R}^2$ be its graph. Show that Γ_f is closed in $\mathbb{R}^2$.

7. Let X be a topological space and $A_\alpha \subset X$ a family of subsets. Show that

$$\bigcup_\alpha \overline{A_\alpha} \subset \overline{\bigcup_\alpha A_\alpha}$$

   and provide an example where equality fails.

8. Let X be a topological space and $\Delta : X \to X \times X$ the diagonal function $\Delta(x) = (x, x)$. Prove that Δ is continuous.